ZETA FUNCTIONS OF CRYSTALLOGRAPHIC GROUPS AND ANALYTIC CONTINUATION

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1. *Introduction*

A finitely generated residually-finite group G has only finitely many subgroups of each finite index. Let the number of subgroups of index n be $a_n(G)$. The study of this sequence and its asymptotic behaviour has attracted considerable interest over the past fifteen years, and the subject has come to be known as *subgroup growth* in contrast to the dual notion of word growth.

One tool which has proved useful in analysing the behaviour of this invariant is the zeta function of a group. This is defined to be the Dirichlet series with coefficients $a_n(G)$:

$$\zeta_G(s) = \sum_{n \in \mathbb{N}} a_n(G) n^{-s}.$$

If the group G is infinite cyclic, then $\zeta_{\mathbb{Z}}(s) = \zeta(s)$, the classical Riemann zeta function. More generally, a free abelian group of rank n has $\zeta_{\mathbb{Z}^n}(s) = \prod_{i=1}^{n} \zeta(s-i+1)$. The study of these Dirichlet series was initiated in [**8**] and [**19**] for finitely generated nilpotent groups. One advantage of working with nilpotent groups is that the zeta function enjoys an Euler product in terms of natural local factors defined for each prime p:

$$\zeta_G(s) = \prod_{p \text{ prime}} \zeta_{G,p}(s)$$

where

$$\zeta_{G,p}(s) = \sum_{i=0}^{\infty} a_{p^i}(G) p^{-is}.$$

We can equally well restrict our zeta functions to count just normal subgroups of finite index. In this case we will denote the zeta function by $\zeta_G^{\triangleleft}(s)$ and its corresponding local factors by $\zeta_{G,p}^{\triangleleft}(s)$.

We can consider the zeta function as a purely formal gadget. However for groups for which the coefficients $a_n(G)$ grow polynomially, this function converges on some right half of the complex plane. Such groups, known as PSG groups (for polynomial subgroup growth) have been classified as the virtually soluble groups of finite rank (see [**14**]). One of the key features of classical zeta functions of number theory is the possibility of extending the zeta function meromorphically beyond the abscissa of convergence. In this paper we prove the following.

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THEOREM 1.1. *Let G be a finite extension of a free abelian group of finite rank. Then $\zeta_G(s)$ and $\zeta_G^{\triangleleft}(s)$ can be extended to meromorphic functions on the whole complex plane.*

This theorem should be contrasted with non-abelian examples where natural boundaries abound (see [**6**]). The theorem, proved in § 2, extends the work of Bushnell and Reiner [**1**, **2**] to show that the zeta function of a finite extension of an abelian group is always made out of the classical examples of Hey [**9**] of zeta functions of central simple algebras.

A number of people have asked what relationship the zeta functions of groups have to the work of Bushnell and Reiner; we hope that § 2 will provide an answer.

One of the interesting questions about these zeta functions which has not been considered is the explicit effect on the function of extending or descending by a finite group. As an example compare the infinite cyclic group $\mathbb{Z}$ where $\zeta_{\mathbb{Z}}(s) = \zeta(s)$ and the infinite dihedral group D_∞ where $\zeta_{D_\infty}(s) = 2^{-s}\zeta(s) + \zeta(s-1)$. So extending by a finite group (here C_2) therefore has quite a subtle effect on the lattice of subgroups even to the extent of changing the rate of polynomial growth.

In this paper we have calculated concrete examples to show the effect of extending $\mathbb{Z}^2$ by a finite group. It reveals how sensitive the zeta function is as the nature of the poles varies dramatically.

We have focused on the plane crystallographic groups. There are seventeen isomorphism types of these objects, which are sometimes called the *Fedorov* or *wallpaper* groups. Each of them is an extension of a free abelian group of rank 2 by a small finite soluble group. These groups have fairly standard names among the crystallographic community and we conform to these. For a discussion of presentations of these groups, and their mutual involvement, see [**3**]. It is an important fact that subgroups of finite index in plane crystallographic groups are plane crystallographic. This reduces the amount of work involved in our calculations.

The proof of Theorem 1.1 provides quite an explicit approach to calculate these zeta functions but relies on ignoring the effect of finitely many 'bad' primes. This is acceptable in proving a result like Theorem 1.1. However in the case of the plane crystallographic groups, we exploit a slightly more sensitive method to calculate these functions which provides a complete calculation of the zeta functions without needing to throw away anything. This method has its origins in [**5**] where the first author showed how to prove rationality of the local zeta functions of finite extensions of groups and the calculations here can be considered as an implementation of the theoretical ideas developed in [**5**]. The calculations are somewhat intricate, though repetitive. We therefore present the results and provide sample calculations to illustrate the method. The interested reader should consult the second author's thesis for more details [**15**].

Despite the considerable theoretical development concerning these zeta functions (see, for example, [**5**, **12**] etc.), the only explicit examples of these functions prior to the examples presented here are some finitely generated torsion-free nilpotent groups of class 2 (see [**8**, **19**, **7**]) and the congruence subgroups in $\mathrm{SL}_2(\mathbb{Z}_p)$ (see [**11**]). We therefore hope that our examples will contribute to this list.

Our examples already provide some interesting corollaries and answers to some previously open questions.

The first question concerns finding examples of groups with the same zeta function, which will give us some indication as to how good an invariant the zeta function is. We make the following definition after Sunada *et al.*

DEFINITION 1.2. Call two groups G and H *isospectral* if $\zeta_G(s) = \zeta_H(s)$.

Following Sunada's example we might ask 'Can we hear the shape of a drum?', that is, what is the zeta function telling us about the shape of the group?

In [**8**] examples of zeta functions are calculated to show that if two finitely generated torsion-free nilpotent groups of class 2 and Hirsch length 7 or smaller are isospectral then they must be commensurable, that is, coincide on some subgroup of finite index.

However, by cardinality considerations, we always know that there are uncountably many groups with the same zeta function. For example, there are uncountably many p-adic analytic groups all of whose zeta functions are rational by [**5**] but only countably many rational functions. In [**10**] Ilani presented a countable number of examples of isospectral pro-p groups: let P_k denote the closed subgroup of the multiplicative group $1 + p\mathbb{Z}_p$ generated by $(1+p)^{p^k}$ and put $M_k = P_k \times \mathbb{Z}_p$ where P_k acts on $\mathbb{Z}_p$ by multiplication. Then M_k for $k \in \mathbb{N}$ are non-isomorphic isospectral pro-p groups with zeta function $\zeta_p(s)\zeta_p(s-1)$. This follows because all subgroups of finite index are two-generated.

Two groups with the same profinite completion will obviously have the same zeta function because the lattice of subgroups of finite index in a group coincides with that of its profinite completion. So it is of interest to observe examples of groups with non-isomorphic profinite completions which nonetheless have the same zeta function. Our analysis of the crystallographic groups reveals two such groups.

THEOREM 1.3. *The groups* $\mathbf{p1} = \langle x, y \,|\, [x, y] \rangle$ *and* $\mathbf{pg} = \langle x, y, t \,|\, [x, y], t^2 = y, x^t = x^{-1} \rangle$ *are isospectral but have non-isomorphic lattices of subgroups of finite index.*

Note that having different lattices of subgroups of finite index is even stronger than having non-isomorphic profinite completions. For example, there are two non-isomorphic groups of order 256 which have isomorphic subgroup lattices (E. A. O'Brien, personal communication).

Avinoam Mann brought to our attention the following explanation of the isospectral behaviour of Theorem 1.3 which is originally due to Mednykh [**16**]. The groups **p1** and **pg** are the oriented and unoriented versions of a surface group of genus 1. More generally the oriented and unoriented surface groups of genus n are generated by $x_1, y_1, \ldots, x_n, y_n$ subject to a single relator. In the case of oriented groups this is $[x_1, y_1] \ldots [x_n, y_n] = 1$; and for unoriented groups the relation is $x_1^2 y_1^2 \ldots x_n^2 y_n^2 = 1$. These two groups are always isospectral for the following reason: subgroups of finite index are determined by homomorphisms into S_n which in turn are determined by the number of solutions in S_n of the equation given by the relator of each group. These two equations always have the same number of solutions because the irreducible characters of S_n are all real. More specifically for **p1** and **pg** the number of elements in S_n commuting with any element is the same as the number of elements inverting it. Note that when the genus is greater than 1, subgroups grow too fast for the zeta function to converge.

