TRANSITION MATRICES FOR YOUNG'S REPRESENTATIONS OF THE SYMMETRIC GROUP

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ABSTRACT

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2. DIHEDRAL GROUP D_N

Let $\mathsf{G} = D_n = \langle r, f : r^n = f^2 = 1, f^{-1}rf = r^{-1}\rangle$ and $\mathsf{H} \leq \mathsf{G}$.

2.1 n is odd

Proposition 2.1.1. *(Conrad) Let n be odd, then every subgroup of D_n is conjugate to one of the following:*

- $\langle r^d\rangle$, *where d is a divisor of n.*
- $\langle r^d, f\rangle$, *where d is a divisor of n.*

The character table of D_n is given by:

g	e	$r^i(1 \leq i \leq (n-1)/2))$	f
χ_{triv}	1	1	1
χ_{sign}	1	1	-1
ψ_j	2	$e^{2\pi ij/n} + e^{-2\pi ij/n}$	0

Proposition 2.1.2. *Let χ be the character of the permutation module $\mathbb{C}[\mathsf{H}\backslash\mathsf{G}]$. Then χ decomposes as followed:*

1. $\mathsf{H} = \langle r^d, f\rangle$, *then* $\chi = \chi_{triv} + \sum_j \psi_j, \quad j \in \left\{\frac{n}{d}, \frac{2n}{d}, ..., \frac{d-1}{2}\frac{n}{d}\right\}$
2. $\mathsf{H} = \langle r^d\rangle$, *then* $\chi = \chi_{triv} + \chi_{sign} + 2\sum_j \psi_j, \quad j \in \left\{\frac{n}{d}, \frac{2n}{d}, ..., \frac{d-1}{2}\frac{n}{d}\right\}$

Proof. 1. $\mathsf{H} = \langle r^d, f\rangle$.

First, we count fixed points when $g \in G$ acts on $\mathbb{C}[\mathsf{H}\backslash\mathsf{G}]$. The elements of $\mathsf{H}\backslash\mathsf{G}$ are $\{Hr^a \mid 0 \leq a \leq d-1\}$. For $0 \leq i \leq n-1$, we have

$$(Hr^a)r^i = Hr^a \implies r^i \in H$$

Moreover,

$$(Hr^a)f = Hr^a \implies a = 0$$

Hence,

$$\begin{cases} \chi(r^i) = \begin{cases} d, & \text{if } i \in \{d, 2d..., n\} \\ 0, & \text{otherwise} \end{cases} \\ \chi(f) = 1, \end{cases}$$

Now, we can use inner product of characters to decompose χ:

- $\langle \chi, \chi_{triv}\rangle = 1$

- $\langle \chi, \chi_{sign} \rangle = 0$
- $$\begin{aligned}\langle \chi, \psi_j \rangle &= \frac{1}{|G|}\sum_{g\in G}\chi(g)\psi_j(g)\\ &= \frac{1}{2n}[2d(1+\psi_j(r^d)+\psi_j(r^{2d})+...+\psi_j(r^{\frac{1}{2}(\frac{n}{d}-1)d})]\\ &= \frac{1}{2n}[2d(1+e^{2\pi jd/n}+e^{-2\pi jd/n}+e^{4\pi jd/n}+e^{-4\pi jd/n}+...+e^{\pi(\frac{n}{d}-1)d^2/n}+e^{-\pi(\frac{n}{d}-1)d^2/n})]\end{aligned}$$

  If j is a multiple of $\frac{n}{d}$, then $e^{2\pi kjd/n}+e^{-2\pi kjd/n}=2$ This means

  $$\langle \chi, \psi_j \rangle = \frac{1}{2n}[2d(1+2\frac{1}{2}(\frac{n}{d}-1)d)] = 1$$

  Moreover, j is bounded above by an integer k_{max} such that

  $$k_{max} \le \frac{d}{n}\Big(\frac{n-1}{2}\Big) \implies k = \frac{d-1}{2}$$

  If j is not a multiple of $\frac{n}{d}$, then $\langle \chi, \psi_j \rangle = 0$ as

  $$X = 1+e^{2\pi jd/n}+e^{-2\pi jd/n}+e^{4\pi jd/n}+e^{-4\pi jd/n}+...+e^{\pi(\frac{n}{d}-1)d^2/n}+e^{-\pi(\frac{n}{d}-1)d^2/n}=0$$