The group **pg** also provides an example of a non-nilpotent group whose zeta function enjoys an Euler product.

THEOREM 1.4. *The group* $\mathbf{pg} = \langle x, y, t \,|\, [x, y], t^2 = y, x^t = x^{-1} \rangle$ *is not nilpotent but* $\zeta_{\mathbf{pg}}(s) = \prod_{p \text{ prime}} \zeta_{\mathbf{pg},p}(s)$.

This Euler product is somewhat accidental and follows from Theorem 1.3 since the zeta function coincides with the zeta function of the abelian group **p1** which does have an Euler product. This discovery prompted a search for other examples of the same phenomenon, and we found **pm** and $\mathbf{p1} \times C_2$ (though the last group is not plane crystallographic). The proofs of Theorems 1.3 and 1.4 follow by inspection of the results contained in § 4.

The non-nilpotent examples should be contrasted with the following result for finite groups.

PROPOSITION 1.5. *If G is finite then $\zeta_G(s) = \prod_{p \text{ prime}} \zeta_{G,p}(s)$ if and only if G is nilpotent.*

Proof. Let $|G| = p_1^{n_1} \ldots p_r^{n_r}$. Looking at the subgroups of index $|G|$, we see from the Euler product that $a_{p_i^{n_i}}(G) = 1$ for each i. Therefore the number of Sylow p_i-subgroups is

$$a_{|G|/p_i^{n_i}}(G) = \prod_{j \neq i} a_{p_j^{n_j}}(G) = 1.$$

Hence there is a unique Sylow p_i-subgroup for each $i = 1, \ldots, r$, which implies that G is nilpotent.

We would still be interested in providing an example of a profinite group to answer the following question (in the negative, presumably).

QUESTION. If $\zeta_G(s) = \prod_{p \text{ prime}} \zeta_{G,p}(s)$ then is G virtually pro-nilpotent?

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2. *Meromorphic continuation*

One of the important properties that one would like to prove about our zeta functions is that they can be meromorphically continued to the whole complex plane. In general this is not possible as explained in [**6**]. However in this section we prove Theorem 1.1, that the zeta function of a finite extension of a free abelian group does admit meromorphic continuation.

It is instructive to consider the corresponding situation for the zeta function of a number field K. If K/k is an abelian extension, we can write the zeta function $\zeta_K(s)$ of K as a product of L-functions of the base field k:

$$\zeta_K(s) = \zeta_k(s) \prod_{\chi \neq 1} L(s, \chi). \tag{2.1}$$

Here the second product is taken over those characters χ of the idele classes of k which are trivial on the class group H corresponding to the abelian extension K/k. The evaluation of this product at $s = 1$ gives the class number formula. Note that L-functions are also at the heart of proving Dirichlet's Theorem on primes in arithmetical progressions.

What makes the product (2.1) so attractive is that the right-hand side is entirely defined in terms of the field k. This is the strength of class field theory at work which relates abelian extensions of k to the structure of the idele class group of k.

Although we cannot hope for anything as beautiful as this in the theory of zeta functions of groups, it nevertheless may be possible to write the zeta function of a finite extension of a group as some sort of twisted zeta function of the base group, perhaps using characters of the finite quotient. This is possible in some sense when the base group is an abelian group, as we now illustrate.

We first make a definition of various partial zeta functions as in [**5**]. Let E be a group with a finite index normal subgroup T and put $P = E/T$. For each subgroup $T \leqslant E_* \leqslant E$, put

$$\zeta_E^{E_*}(s) = \sum_{H \in \mathscr{H}(E_*)} |E_* : H|^{-s},$$

$$\zeta_{E,p}^{E_*}(s) = \sum_{H \in \mathscr{H}_p(E_*)} |E_* : H|^{-s},$$

$$\zeta_E^{E_*, \lhd}(s) = \sum_{H \in \mathscr{H}^{\lhd}(E_*)} |E_* : H|^{-s},$$

$$\zeta_{E,p}^{E_*, \lhd}(s) = \sum_{H \in \mathscr{H}_p^{\lhd}(E_*)} |E_* : H|^{-s},$$

where

$$\mathscr{H}(E_*) = \{H \leqslant E\colon TH = E_*\},$$

$$\mathscr{H}_p(E_*) = \{H \in \mathscr{H}(E_*)\colon H \text{ has } p\text{-power index in } E_*\},$$

$$\mathscr{H}^{\lhd}(E_*) = \{H \lhd E\colon TH = E_*\},$$

$$\mathscr{H}_p^{\lhd}(E_*) = \{H \in \mathscr{H}^{\lhd}(E_*)\colon H \text{ has } p\text{-power index in } E_*\}.$$

2.1. *Counting subgroups*

Since

$$\zeta_E(s) = \sum_{T \leqslant E_* \leqslant E} |E : E_*|^{-s} \zeta_E^{E_*}(s),$$

$$\zeta_{E,p}(s) = \sum_{T \leqslant E_* \leqslant_p E} |E : E_*|^{-s} \zeta_{E,p}^{E_*}(s),$$

to prove meromorphic continuation it suffices to prove it for the partial zeta functions $\zeta_E^{E_*}(s)$. (Here $\leqslant_p$ means a subgroup of p-power index.) We have a similar situation for the normal zeta function $\zeta_E^{\triangleleft}(s)$ which we shall come back to later in § 2.2. For now we focus on $\zeta_E^{E_*}(s)$. We may suppose, without loss of generality, that $E_* = E$.

The following proposition shows how, when T is abelian, we can write the zeta function of a finite extension as a sort of L-function of the base group.

Proposition 2.1. *We have*

$$\zeta_E^E(s) = \sum_{T_* \in \mathscr{H}^{\triangleleft}(T)} |T : T_*|^{-s} |\mathrm{Der}(P, T/T_*)| \delta_{T_*}$$

where $\delta_{T_*} = 1$ *if* T/T_* *has a complement in* E/T_* *and* $\delta_{T_*} = 0$ *otherwise. (Recall that* $\mathscr{H}^{\triangleleft}(T) = \{T_* \leqslant T \colon T_* \triangleleft E\}$.)

Proof. If $H \leqslant E$ and $HT = E$ then $T_* = H \cap T$ is a normal subgroup of E. For each $T_* \in \mathscr{H}^{\triangleleft}(T)$ we want to count how many complements H there are for T/T_* in E/T_*. If there is one such complement then there are precisely $|\mathrm{Der}(P, T/T_*)|$ such complements. For a proof of this we refer to Proposition 1 of Chapter 3 of [**18**]. However, since the finite extension of T may not split, it is not always the case that complements exist. We shall see below that we can ignore finitely many primes, in which case we are always guaranteed a complement so the factor δ_{T_*} will not trouble us for long.

We begin by showing that $\zeta_E^E(s)$ has an Euler product, even though $\zeta_E(s)$ generally does not. Since E is not nilpotent, it is not obvious at first sight that $\zeta_E^E(s)$ should have an Euler product.

Proposition 2.2.

$$\zeta_E^E(s) = \prod_{p \text{ prime}} \zeta_{E,p}^E(s).$$

Proof. We can replace E by its profinite completion $\widehat{E}$ since the lattice of subgroups remains the same. Note that $\widehat{E}$ is a finite extension of $\widehat{T} = (\widehat{\mathbb{Z}})^d = (\prod_p \mathbb{Z}_p)^d = \prod_p \widehat{T}_p$ by P. Define $T_{p'} = \prod_{q \neq p} \widehat{T}_q$. If H is a subgroup of index $n = p_1^{a_1} \dots p_k^{a_k}$ in $\widehat{E}$ such that $H \cdot \widehat{T} = \widehat{E}$ then $H \cdot T_{p_i'}$ is a subgroup of index $p_i^{a_i}$ in $\widehat{E}$ and $H \cdot T_{p_i'} \cdot \widehat{T} = \widehat{E}$. Hence

$$H \mapsto (H \cdot T_{p_1'}, \dots, H \cdot T_{p_k'}) \tag{2.2}$$

defines a map between the set of subgroups H of index n in $\widehat{E}$ such that $H \cdot \widehat{T} = \widehat{E}$ and the product of the k sets of subgroups $(H_1, \dots, H_k)$ of indices, respectively, $p_1^{a_1}, \dots, p_k^{a_k}$ in $\widehat{E}$ and $H_i \widehat{T} = \widehat{E}$. This map is injective since $H = \bigcap_{p_i} H \cdot T_{p_i'}$. We must show finally that this is also a surjection, that is, if $H_i \widehat{T} = \widehat{E}$ for $i = 1, \dots, k$ then $H\widehat{T} = \widehat{E}$ where $H = H_1 \cap \dots \cap H_k$. Now

$$\begin{aligned} |\widehat{E} : (H_1 \cap \dots \cap H_k)\widehat{T}| &= |H_1 \widehat{T} : (H_1 \cap \dots \cap H_k)\widehat{T}| \\ &= |H_1 : (H_1 \cap \dots \cap H_k)\widehat{T} \cap H_1|. \end{aligned}$$

This last index divides $|H_1 : (H_1 \cap \ldots \cap H_k)|$, which in turn divides

$$|H_1 \cdot (H_2 \cap \ldots \cap H_k) : H_1|$$

which is a power of p_1. But a similar analysis with H_2 implies then that

$$(H_1 \cap \ldots \cap H_k)\widehat{T} = \widehat{E}.$$

Hence the map in (2.2) is a bijection which implies the Euler product claimed in the statement of Proposition 2.2.

We show now that the zeta function of E is a variation on a classical zeta function first considered by Hey and Eichler and later by Solomon, Bushnell and Reiner.

We recall the definition of Solomon's zeta function. Throughout, R is a Dedekind ring with quotient field K. In the global case, K is an algebraic number field and R is the ring of integers of K. In the local case, K is the completion of a number field at some non-Archimedean place and R is the valuation ring in K. Let A denote a finite-dimensional semisimple K-algebra. We denote by Λ an order in A. This means that Λ is a subring of A, finitely generated as an R-module and containing a K-basis of A.

Solomon defined the zeta function of Λ to be

$$\zeta_\Lambda(s) = \sum_{X \subset \Lambda} |\Lambda : X|^{-s},$$

where the sum is taken over left ideals X of finite index in Λ. We can generalise this to considering V a finitely generated left A-module on ν generators containing a full left Λ-lattice M (where full means that $M \otimes K = V$). Then we set

$$\zeta_M(s) = \sum_{X \subset M} |M : X|^{-s}$$

where X ranges over all Λ-sublattices of M.

For each prime p set $A_p = \mathbb{Q}_p P$ with order $\Lambda_p = \mathbb{Z}_p P$. Then the induced action of P on the abelian group T defines a left A_p-module $V_p = T \otimes \mathbb{Q}_p$ with Λ_p-lattice $M_p = T \otimes \mathbb{Z}_p = \widehat{T}_p$. Let E_p be the group $\widehat{E}/T_{p'}$ where $T_{p'} = \prod_{q \neq p} \widehat{T}_q$ was defined in the proof of Proposition 2.2. Then E_p is an extension of the abelian group $M_p = T \otimes \mathbb{Z}_p$ by P. The following proposition shows how our zeta function is a weighted version of Solomon's zeta function.

Proposition 2.3. *If p is a prime then*

$$\begin{aligned}\zeta^E_{E,p}(s) = \zeta^{E_p}_{E_p}(s) &= \sum_{X_p \subset M_p} |M_p : X_p|^{-s} |\mathrm{Der}(P, M_p/X_p)| \delta_{X_p} \\ &= \sum_{X_p \subset M_p} |M_p : X_p|^{-s} |\mathrm{Der}(P, M_p/X_p)| \quad \textit{if } (p, |P|) = 1,\end{aligned}$$

where X_p ranges over Λ_p-sublattices of M_p.

Proof. The first equality is a straightforward consequence of the fact that a subgroup of p-power index in $\widehat{E}$ contains $T_{p'}$ combined with the fact that all subgroups of E_p have p-power index in E_p if their join with M_p is E_p. The second equality follows from Proposition 2.1 and the fact that if $X_p \in \mathscr{H}^{\lhd}(M_p)$ (that is, a subgroup of M_p normal in E_p) then it is a Λ_p-sublattice of M_p and conversely. If p is coprime to $|P|$ then the extension M_p by P splits and hence M_p/X_p has a complement in E_p/X_p for every Λ_p-sublattice X_p, that is, $\delta_{X_p} = 1$ (see Theorem 7.7 of [**17**]).

REMARK 1. In the case that the extension E of T by P splits, we can take the $\mathbb{Q}$-algebra A to be the group algebra $\mathbb{Q}P$ with order $\mathbb{Z}P$. The induced action of P on the abelian group T defines a left A-module $V = T \otimes \mathbb{Q}$ with Λ-lattice $M = T$, and we can write the global zeta function as a weighted version of Solomon's zeta function

$$\zeta_E^E(s) = \sum_{X \subset M} |M : X|^{-s} |\mathrm{Der}(P, M/X)|$$

where X ranges over Λ-sublattices of M.

So by Proposition 2.2,

$$\zeta_E^E(s) = \left(\prod_{(p,|P|)\neq 1} Q_p(p^{-s}) \right) \cdot \prod_{(p,|P|)=1} \sum_{X_p \subset M_p} |M_p : X_p|^{-s} |\mathrm{Der}(P, M_p/X_p)|, \quad (2.3)$$

where $Q_p(X)$ is a rational function by [**5**]. Since $Q(s) = (\prod_{(p,|P|)\neq 1} Q_p(p^{-s}))$ is a meromorphic function on the whole complex plane, we suppose for the rest of the section that $(p, |P|) = 1$ and therefore we can focus our attention on $\sum_{X_p \subset M_p} |M_p : X_p|^{-s} |\mathrm{Der}(P, M_p/X_p)|$.

By [**1**, § 3.3], if Λ_p is decomposable into a direct sum of orders $\Lambda_p = \Lambda_1 \oplus \ldots \oplus \Lambda_r$ then the Λ_p-lattice M_p and a sublattice X_p will have corresponding decompositions: $M_p = M_1 \oplus \ldots \oplus M_r$ (where $M_i = \Lambda_i M_p$), $X_p = X_1 \oplus \ldots \oplus X_r$ and

$$|\mathrm{Der}(P, M_p/X_p)| = \prod_{i=1}^{r} |\mathrm{Der}(P, M_i/X_i)|.$$

Hence

$$\sum_{X_p \subset M_p} |M_p : X_p|^{-s} |\mathrm{Der}(P, M_p/X_p)| = \prod_{i=1}^{r} \sum_{X_i \subset M_i} |M_i : X_i|^{-s} |\mathrm{Der}(P, M_i/X_i)|.$$

Now the algebra A decomposes into a product of simple algebras $A_1 \oplus \ldots \oplus A_r$ which in turn gives a decomposition of $A_p = A_{p,1} \oplus \ldots \oplus A_{p,r}$. Note that $A_{p,i}$ will not necessarily be simple but we return to this point in a bit. The problem is that the order Λ_p may not have a corresponding decomposition. However if the order is maximal then it does: $\Lambda_p = \Lambda_1 \oplus \ldots \oplus \Lambda_r$ where Λ_i is a maximal order in $A_{p,i}$. Now for almost all primes p with $(p, |P|) = 1$, Λ_p is in fact a maximal order in the algebra A_p (see [**4**, Proposition 27.1]).

Therefore for $(p, |P|) = 1$ we may reduce to the case where we may suppose that A is a simple $\mathbb{Q}$-algebra and hence

$A = \mathrm{M}_m(D)$,

D is the division algebra with centre F,

F is the finite extension of $\mathbb{Q}$ with valuation ring R,

$\dim_F D = e^2$,

$V = W^k$ where W is the simple left A-module D^m.

The task is to identify almost all the local factors

$$\sum_{X_p \subset M_p} |M_p : X_p|^{-s} |\mathrm{Der}(P, M_p/X_p)|$$

as something classical which we can put together to get something that we can meromorphically continue.