  Let $\omega = e^{2\pi jd/n}$. Then

  $$X = 1+\omega+\omega^{n-1}+\omega^2+\omega^{n-2}+...+\omega^{-\frac{1}{2}(\frac{n}{d}-1)}+\omega^{\frac{1}{2}(\frac{n}{d}-1)} = 1+\omega+\omega^2+...+\omega^{\frac{n}{d}-1}$$

  However, $\omega^{n/d}=1$. This means, $X=0$

2. $\mathsf{H} = \langle r^d \rangle$.
   The elements of $\mathsf{H}\backslash\mathsf{G}$ are $\{Hr^a \mid 0 \le a \le d-1\} \cup \{Hfr^a \mid 0 \le a \le d-1\}$. For $0 \le i \le n-1$, we have

   $$(Hr^a)r^i = Hr^a \implies r^i \in H$$

   Moreover,

   $$(Hfr^a)r^i = Hfr^a \implies r^{-i} \in H$$

   And there does not exist any j such that

   $$Hr^a f = Hr^a \quad \text{or} \quad (Hr^a f)f = Hr^a f$$

   Hence, the character χ is given by:

   $$\begin{cases}\chi(r^i) = \begin{cases} d, & \text{if } i \in \{d, 2d..., n\}\\ 0, & \text{otherwise}\end{cases}\\ \chi(f) = 1,\end{cases}$$

□

For example: Let $n = 15$. The character table is given by, where $\omega^i = e^{2\pi i/15} + e^{-2\pi i/15}$

g	e	r	r^2	r^3	r^4	r^5	r^6	r^7	b
χ_{triv}	1	1	1	1	1	1	1	1	1
χ_{sign}	1	1	1	1	1	1	1	1	-1
ψ_3	2	ω^3	ω^6	ω^9	ω^{12}	2	ω^3	ω^6	0
ψ_5	2	ω^5	ω^{10}	2	ω^5	ω^{10}	2	ω^5	0
ψ_6	2	ω^6	ω^{12}	ω^3	ω^9	2	ω^6	ω^{12}	0
$\chi_{\langle r^3,f\rangle}$	3	0	0	3	0	0	3	0	1
$\chi_{\langle r^5,f\rangle}$	5	0	0	5	0	0	5	0	1
$\chi_{\langle r^3\rangle}$	6	0	0	6	0	0	6	0	0
$\chi_{\langle r^5\rangle}$	10	0	0	10	0	0	10	0	0

Consider $H = \langle r^3, f\rangle$. The coset representatives are H, Hr, Hr^2. First we find the projection of χ onto ψ_5. Note that

$$P_5 = \sum_{g\in D_n} \psi_5(g^{-1})g = \sum_{i=0}^{n-1} \omega^{5i} r^i$$

Now projecting P_5 onto H, Hr, Hr^2 gives:

$$\begin{aligned} HP_5 &= \sum_{i=0}^{n-1} \omega^{5i} Hr^i = 5(2H + \omega^5 Hr + \omega^{10} Hr^2) \\ HrP_5 &= \sum_{i=0}^{n-1} \omega^{5i} Hr^i = 5(\omega^{10} H + 2Hr + \omega^5 Hr^2) \\ Hr^2P_5 &= \sum_{i=0}^{n-1} \omega^{5i} Hr^i = 5(\omega^5 H + \omega^{10} Hr + 2Hr^2) \end{aligned}$$

Using row reduce, we obtain the basis:

$$\begin{aligned} v_1 &= H - Hr^2 \\ v_2 &= Hr - Hr^2 \end{aligned}$$

Let $\nu = H - Hr^2$. Then, the lift of ν onto G is given by

$$\begin{aligned} \overline{\nu} =& 1 + r^3 + r^6 + r^9 + r^{12} + f + fr^3 + fr^6 + fr^9 + fr^{12} - \\ & r^2 - r^5 - r^8 - r^{11} - r^{14} - fr^2 - fr^5 - fr^8 - fr^{11} - fr^{14} \end{aligned}$$

$$\text{Then } g\overline{\nu} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

2.2 n is even

Proposition 2.2.1. *(Conrad) Let n be even, then every subgroup of D_n is conjugate to one of the following:*