We proceed now in an analogous manner to the proof in § 4.2 of [**1**]. For each prime p, we have a further decomposition of the local algebra $A_p \cong \prod_{\mathfrak{p}|p} A_{\mathfrak{p}}$ where the product is taken over all prime ideals $\mathfrak{p}$ of R lying over p. Each $A_{\mathfrak{p}}$ is a central simple $F_{\mathfrak{p}}$-algebra and is isomorphic to a full ring of integers $\mathrm{M}_{m_{\mathfrak{p}}}(D_{\mathfrak{p}})$ over some central simple $F_{\mathfrak{p}}$-division algebra $D_{\mathfrak{p}}$ of dimension $e_{\mathfrak{p}}^2$, say. Note that $e_{\mathfrak{p}} m_{\mathfrak{p}} = em$ which is the degree of both $A_{\mathfrak{p}}$ over $F_{\mathfrak{p}}$ and A over F. There are in fact only finitely many p such that for some prime $\mathfrak{p}$ lying over p, $e_{\mathfrak{p}} > 1$; or in other words the division algebra D splits for almost all primes p. Also the module $V_{\mathfrak{p}} = A_{\mathfrak{p}} \cdot V$ is $W_{\mathfrak{p}}^{k_{\mathfrak{p}}}$ where $W_{\mathfrak{p}} = D_{\mathfrak{p}}^{m_{\mathfrak{p}}}$, the simple left $A_{\mathfrak{p}}$-module. Since $V_{\mathfrak{p}}$ is also the completion of V at $\mathfrak{p}$, comparison of dimensions shows that $k_{\mathfrak{p}} e_{\mathfrak{p}} = ke$. When $e_{\mathfrak{p}} = 1$ therefore we get $k_{\mathfrak{p}} = ke$. We have a corresponding decomposition of $M_p \cong \prod_{\mathfrak{p}|p} M_{\mathfrak{p}}$ where $M_{\mathfrak{p}} = M_p \otimes R_{\mathfrak{p}}$ and $R_{\mathfrak{p}}$ is the ring of integers of $F_{\mathfrak{p}}$. As above, our function can be decomposed further as a product over all the primes $\mathfrak{p}$ lying over p:

$$\sum_{X_p \subset M_p} |M_p : X_p|^{-s} |\mathrm{Der}(P, M_p/X_p)| = \prod_{\mathfrak{p}|p} \sum_{X_{\mathfrak{p}} \subset M_{\mathfrak{p}}} |M_{\mathfrak{p}} : X_{\mathfrak{p}}|^{-s} |\mathrm{Der}(P, M_{\mathfrak{p}}/X_{\mathfrak{p}})|.$$

LEMMA 2.4. *If* $(p, |P|) = 1$ *then*

$$|\mathrm{Der}(P, M_p/X_p)| = |M_p : (C_{M_p}(P) + X_p)| = \prod_{\mathfrak{p}|p} |M_{\mathfrak{p}} : (C_{M_{\mathfrak{p}}}(P) + X_{\mathfrak{p}})|.$$

Proof. There are two things to show here:

(1) $|\mathrm{Der}(P, M_p/X_p)| = |M_p/X_p : C_{M_p/X_p}(P)|$ and

(2) $C_{M_p/X_p}(P) = (C_{M_p}(P) + X_p)/X_p$.

Proof of (1). Each derivation corresponds to a complement for M_p/X_p in E_p/X_p since the extension splits now. Such complements are conjugate. Let Q be a complement. Then $|\mathrm{Der}(P, M_p/X_p)|$ is equal to the number of conjugates of Q in E_p/X_p, which is $|E_p/X_p : N_{E_p/X_p}(Q)| = |M_p/X_p : N_{M_p/X_p}(Q)|$ since $E_p/X_p = M_p Q$. If $m \in N_{M_p/X_p}(Q)$ then $[m, q] \in M_p/X_p \cap Q = 1$. Hence $N_{M_p/X_p}(Q) = C_{M_p/X_p}(P)$.

Proof of (2). Suppose that $a \in M_p$ and $a^g \equiv a \bmod X_p$ for all $g \in P$. Then $\sum_{g \in P} a^g \equiv |P| a \bmod X_p$. Now $\sum_{g \in P} a^g \in C_{M_p}(P)$. Since $(p, |P|) = 1$, we have $a \equiv |P| a \equiv \sum_{g \in P} a^g \bmod X_p$.

This completes the proof of the lemma.

Now since $C_M(P) = C_V(P) \cap M$ and $C_V(P) = C_W(P)^k$, where $C_W(P)$ is an A-submodule of the simple module W, either $C_M(P) = M$ or $C_M(P) = 0$. Put

$$\varepsilon = \begin{cases} 0 & \text{if } C_W(P) = W, \\ 1 & \text{if } C_W(P) = 0. \end{cases}$$

Then $C_M(P) = M$ or 0 according to $\varepsilon = 0$ or 1. Since $C_{M_\mathfrak{p}}(P) = C_{M\otimes R_\mathfrak{p}}(P) = C_M(P) \otimes R_\mathfrak{p}$, we have, for $(p, |P|) = 1$,

$$\sum_{X_\mathfrak{p} \subset M_\mathfrak{p}} |M_\mathfrak{p} : X_\mathfrak{p}|^{-s} |M_\mathfrak{p} : (C_{M_\mathfrak{p}}(P) + X_\mathfrak{p})| = \sum_{X_\mathfrak{p} \subset M_\mathfrak{p}} |M_\mathfrak{p} : X_\mathfrak{p}|^{\varepsilon - s}.$$

We may now apply the formula due to Hey (see, for example, Formula (16) of [**1**]) to calculate $\sum_{X_\mathfrak{p} \subset M_\mathfrak{p}} |M_\mathfrak{p} : X_\mathfrak{p}|^{\varepsilon - s}$ since this is just the zeta function of a $\Lambda_\mathfrak{p}$-lattice $M_\mathfrak{p}$ inside an $A_\mathfrak{p}$-module where $\Lambda_\mathfrak{p}$ is a maximal order:

$$\sum_{X_\mathfrak{p} \subset M_\mathfrak{p}} |M_\mathfrak{p} : X_\mathfrak{p}|^{\varepsilon - s} = \prod_{j=0}^{k_\mathfrak{p} - 1} \zeta_{R_\mathfrak{p}}(m_\mathfrak{p} e_\mathfrak{p}(s - \varepsilon) - e_\mathfrak{p} j).$$

Referring back to formula (2.3) and assuming still for the moment that A is a simple algebra, we have

$$\begin{aligned} \zeta_E^E(s) &= Q(s) \cdot \prod_{(p,|P|)=1} \prod_{\mathfrak{p}|p} \prod_{j=0}^{k_\mathfrak{p}-1} \zeta_{R_\mathfrak{p}}(m_\mathfrak{p} e_\mathfrak{p}(s-\varepsilon) - e_\mathfrak{p} j) \\ &= S(s) \cdot \prod_{j=0}^{ke-1} \zeta_R(me(s-\varepsilon) - j), \end{aligned}$$

where

$$\begin{aligned} S(s) = Q(s) \cdot &\Bigg(\prod_{(p,|P|)\neq 1} \prod_{j=0}^{ke-1} \zeta_{R_\mathfrak{p}}(me(s-\varepsilon) - j) \Bigg)^{-1} \\ &\times \Bigg(\prod_{\substack{\mathfrak{p}|p,\,(p,|P|)=1 \\ e_\mathfrak{p}>1}} \prod_{\substack{j=0 \\ j \neq 0 \bmod e_\mathfrak{p}}}^{k_\mathfrak{p} e_\mathfrak{p} - 1} \zeta_{R_\mathfrak{p}}(m_\mathfrak{p} e_\mathfrak{p}(s-\varepsilon) - e_\mathfrak{p} j) \Bigg)^{-1}. \end{aligned}$$

Since there are only finitely many primes p with $e_\mathfrak{p} > 1$ for some prime $\mathfrak{p}$ lying over p and each of the terms in these products is a rational function, $S(s)$ admits meromorphic continuation to the whole of $\mathbb{C}$. But since $\zeta_R(me(s - \varepsilon) - j)$ is just the Dedekind zeta function of the number field F which admits meromorphic continuation to the whole of $\mathbb{C}$, we may deduce that $\zeta_E^E(s)$ also admits meromorphic continuation to the whole of $\mathbb{C}$.

To complete the proof we return to the case that A is not simple. Recall that E is a finite extension of the free abelian group T by a finite group P. To calculate $\zeta_E^E(s)$ explicitly up to a finite product of rational functions we are required to gather the following data:

(1) decompose $A = \mathbb{Q}P$ into simple components $A = A_1 \oplus \ldots \oplus A_r$;

(2) let F_i denote the centre of the simple algebra A_i and R_i its ring of integers; A_i is isomorphic to a full ring of matrices of rank m_i over some central F_i-division algebra D_i;

(3) let $n_i^2 = \dim_{F_i}(A_i) = m_i^2 e_i^2$ where $e_i^2 = \dim_{F_i}(D_i)$;

(4) let $V = T \otimes \mathbb{Q}$ be the module for A induced by the action of P on T; then $V = V_1 \oplus \ldots \oplus V_r$ where $V_i = A_i V = (W_i)^{k_i}$ and W_i is the A_i-module $(D_i)^{m_i}$;

(5) set $\varepsilon_i = 0$ or 1 according to whether $C_{W_i}(P) = C_{W_i}(A) = W_i$ or 0; then

$$\zeta_E^E(s) = S(s) \prod_{i=1}^{r} \prod_{j=0}^{k_i e_i - 1} \zeta_{R_i}(n_i(s - \varepsilon_i) - j) \tag{2.4}$$

where $S(s)$ is a product over a finite number of primes p of rational functions in p^{-s}.

The conclusion of all this is that $\zeta_E^E(s)$ admits continuation to a meromorphic function on the whole complex plane. This completes the proof of Theorem 1.1.

Note that from the formula (2.4) we can write down an explicit expression for the zeta function up to a finite number of Euler factors. We shall explain in § 3 a subtler procedure which will enable us to calculate the full zeta function in particular cases. As we shall see in § 4 it is the bad primes which distinguish split and non-split cases of an abelian group T extended by a finite group P.

2.2. *Counting normal subgroups*

As in the previous section we can focus our attention on proving meromorphic continuation for $\zeta_E^{E_*,\lhd}(s)$ since

$$\zeta_E^{\lhd}(s) = \sum_{T \leqslant E_* \lhd E} |E : E_*|^{-s} \zeta_E^{E_*,\lhd}(s),$$

$$\zeta_{E,p}^{\lhd}(s) = \sum_{T \leqslant E_* \lhd_p E} |E : E_*|^{-s} \zeta_{E,p}^{E_*,\lhd}(s).$$

In a similar fashion we also have an Euler product, which means that we can concentrate on the local factors for almost all primes.

Proposition 2.5.

$$\zeta_E^{E_*,\lhd}(s) = \prod_{p \text{ prime}} \zeta_{E,p}^{E_*,\lhd}(s).$$

The proof is the same as in Proposition 2.2 with the extra note added that H is normal if and only if each $H \cdot T_{p'}$ is normal for each prime p.

Let E_p be the group $\widehat{E}/T_{p'}$ where $T_{p'} = \prod_{q \neq p} \widehat{T}_q$ was defined in the proof of Proposition 2.2. Then E_p is an extension of the abelian group $M_p = T \otimes \mathbb{Z}_p$ by P. Again we can reduce to considering the group E_p since the following holds.

Proposition 2.6. *We have* $\zeta_{E,p}^{E_*,\lhd}(s) = \zeta_{E_p}^{E_{p*},\lhd}(s)$ *where* $E_{p*} = \widehat{E}_*/T_{p'}$.

However, the one thing that we cannot assume as in the previous section is that $E_p = E_{p*}$, since if H is normal in E_{p*}, it does not imply of course that H is normal in E_p. If H is normal in E_p then certainly $X_p = M_p \cap H$ and $E_{p*} = M_p H$ are normal in E_p. In particular, X_p is a Λ_p-sublattice of M_p where $\Lambda_p = \mathbb{Z}_p P$, an order inside the group algebra $A_p = \mathbb{Q}_p P$. We need to know how to tell, given E_{p*} normal in E_p and a Λ_p-sublattice X_p, whether there exists a normal complement H for M_p/X_p in E_{p*}/X_p with H normal in E_p. Once we reduce the situation to considering primes p coprime to the order of P this is done by the following.

PROPOSITION 2.7. *Suppose that* $(p, |P|) = 1$. *Let* $M_p \leqslant E_{p*}$ *be normal in* E_p *and* X_p *be a* Λ_p*-sublattice of* M_p. *Then there exists a normal complement* H *for* M_p/X_p *in* E_{p*}/X_p *with* H *normal in* E_p *if and only if* $[E_{p*}, M_p] \leqslant X_p$. *If such a complement exists, it is unique.*

Proof. Note first that since we have the coprime condition, we always have a complement H for M_p/X_p in E_{p*}/X_p and all such complements are conjugate in E_{p*} (see [**17**, Theorem 7.77]). If H is normal in E_p then since M_p is also normal in E_p, $[H, M_p] \leqslant M_p \cap H = X_p$. Hence $[E_{p*}, M_p] = [HM_p, M_p] \leqslant X_p$. Also H is a unique complement since it is normal in E_p and all complements are conjugate.

For the converse we want to prove that $N_{E_p}(H) = E_p$. If $g \in E_p$ then $H^g \leqslant E_{p*}$ is also a complement for M_p/X_p in E_{p*}/X_p. Since complements are conjugate in E_{p*}, there exists $k \in E_{p*}$ such that $H^{gk} = H$, that is, $gk \in N_{E_p}(H)$. Hence $E_p = E_{p*}N_{E_p}(H) = M_pN_{E_p}(H)$ since $E_{p*} = M_pH$ and $H \leqslant N_{E_p}(H)$. This implies then that

$$|E_p : N_{E_p}(H)| = |M_p : M_p \cap N_{E_p}(H)|.$$

But since $[H, M_p] \leqslant [E_{p*}, M_p] \leqslant X_p \leqslant H$, we have $M_p \cap N_{E_p}(H) = M_p$. Hence $|E_p : N_{E_p}(H)| = 1$, that is, $E_p = N_{E_p}(H)$. This completes the proof.

COROLLARY 2.8. *Suppose that* $(p, |P|) = 1$. *Then*

$$\zeta_{E,p}^{E_*,\lhd}(s) = \zeta_{E_p}^{E_{p*},\lhd}(s) = \sum_{\substack{X_p \subset M_p \\ [E_{p*}, M_p] \leqslant X_p}} |M_p : X_p|^{-s}$$

where X_p *ranges over* Λ_p*-sublattices of* M_p.

Note that if $E_{p*} = M_p$ then this is just the Solomon zeta function $\zeta_{M_p}(s)$ of the lattice M_p. More generally we have the following.

COROLLARY 2.9. *Suppose that* $(p, |P|) = 1$. *Then*

$$\zeta_{E,p}^{E_*,\lhd}(s) = \sum_{\overline{X_p} \subset \overline{M_p}} |\overline{M_p} : \overline{X_p}|^{-s} = \zeta_{\overline{M_p}}(s),$$

where $\overline{X_p}$ *ranges over* Λ_p*-sublattice of* $\overline{M_p}$, *and* $\overline{M_p} = M_p/[E_{p*}, M_p]$. (*Recall that we are assuming that* E_{p*} *is normal in* E_p *so that* $[E_{p*}, M_p]$ *itself is a* Λ_p*-sublattice of* M_p.)

COROLLARY 2.10. *The function* $\zeta_E^{\lhd}(s)$ *admits meromorphic continuation to the whole complex plane.*

Proof. This now follows the same argument as in the previous section without the added complication of the weighted term we obtain from the derivations when we count all subgroups. Note also that $[E_{p*}, M_p] = [E_*, T] \otimes \mathbb{Z}_p$. Hence, except for a finite number of primes, $\zeta_E^{E_*,\lhd}(s)$ is essentially $\zeta_{\overline{M}}(s)$ where $\overline{M} = T/[E_*, T]$. We will see this in practice in § 6 when we interpret the calculations of $\zeta_E^{\lhd}(s)$ for the wallpaper groups made in § 4.2 in terms of Solomon's zeta function.