- $\langle r^d \rangle$, *where d is a divisor of n.*
- $\langle r^d, f \rangle$, *where d is a divisor of n.*
- $\langle r^d, rf \rangle$, *where d is a divisor of n.*

The character table of D_n is given below:

g	e	r^m	$r^i(1 \leq i \leq m-1)$	f	rf
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^m$	$(-1)^i$	1	-1
χ_4	1	$(-1)^m$	$(-1)^i$	-1	1
ψ_j	2	$2(-1)^j$	$e^{2\pi ij/n} + e^{-2\pi ij/n}$	0	0

Proposition 2.2.2. *Let χ be the character of the permutation module $\mathbb{C}[\mathsf{H}\backslash\mathsf{G}]$. Then χ decomposes as followed:*

1. $\mathsf{H} = \langle r^d, f \rangle$
   - *d is odd, then* $\chi = \chi_1 + \sum_j \psi_j, \quad j \in \left\{ \frac{n}{d}, \frac{2n}{d}, \ldots, \frac{d-1}{2}\frac{n}{d} \right\}$
   - *d is even, then* $\chi = \chi_1 + \chi_3 + \sum_j \psi_j, \quad j \in \left\{ \frac{n}{d}, \frac{2n}{d}, \ldots, \frac{d-2}{2}\frac{n}{d} \right\}$
2. $\mathsf{H} = \langle r^d, rf \rangle$
   - *d is odd, then* $\chi = \chi_1 + \sum_j \psi_j, \quad j \in \left\{ \frac{n}{d}, \frac{2n}{d}, \ldots, \frac{d-1}{2}\frac{n}{d} \right\}$

- *d is even, then* $\chi = \chi_1 + \chi_4 + \sum_j \psi_j, \quad j \in \left\{\frac{n}{d}, \frac{2n}{d}, ..., \frac{d-2}{2}\frac{n}{d}\right\}$

3. $\mathsf{H} = \langle r^d \rangle$

   - *d is odd, then* $\chi = \chi_1 + \chi_2 + 2\sum_j \psi_j, \quad j \in \left\{\frac{n}{d}, \frac{2n}{d}, ..., \frac{d-1}{2}\frac{n}{d}\right\}$
   - *d is even, then* $\chi = \chi_1 + \chi_2 + \chi_3 + \chi_4 + 2\sum_j \psi_j, \quad j \in \left\{\frac{n}{d}, \frac{2n}{d}, ..., \frac{d-2}{2}\frac{n}{d}\right\}$

Proof. 1. $\mathsf{H} = \langle r^d, f \rangle$.

The elements of $\mathsf{H}\backslash\mathsf{G}$ are $\{Hr^a \mid 0 \le a \le d-1\}$. For $0 \le i \le n-1$, we have

$$(Hr^a)r^i = Hr^a \implies r^i \in H$$

Moreover,

$$(Hr^a)f = Hr^a \implies (Hr^a)^2 = H$$

If d is even, then $a = 0, d/2$; however, if d is odd, then $a = 0$. Moreover, consider,

$$H(r^a)(rf) = Hr^a \implies Hr^{d-a} = Hr^{a+1} \implies a = \frac{d-1}{2} \quad \text{and} \quad a \in \mathbb{N}$$

This is only possible if d is odd. Hence, $\chi(rf) = 0$ if n is even and $\chi(rf) = 1$ if n is odd. Hence, if d is odd, we have

$$\begin{cases} \chi(r^i) = \begin{cases} d, & \text{if } i \in \{d, 2d..., n\} \\ 0, & \text{otherwise} \end{cases} \\ \chi(f) = 1 \\ \chi(rf) = 1 \end{cases}$$

If d is even, then

$$\begin{cases} \chi(r^i) = \begin{cases} d, & \text{if } i \in \{d, 2d..., n\} \\ 0, & \text{otherwise} \end{cases} \\ \chi(f) = 2 \\ \chi(rf) = 0 \end{cases}$$

□

2.2.1 Where to find the smallest frame?

Consider D_{15}.