3. *Method to calculate 'bad' primes*

The first place where the zeta function of a finite extension of a group was systematically considered in terms of the zeta function of the base group was in [**5**]. It was shown in § 2.2 of that paper how one can extend knowledge of the rationality of the zeta function of a uniform pro-p group G_1 to prove the rationality of any finite extension G of G_1.

This result can then be applied to calculate the local zeta function of an extension Γ of an abstract finitely generated group Γ_1 whose pro-p completion is uniform and whose subgroups of p-power index are all subnormal. This is explained in Theorem 3.3 of [**5**]. For example, this includes finite extensions of nilpotent groups. More generally, as explained in the last paragraph of [**5**], it also includes all finite extensions of finitely generated groups of finite rank. As proved by Lubotzky and Mann [**13**], finitely generated, residually finite groups of finite rank are actually finite extensions of soluble groups of finite rank. This is precisely the class of groups for which the global zeta function defines an analytic function on some right half of the complex plane (see [**14**]).

In the following two sections an adaptation of this method is applied to calculate explicitly the zeta functions of the wallpaper groups, certain finite extensions of the free abelian group of rank 2.

We take E to be a finite extension of a free abelian group T of rank 2 on x and y. As is well known, every subgroup T_* of finite index in T is free on elements $x^a y^b$ and y^c where $a, c > 0$ and $0 \leqslant b < c$, and these exponents are uniquely determined subject to the given constraints. The index $|T : T_*| = ac$. The elements $x^a y^b$ and y^c are a 'good basis' for T_* in the terminology of [**8**] and [**5**].

Suppose $T \leqslant E_* \leqslant E$. In order to enumerate all subgroups of finite index in E, it suffices to list all groups T_* and E_*, and find all H such that $HT = E_*$ and $H \cap T = T_*$. This will enable us to calculate $\zeta_E^{E_*}(s)$ and hence $\zeta_E = \sum_{T \leqslant E_* \leqslant E} |E : E_*|^{-s} \zeta_E^{E_*}(s)$. For such an H to exist, it is a necessary condition that H normalize T_* so $T_* \trianglelefteq E_*$. To test this we may choose elements X of E_* which generate E_* modulo T. One may determine whether or not $T_* \trianglelefteq E_*$ by conjugating the good basis for T_* by elements of X and testing for membership in T_*. Since T_* has finite index in E_*, it is not necessary to check conjugation by inverses of elements of X. We now assume that $T_* \trianglelefteq E_*$. If H is a subgroup of E whose meet and join with T are (respectively) T_* and E_*, then there is a natural isomorphism between $E_*/T = HT/T$ and H/T_*. This isomorphism sends the preferred generators X of E_*/T to preferred generators of H/T_*.

Interestingly enough, this process is reversible. Suppose we have elements Y of E_*/T_* in a distinguished bijective correspondence with X and such that corresponding elements coincide modulo T. Find a set of defining relations R for E_*/T on the generating set X. Let $H = \langle Y, T_* \rangle$ and suppose that the elements of Y satisfy the relations R modulo T_*, where each element of Y replaces the corresponding element of X in each relation. It follows that H/T_* is a quotient of E_*/T. However, $H/(H \cap T)$ is isomorphic to E_*/T and simultaneously is a quotient of H/T_* and thus is a quotient of E_*/T. We conclude that $H/(H \cap T)$ is not a proper quotient of H/T_* by finiteness (or because polycyclic by finite groups are Hopfian). Thus $T_* = H \cap T$ and $E_* = HT$. Moreover, the generators Y are (modulo T_*) the ones induced from X in the previous paragraph.

In § 2.2 of [**5**] the preferred set of generators X is taken to be a transversal

$\{x_1, \ldots, x_n\}$ for T in E_* and then $Y = \{t_1 x_1, \ldots, t_n x_n\}$ is a right transversal for T_* in H where $\{t_1, \ldots, t_n\}$ is a 'transversal basis' for H (see Definition 2.10 of [**5**]). In Lemma 2.12 of [**5**], conditions are given for when a set $\{t_1, \ldots, t_n\}$ defines a transversal basis for some subgroup which corresponds to checking above that the elements of Y satisfy the relations R modulo T_*. When it comes to explicit calculations this choice of a transversal for the generating set is inefficient and a more judicious choice is made instead.

In order to enumerate all the normal subgroups of finite index in E we have to make a minor modification. If H is a normal subgroup of finite index in E, then HT, the smallest overgroup of T containing H, will itself be normal in E, and $H \cap T$ must also be normal in E.

Thus when looking for T_* and E_* we may restrict attention to the case that they are both normal in E. This may be verified by suitable conjugation and membership testing. When this is done, we may apply the standard method to find subgroups H with appropriate meet and join with T, but this will not guarantee that $H \trianglelefteq E$. A specific check must be made that a generating set of E will conjugate generators of H into H. We may do this modulo T_* if we wish, so the issue can be settled inside the finite group E/T_*. The same issue arises in § 2.3.1 of [**5**] where extra conditions are needed to ensure that the set $\{t_1, \ldots, t_n\}$ defines a transversal basis for some normal subgroup of E.

4. *Results*

4.1. *The zeta functions of the wallpaper groups*

The zeta functions of the wallpaper groups are now listed. We have arranged as far as possible that $\zeta_E^E(s)$ (which each time will represent the new calculation being made in each example) appears as the first term in the formula. However some collecting of terms has been done so that the first expression also includes terms from intermediate subgroups E_* which give rise to zeta functions of a similar structure. This results, for example, in the non-nilpotent groups whose zeta functions nonetheless have Euler products, like **pm** and **pg**.

p1 $= \langle x, y \,|\, [x, y] \rangle$ has zeta function

$$\zeta(s)\zeta(s-1).$$

p2 $= \langle x, y, r \,|\, [x, y], r^2, x^r = x^{-1}, y^r = y^{-1} \rangle$ has zeta function

$$\zeta(s-1)\,\zeta(s-2) + 2^{-s}\zeta(s)\,\zeta(s-1).$$

pm $= \langle x, y, m \,|\, [x, y], m^2, x^m = x, y^m = y^{-1} \rangle$ has zeta function

$$(1 + 2^{-s+2})\,\zeta(s)\,\zeta(s-1).$$

pg $= \langle x, y, t \,|\, [x, y], t^2 = x, y^t = y^{-1} \rangle$ has zeta function

$$\zeta(s)\,\zeta(s-1).$$

p2mm $= \langle x, y, p, q \,|\, [x, y], [p, q], p^2, q^2, x^p = x, x^q = x^{-1}, y^p = y^{-1}, y^q = y \rangle$ has zeta function

$$(1 + 2^{-s+3} + 2^{-2s+2})\,\zeta(s-1)\,\zeta(s-1) + (2^{-s+1} + 7 \cdot 2^{-2s})\,\zeta(s)\,\zeta(s-1) + 2^{-s}\zeta(s-1)\,\zeta(s-2).$$

p2mg $= \langle x, y, m, t \,|\, [x, y], t^2, m^2 = y, x^t = x, x^m = x^{-1}, y^t = y^{-1}, m^t = m^{-1} \rangle$ has zeta function

$$(1 - 2^{-2s+2})\,\zeta(s-1)\,\zeta(s-1) + (2^{-s+1} + 3 \cdot 2^{-2s})\,\zeta(s)\,\zeta(s-1) + 2^{-s}\zeta(s-1)\,\zeta(s-2).$$

p2gg $= \langle x, y, u, v \,|\, [x, y], u^2 = x, v^2 = y, x^v = x^{-1}, y^u = y^{-1}, (uv)^2 \rangle$ has zeta function

$$(1 - 2^{-s+1})^2\zeta(s-1)\,\zeta(s-1) + (2^{-s+1} - 2^{-2s})\,\zeta(s)\,\zeta(s-1) + 2^{-s}\zeta(s-1)\,\zeta(s-2).$$

cm $= \langle x, y, t \,|\, [x, y], t^2, y^t = y^{-1}, x^t = xy \rangle$ has zeta function

$$(1 + 2^{-2s+2})\,\zeta(s)\,\zeta(s-1).$$

c2mm $= \langle x, y, m, r \,|\, [x, y], m^2, r^2, y^m = y^{-1}, x^m = xy, y^r = y^{-1}, x^r = x^{-1}, r^m = r^{-1} \rangle$ has zeta function

$$(1 + 2^{-2s+3})\,\zeta(s-1)\,\zeta(s-1) + (2^{-s+1} - 2^{-2s} + 2^{-3s+3})\,\zeta(s)\,\zeta(s-1) + 2^{-s}\zeta(s-1)\,\zeta(s-2).$$

p4 $= \langle x, y, r \,|\, [x, y], r^4, y^r = x^{-1}, x^r = y \rangle$ has zeta function

$$\zeta(s-1)\,L(s-1, \chi_4) + 2^{-2s}\zeta(s)\,\zeta(s-1) + 2^{-s}\zeta(s-1)\,\zeta(s-2),$$

where χ_4 is the extended primitive residue class character χ_4: $\mathbb{Z} \to (\mathbb{Z}/4\mathbb{Z})^*$ with

$$\chi_4(a) = \begin{cases} 1 & \text{if } a \equiv 1 \bmod 4, \\ -1 & \text{if } a \equiv 3 \bmod 4, \\ 0 & \text{otherwise,} \end{cases}$$

and $L(s, \chi)$ denotes the Dirichlet L-function of χ,

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)\,n^{-s}.$$

Notice that $\zeta(s)L(s, \chi_4)$ is the Dedekind zeta function of the number field $\mathbb{Q}(\omega_4)$ where ω_4 is a primitive 4th root of unity. We shall see why this should be in § 6.

p4mm $= \langle x, y, r, m \,|\, [x, y], r^4, m^2, y^r = x^{-1}, x^r = y, x^m = y, r^m = r^{-1} \rangle$ has zeta function

$$\begin{aligned}(1 + 4 \cdot 2^{-s} + 4 \cdot 2^{-2s})\,\zeta(2s-2) &+ (4 \cdot 2^{-2s} + 5 \cdot 2^{-3s} + 8 \cdot 2^{-4s})\,\zeta(s)\,\zeta(s-1) \\ &+ (2 \cdot 2^{-s} + 8 \cdot 2^{-2s} + 4 \cdot 2^{-3s})\,\zeta(s-1)\,\zeta(s-1) \\ &+ 2^{-2s}\zeta(s-1)\zeta(s-2) + 2^{-s}\zeta(s-1)L(s-1, \chi_4).\end{aligned}$$

(The L-function $L(s-1, \chi_4)$ is explained above.)

p4gm $= \langle x, y, r, t \,|\, [x, y], r^4, t^2, y^r = x^{-1}, x^r = y, x^t = y, r^t = r^{-1}x^{-1} \rangle$ has zeta function

$$(1 - 4 \cdot 2^{-2s})\,\zeta(2s-2) + (4 \cdot 2^{-2s} - 3 \cdot 2^{-3s} + 8 \cdot 2^{-4s})\,\zeta(s)\,\zeta(s-1)$$
$$+ (2 \cdot 2^{-s} - 4 \cdot 2^{-2s} + 12 \cdot 2^{-3s})\,\zeta(s-1)\,\zeta(s-1)$$
$$+ 2^{-2s}\zeta(s-1)\,\zeta(s-2) + 2^{-s}\zeta(s-1)\,L(s-1, \chi_4).$$

p3 $= \langle x, y, r \,|\, [x, y], r^3, x^r = x^{-1}y, y^r = x^{-1} \rangle$ has zeta function

$$\zeta(s-1)\,L(s-1, \chi_3) + 3^{-s}\zeta(s)\,\zeta(s-1),$$

where χ_3 is the extended primitive residue class character $\chi_3\colon \mathbb{Z} \to (\mathbb{Z}/3\mathbb{Z})^*$ with

$$\chi_3(a) = \begin{cases} 1 & \text{if } a \equiv 1 \bmod 3, \\ -1 & \text{if } a \equiv 2 \bmod 3, \\ 0 & \text{otherwise.} \end{cases}$$

As we saw in the example of **p4**, $\zeta(s)L(s, \chi_3)$ is the Dedekind zeta function of the number field $\mathbb{Q}(\omega_3)$ where ω_3 is a primitive 3rd root of unity.

p31m $= \langle x, y, r, t \,|\, [x, y], r^2, t^2, (tr)^3, x^r = x, y^t = y, x^t = x^{-1}y, y^r = xy^{-1} \rangle$ has zeta function

$$(1 + 3^{-s})\,\zeta(2s-2) + 2^{-s}\zeta(s-1)\,L(s-1, \chi_3)$$
$$+ (3 \cdot 3^{-s} - 2 \cdot 6^{-s} + 12 \cdot 12^{-s})\,\zeta(s)\,\zeta(s-1).$$

p3m1 $= \langle x, y, r, m \,|\, [x, y], r^3, m^2, r^m = r^{-1}, x^r = x^{-1}y, y^r = x^{-1}, x^m = x^{-1}, y^m = x^{-1}y \rangle$ has zeta function

$$(1 + 3^{-s+2})\,\zeta(2s-2) + 2^{-s}\zeta(s-1)\,L(s-1, \chi_3)$$
$$+ (3 \cdot 3^{-s} - 2 \cdot 6^{-s} + 12 \cdot 12^{-s})\,\zeta(s)\,\zeta(s-1).$$

p6 $= \langle x, y, r, \,|\, [x, y], r^6, x^r = y, y^r = x^{-1}y \rangle$ has zeta function

$$(1 + 3^{-s+1})\,\zeta(s-1)\,L(s-1, \chi_3) + 3^{-s}\zeta(s-1)\,\zeta(s-2) + 6^{-s}\zeta(s)\,\zeta(s-1).$$

p6mm $= \langle x, y, r, m \,|\, [x, y], r^6, m^2, y^r = x^{-1}y, x^r = y, x^m = x^{-1}, y^m = x^{-1}y, r^m = r^{-1}y \rangle$ has zeta function

$$(1 + 2 \cdot 2^{-s} + 3 \cdot 3^{-s} + 10 \cdot 6^{-s})\,\zeta(2s-2) + (2^{-s} + 4^{-s})\,\zeta(s-1)\,L(s-1, \chi_3)$$
$$+ 3 \cdot 3^{-s}(1 + 8 \cdot 4^{-s})\,\zeta(s-1)\,\zeta(s-1) + 6^{-s}\zeta(s-1)\,\zeta(s-2)$$
$$+ (6 \cdot 6^{-s} - 5 \cdot 12^{-s} + 24 \cdot 24^{-s})\,\zeta(s)\,\zeta(s-1).$$

4.2. *The normal zeta functions of the wallpaper groups*

$$\zeta^{\lhd}_{\mathbf{p1}}(s) = \zeta(s)\,\zeta(s-1),$$
$$\zeta^{\lhd}_{\mathbf{p2}}(s) = 1 + 6 \cdot 2^{-s} + 4 \cdot 4^{-s} + 2^{-s}\zeta(s)\,\zeta(s-1),$$
$$\zeta^{\lhd}_{\mathbf{pm}}(s) = (1 + 5 \cdot 2^{-s} + 2 \cdot 4^{-s})\,\zeta(s) + 2^{-s}(1 + 2^{-s})\,\zeta(s)^2,$$

$$\zeta^{\triangleleft}_{\mathbf{pg}}(s) = (1 + 2^{-s} - 2 \cdot 4^{-s})\,\zeta(s) + 2^{-s}(1 + 2^{-s})\,\zeta(s)^2,$$

$$\zeta^{\triangleleft}_{\mathbf{p2mm}}(s) = 1 + 13 \cdot 2^{-s} + 20 \cdot 4^{-s} + 4 \cdot 8^{-s} + (2 \cdot 2^{-s} + 10 \cdot 4^{-s} + 4 \cdot 8^{-s})\,\zeta(s) + (4^{-s} + 8^{-s})\,\zeta(s)^2,$$