V	V_{triv}	V_{sign}	V_1	V_2	V_3	V_4	V_5	V_6	V_7
H	$\langle r^3, f \rangle$	$\langle r^3 \rangle$	$\langle f \rangle$	$\langle f \rangle$	$\langle r^5, f \rangle$	$\langle f \rangle$	$\langle r^3, f \rangle$	$\langle r^5, f \rangle$	$\langle f \rangle$
$\lvert H \rvert$	10	5	2	2	6	2	10	6	2
$\lvert$frame$\rvert$	3	6	15	15	5	15	3	5	15

Consider D_{24}.

V	W_1	W_2	W_3	W_4
H	$\langle r^2, f\rangle$	$\langle r^2\rangle$	$\langle r^2, f\rangle$	$\langle r^2, rf\rangle$
$\lvert H\rvert$	24	12	24	12
$\lvert$frame$\rvert$	2	4	2	4

V	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}
H	$\langle f\rangle$	$\langle r^{12}, f\rangle$	$\langle r^8, f\rangle$	$\langle r^6, f\rangle$	$\langle f\rangle$	$\langle r^4, f\rangle$	$\langle f\rangle$	$\langle r^3, f\rangle$	$\langle r^8, f\rangle$	$\langle r^{12}, f\rangle$	$\langle f\rangle$
$\lvert H\rvert$	2	4	6	8	2	12	2	16	6	4	2
$\lvert$frame$\rvert$	24	12	8	6	24	4	24	3	8	12	24

To construct the smallest frame, we want the largest subgroup.

Proposition 2.2.3. *Let V_j be a 2-dimensional submodule of D_n with character χ_j as defined above. The largest subgroup such that $V_j \subseteq \mathbb{C}[\mathsf{H}\backslash\ D_n]$ is $H = \langle r^k, f\rangle$, where $k = n/\gcd(n, j)$.*

2.3 Frame Examples

2.3.1 Dihedral Group

Consider D_3 and $H = \langle f\rangle$. Then, $\mathbb{C}[\mathsf{H}\backslash\mathsf{D}_n] = sp\{H, H_r, H_r^2\}$. Let χ be the character of $\mathbb{C}[\mathsf{H}\backslash\mathsf{D}_n]$. Then, $\chi = \chi_{triv} + \psi_1$. The character table is given by

g	e	r	f
χ_{triv}	1	1	1
χ_{sign}	1	1	-1
ψ_1	2	-1	0
χ	3	0	1

Now the projector of $\mathbb{C}[\mathsf{H}\backslash\mathsf{D}_n]$ onto V_1 is given by:

$$P_1 = \sum_{g\in D_3} \psi_1(g^{-1})g = 2e - (r + r^2)$$

The Frame for $\mathbb{C}[\mathsf{H}\backslash\mathsf{D}_n]$ is given by

$$\{\overline{v_t} = P_1(Ht)|t \in R_H\} = \{\overline{v_e}t|t \in R_H\}$$

Hence,

$$\overline{v_e} = P_1(H) = 2H - (Hr + Hr^2)$$
$$\overline{v_r} = P_1(H) = 2Hr - (H + Hr^2)$$
$$\overline{v_{r^2}} = P_1(H) = 2Hr^2 - (H + Hr)$$

Let v_t be the lift of $\overline{v_t}$ onto $\mathbb{C}[D_n]$. Let $\Phi_t = \{gv_t | g \in G\} = \{sv_t | s \in L_H\}$. Note that $L_H = \{1, r, r^2\}$.

Hence,

$$\Phi_{r^2} = \{e.v_{r^2}, rv_{r^2}, r^2 v_{r^2}\}$$

, where

$$\begin{aligned} ev_{r^2} &= 2r^2 + 2fr^2 - e - f - r - fr \\ rv_{r^2} &= 2e + 2fr - r - f - r^2 - fr^2 \\ r^2 v_{r^2} &= 2f + 2r - e - fr - fr^2 - r^2 \end{aligned}$$

2.3.2 Symmetric Group

Let $G = S_n$ and $H = S_{n-1}$. Then,

$$H \backslash \mathsf{G} = \{H(i,n) | 1 \leq i \leq n\}$$

, where $H(i,n) = \{\sigma \in S_n | \sigma(i) = n\}$

Now note that $\mathbb{C}[\mathsf{H}\backslash\mathsf{G}]$ decomposes into the trivial module $V_{trivial}$ and the zero-sum module V_{0-sum}. Let $\overline{v}_i = H(i,n)$. The frame for V_{0-sum} in $H\backslash\mathsf{G}$ is given by

$$\overline{\Phi} = \{\overline{w_i} | 1 \leq i \leq n\}$$

, where $\overline{w_i} = nv_i - \sum_{j=1}^{n} v_j$.

Note that

$$\overline{w}_i \sigma = (n\overline{v_i} - \sum_{j=1}^{n} \overline{v_j})\sigma = n\overline{v}_{\sigma^{-1}(i)} - \sum_{j=1}^{n} v_{\sigma^{-1}(j)} = \overline{w}_{\sigma^{-1}(i)}$$

Hence, we can write the frame simply as

$$\overline{\Phi} = \{\overline{w_1}\sigma | \sigma \in S_n\}$$

Now, we lift $\overline{w}_i$ onto $\mathbb{C}[\mathsf{G}]$.

$$w_i = n \sum_{\sigma \in S_n, \sigma(i)=n} \sigma - \sum_{\sigma \in S_n} \sigma$$

Now to find the frame,

$$\Phi_i = \{\sigma w_i | \sigma \in S_n\} = \{\sigma w_i | \sigma \in L_H\}$$

Note that $L_H = \{(j,n) \in S_n | 1 \leq j \leq n\}$. Hence,

$$(j,n)w_i = n \sum_{\sigma \in S_n, \sigma(i)=n} (j,n)\sigma - \sum_{\sigma \in S_n} (j,n)\sigma = n \sum_{\sigma \in S_n, \sigma(i)=j} \sigma - \sum_{\sigma \in S_n} \sigma$$

Hence,

$$\overline{\Phi_i} = \left\{ u_{ij} = n \sum_{\sigma \in S_n, \sigma(i)=j} \sigma - \sum_{\sigma \in S_n} \sigma | 1 \leq j \leq n \right\}$$

Example

Let $G = S_3$. Then,

$$\overline{\Phi_2} = \{u_{21}, u_{22}, u_{23}\}$$

, where

$$\begin{aligned} u_{21} &= 2(132) + 2(12) - (23) - (12) - (1) - (123) \\ u_{22} &= 2(1) + 2(13) - (23) - (132) - (12) - (123) \\ u_{23} &= 2(23) + 2(123) - (1) - (12) - (132) - (13) \end{aligned}$$

Note that $\overline{\Phi_2} \cong \overline{\Phi_{r^2}}$ by letting, $r \to (123)$ and $f \to (23)$.

2.4 Generalized Symmetric Group

(Tom) The irreducible representations of $G_{n,r}$ are indexed by *r-partitions* of n, which are ordered r-tuples of the form $\underline{\gamma} = (\gamma^{(1)} \ldots, \gamma^{(r)})$ such that each $\gamma^{(i)}$ is a partition and $|\gamma^{(1)}| + \cdots + |\gamma^{(r)}| = n$.

An ordered set partition of shape $\underline{\gamma}$ is a set partition of $\{1, 2, .., n\}$ into blocks of $\gamma_j^{(i)}$.

2.4.1 Young Subgroup

let $\Pi_{\underline{\gamma}}$ denotes the set of ordered set of partitions of shape $\underline{\gamma}$.

Definition 2.4.1. *A Type-1 Young subgroup of shape $\underline{\gamma}$ is a subgroup G_π for any $\pi \in \Pi_{\underline{\gamma}}$ such that it is a stabilizer for $\pi^{(1)}$ and a stabilizer for $\pi^{(k)}$, $k > 1$, up to ξ^r.*

In other words,

$$G_\pi = \bigoplus_{j=1}^{\ell(\gamma^i)} S_{\pi_j^{(1)}} \oplus \bigoplus_{i=2}^{r} \bigoplus_{j=1}^{\ell(\gamma^i)} G_{\pi_j^{(i)}}$$

, where $G_{\pi_j^{(i)}}$ is the wreath product of the symmetric group $S_{\pi_j^{(1)}}$ with C_r.

Proposition 2.4.2. *(Himmet Can Prop 3.20 300) Let $M_{\underline{\gamma}} = \mathbb{C}[G_\pi \backslash H]$. Then the mutiplicity of the irreducible $G_{\underline{\gamma}}$ appearing in $M_{\underline{\gamma}}$ is exactly one.*

Hyperoctahedral Group

We denote $G_{n,2} = H$. Let $M_{\underline{\gamma}} = \mathbb{C}[G_\pi \backslash H]$. Also let $\underline{\gamma} = (\gamma^{(1)}, \gamma^{(2)}) = (\mu, \nu)$.

Proposition 2.4.3. *(Gessinger and Kinch III.5 12)*

$$M_{(\mu,\nu)} = \bigoplus_{(\alpha,\beta)\vdash n} \sum_{\lambda \subseteq \mu} K_{\lambda,\alpha} K_{(\mu-\lambda)\cup\nu,\beta} H_{\alpha,\beta}$$

,where $\lambda \subseteq \mu$ if $\lambda(i) \leq \mu(i)$ for all i, and $\lambda \cup \mu$ is a concatenation of the two partitions, and rearranging into decreasing sequence.

Proposition 2.4.4. *(Gessinger and Krinch II.3 9) The multipicity of $H_{(\mu,\nu)}$ appearing in $M_{(\mu,\nu)}$ is 1.*

Definition 2.4.5. *A Type-2 Young subgroup of shape $\underline{\gamma}$ is a subgroup G_π for any $\pi \in \Pi_{\underline{\gamma}}$ such that it is a stabilizer for all $\pi^{(i)}$, $1 \leq i \leq r$.*

In other words,

$$G'_\pi = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{\ell(\gamma^i)} G_{\pi_j^{(i)}}$$

Proposition 2.4.6. *(Puttaswamaiah Corrollary 35) Let $M_{\underline{\gamma}} = \mathbb{C}[G'_\pi \backslash H]$. Then the mutiplicity of the irreducible $G_{\underline{\gamma}}$ appearing in $M_{\underline{\gamma}}$ is exactly one.*

2.5 Propositions

Proposition 2.5.1. *Lifting and Projecting commute:*

Proof. The projector onto V is given by

$$P_V = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g$$

- LHS: First we project e_H onto V:

$$\overline{v_0} = P_V(e_H) = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) e_{Hg} = \sum_{i=1}^{m} \sum_{x \in Hg_i} \frac{\chi(x^{-1})}{|G|} e_{Hg_i} = \sum_{i=1}^{m} \lambda_i e_{Hg_i}$$

  , where $\lambda_i = \sum_{x \in Hg_i} \frac{\chi(x^{-1})}{|G|}$ and g_i indexes the right coset representative.

  Now, we lift onto $\mathbb{C}[G]$:

$$v_0 = \sum_{i=1}^{m} \sum_{h \in H} \lambda_i e_{hg_i} = \sum_{g \in G} \lambda_g e_g$$

  , where $\lambda_g = \sum_{x \in Hg} \frac{\chi(x^{-1})}{|G|}$

- RHS: First we lift e_H onto $C[G]$

$$w = \sum_{h \in H} e_h$$

  Now, projecting onto V gives:

$$
\begin{aligned}
w_0 &= \sum_{h\in H}\frac{1}{|G|}\sum_{g\in G}\chi(g^{-1})e_{hg}\\
&= \sum_{h\in H}\frac{1}{|G|}\sum_{i=1}^{m}\sum_{h'\in H}\chi((h'g_i)^{-1})e_{h.h'g_i}\\
&= \frac{1}{|G|}\sum_{i=1}^{m}\sum_{h'\in H}\chi((h'g_i)^{-1})\sum_{h\in H}e_{h.h'g_i}\\
&= \frac{1}{|G|}\sum_{i=1}^{m}\sum_{h'\in H}\chi((h'g_i)^{-1})\sum_{h\in H}e_{hg_i}\\
&= \sum_{i=1}^{m}\sum_{h'\in H}\frac{\chi((h'g_i)^{-1})}{|G|}\sum_{h\in H}e_{hg_i}\\
&= \sum_{i=1}^{m}\sum_{h\in H}\lambda_i e_{hg_i} = \sum_{g\in G}\lambda_g e_g
\end{aligned}
$$

□

Definition 2.5.2. *Let $H \leq G$. Define a lifting of vectors from $\mathbb{C}[G\backslash H]$ onto $\mathbb{C}[G]$ as*

$$\phi_H : e_H \to \sum_{h\in H} e_h$$

Proposition 2.5.3. *Let P_χ be a right G-homormorphism projection onto irreducible module with character χ. Then,*

$$\phi_H P_\chi(H) = P_\chi \phi_H(H)$$

Proof.

$$
\begin{aligned}
\phi_H P_\chi(H) &= \phi_H\Big(\frac{1}{|G|}\sum_{g\in G}\chi(g^{-1})Hg\Big)\\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{h\in H}\chi(g^{-1})hg\\
P_\chi\phi_H(H) &= P_\chi\Big(\sum_{h\in H}h\Big)\\
&= \frac{1}{|G|}\sum_{h\in H}\sum_{g\in G}\chi(g^{-1})hg
\end{aligned}
$$

Since the summation sign commutes, $\phi_H P_\chi(H) = P_\chi \phi_H(H)$

□

2.6 Irreducible Frames are tight

2.6.1 Frame Operator

Definition 2.6.1. *Let J be an indexing set. Let $(f_j)_{j\in J}$ in $\mathcal{H}$.*

1. *The synthesis operator V is a linear map from $\ell_2(J) \to \mathcal{H}$ given by*

$$V := a \to \sum_{j\in J} a_j f_j$$

In matrix form, and representing a by a vector:

$$V = \begin{bmatrix} | & | & & | \\ f_1 & f_2 & \dots & f_n \\ | & | & & | \end{bmatrix}$$

2. *The analysis operator V^T is a linear map from $\mathcal{H} \to \ell_2(J)$ given by*

$$V^T := f \to (\langle f, f_j \rangle)_{j\in J}$$

In matrix form:

$$V^T = \begin{bmatrix} — & f_1^T & — \\ — & f_2^T & — \\ & \vdots & \\ — & f_n^T & — \end{bmatrix}$$

3. *The frame operator S is defined by $S = VV^T$. Hence*

$$S : f \to \sum_{j\in J} \langle f, f_j \rangle f_j$$

Note that $trace(S) = \sum_{j\in J} ||f_j||^2$. This is because

$$trace(S) = trace(VV^T) = trace(V^T V) = \sum_{j\in J} f_j^T f_j = \sum_{j\in J} \langle f_j, f_j \rangle = \sum_{j\in J} ||f_j||^2$$

Proposition 2.6.2. *A finite sequence $(f_j)_{j\in J}$ is a tight frame with bound A if and only if*

$$S = VV^T = AI_{\mathcal{H}}$$

Proof. $\rightarrow (f_j)_{j\in J}$ is a tight frame. Then for all $f \in \mathcal{H}$

$$f = \frac{1}{A}\sum_{j\in J}\langle f, f_j\rangle f_j$$

Then,

$$Sf = \sum_{j\in J}\langle f, f_j\rangle f_j = Af$$

$\leftarrow$ Let $S = AI_{\mathcal{H}}$.
Then

$$\sum_{j\in J}||f_j||^2 = trace(S) = Adim(\mathcal{H})$$

□

Proposition 2.6.3. *Let $\langle,\rangle$ be G-invariant. The frame operator S commutes with the action of $g \in G$:*

$$S(gf) = gS(f), \forall g \in G, f \in \mathcal{H}$$

Proof. Let $\Phi = (\phi_g)_{g\in G}$ be a group frame with a frame operator S:

$$S(hf) = \sum_{h\in G}\langle hf, \phi_g\rangle\phi_g = h\sum_{g\in G}\langle f, h^{-1}\phi_g\rangle h^{-1}\phi_g = h\sum_{g\in G}\langle f, \phi_{h^{-1}g}\rangle\phi_{h^{-1}g} = hS(f)$$

□

Theorem 2.6.4. *(Waldron 10.5) Let v be any non-zero vector in an irreducible module V. Then $(gv)_{g\in G}$ is a tight frame.*

Let S be the frame operator for $(gv)_{g\in G}$. S has positive eigenvalues, let that be λ, and the corresponding eigenvector w.

$$S(g.w) = gS(w) = g\lambda W = \lambda(g.w)$$

Since eigenspaces are submodules, and V is irreducible, then we must have $(gw)_{g\in G}$ spanning V. Hence S= $\lambda I_{\mathcal{V}}$. By proposition 1.6.2, $(gv)_{g\in G}$ is a tight frame.