$$\zeta^{\triangleleft}_{\mathbf{p2mg}}(s) = 1 + 5 \cdot 2^{-s} + 2 \cdot 4^{-s} + (2 \cdot 2^{-s} + 2 \cdot 4^{-s} - 2 \cdot 8^{-s})\,\zeta(s) + (4^{-s} + 8^{-s})\,\zeta(s)^2,$$

$$\zeta^{\triangleleft}_{\mathbf{p2gg}}(s) = 1 + 2^{-s+1}(1 - 2^{-s})\,\zeta(s) + 2^{-s} + 2^{-2s+1} + 2^{-2s}(1 + 2^{-s})\,\zeta(s)^2,$$

$$\zeta^{\triangleleft}_{\mathbf{cm}}(s) = (1 + 2^{-s})\,\zeta(s) + 2^{-s}(1 - 2^{-s} + 2 \cdot 4^{-s})\,\zeta(s)^2,$$

$$\zeta^{\triangleleft}_{\mathbf{c2mm}}(s) = 1 + 2^{-s+2} + 2^{-s+1}(1 + 2^{-s})\,\zeta(s) + 2^{-s}(1 + 2^{-s+1} + 2^{-2s+1}) + 2^{-2s}(1 - 2^{-s} + 2 \cdot 4^{-s})\,\zeta(s)^2,$$

$$\zeta^{\triangleleft}_{\mathbf{p4}}(s) = 1 + 3 \cdot 2^{-s} + 2 \cdot 4^{-s} + 2 \cdot 8^{-s} + 2^{-2s}\zeta(s)\,L(s, \chi_4),$$

$$\zeta^{\triangleleft}_{\mathbf{p4mm}}(s) = 1 + 7 \cdot 2^{-s} + 7 \cdot 4^{-s} + 6 \cdot 8^{-s} + 2 \cdot 16^{-s} + (8^{-s} + 16^{-s})\,\zeta(2s),$$

$$\zeta^{\triangleleft}_{\mathbf{p4gm}}(s) = 1 + 3 \cdot 2^{-s} + 3 \cdot 4^{-s} + 2 \cdot 8^{-s} + (8^{-s} + 16^{-s})\zeta(2s),$$

$$\zeta^{\triangleleft}_{\mathbf{p3}}(s) = 1 + 3^{-s+1} + 3^{-s}\zeta(s)\,L(s, \chi_3),$$

$$\zeta^{\triangleleft}_{\mathbf{p31m}}(s) = 1 + 3^{-s} + 2^{-s}(1 + 3^{-s}) + 6^{-s}(1 + 3^{-s})\,\zeta(2s),$$

$$\zeta^{\triangleleft}_{\mathbf{p3m1}}(s) = 1 + 2^{-s}(1 + 3^{-s+1}) + 6^{-s}(1 + 3^{-s})\,\zeta(2s),$$

$$\zeta^{\triangleleft}_{\mathbf{p6}}(s) = 1 + 2^{-s} + 3^{-s} + 6^{-s} + 12^{-s} + 6^{-s}\zeta(s)\,L(s, \chi_3),$$

$$\zeta^{\triangleleft}_{\mathbf{p6mm}}(s) = 1 + 3 \cdot 2^{-s} + 4^{-s} + 2 \cdot 6^{-s} + 12^{-s} + 24^{-s} + (12^{-s} + 36^{-s})\,\zeta(2s).$$

5. *Examples*

We give two sample calculations. The first is straightforward without being trivial, and illustrates the method very quickly. The second is a little more intricate, and demonstrates how congruence conditions on primes may arise. Details of all the other calculations may be obtained from the second author and are contained in [**15**].

5.1. *The zeta function of* **pm**

This group can be presented as

$$E = \langle x, y, m \,|\, [x, y], m^2, x^m = x, y^m = y^{-1} \rangle.$$

The translation subgroup T is generated by x and y. The point group P, generated by mT, is cyclic of order 2.

We first count subgroups H of finite index in E such that $HT = E$. Now $T_* = \langle x^c y^d, y^e \rangle$ for suitable natural numbers c, e and $0 \leqslant d < e$ (uniquely). The condition that $T_* \trianglelefteq E$ amounts to $(x^c y^d)^m = x^c y^{-d} \in T_*$ and $(y^e)^m = y^{-e} \in T_*$. We distinguish m, a generator of $E/T = \langle m | m^2 \rangle$, and seek $mx^a y^b$ which satisfies $(m^1 x^a y^b)^2 = x^{2a} \in T_*$.

Now, y^{-e} is contained in T_* regardless of the value that e takes. For x^{2a} and $x^c y^{-d}$ are in T_* if and only if there are integers α_1, α_2, β_1 and β_2 such that $x^{2a} = (x^c y^d)^{\alpha_1}(y^e)^{\beta_1}$ and $x^c y^{-d} = (x^c y^d)^{\alpha_2}(y^e)^{\beta_2}$. Since we are only concerned with $m^1 x^a y^b$ modulo T_*, we may assume that $0 \leqslant a < c$ and $0 \leqslant b, d < e$. Moreover, subject to these restrictions, the various cosets $mx^a y^b T_*$ are distinct.

The following set of necessary and sufficient conditions must hold:

$$\mathfrak{C}_1 = \{2a = c\alpha_1, 0 = d\alpha_1 + e\beta_1, c = c\alpha_2, -d = d\alpha_2 + e\beta_2, 0 \leqslant a < c, 0 \leqslant b, d < e\}$$

for integers α_1, α_2, β_1 and β_2.

We find it convenient to summarize the possible subgroups H via a matrix of exponents

$$\begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & 0 & e \end{pmatrix}.$$

The condition $2a = c\alpha_1$ is satisfied exactly when either $a = \frac{1}{2}c$ or $a = 0$. If $a = \frac{1}{2}c$, then $\alpha_1 = 1$ so that $0 = d + e\beta_1$. Now $d < e$ so $a = \frac{1}{2}c$ forces $d = 0$. If $d \neq 0$, then $-2d = e\beta_2$ yields that $d = \frac{1}{2}e$. We get three cases, in each of which b is free within the range $0 \leqslant b < c$. The possibilities are:

(i) $a = d = 0$ and no extra restrictions on c, e, or

(ii) $a = \frac{1}{2}c$ (and c is even), $d = 0$, or

(iii) $a = 0$, $d = \frac{1}{2}e$ (and e is even).

We form the three corresponding sums to obtain a contribution towards $\zeta_{\mathbf{pm}}(s)$ of

$$\sum_{c,e \in \mathbb{N}} c^{-s}e^{-s}e + \sum_{c \in 2\mathbb{N}, e \in \mathbb{N}} c^{-s}e^{-s}e + \sum_{c \in \mathbb{N}, e \in 2\mathbb{N}} c^{-s}e^{-s}e$$

$$= (1 + 2^{-s} + 2^{-s+1})\,\zeta(s)\,\zeta(s-1)$$
$$= (1 + 3 \cdot 2^{-s})\,\zeta(s)\,\zeta(s-1).$$

The subgroup E_2 is free abelian of rank 2 so $\zeta_{E_2} = \zeta(s)\,\zeta(s-1)$. Now, E_2 has index 2 in E so the contribution from E_2 to the zeta function of E is $2^{-s}\zeta(s)\,\zeta(s-1)$:

$$\zeta_{\mathbf{pm}}(s) = (1 + 4 \cdot 2^{-s})\,\zeta(s)\,\zeta(s-1).$$

5.2. *The zeta function of* **p4**

This group can be presented as

$$E = \mathbf{p4} = \langle x, y, r \,|\, [x, y], r^4, y^r = x^{-1}, x^r = y \rangle.$$

The translation subgroup T is generated by x and y. The point group P, generated by rT, is cyclic of order 4. Possible groups E_* are

(i) $E_1 (=E)$,

(ii) $E_2 = \langle x, y, r^2 \rangle$, and

(iii) $E_4 = \langle x, y \rangle$,

where the subscript denotes the index in E. Finite index subgroups H such that $HT = E_1$ can be generated as

$$H = \langle r^1 x^a y^b, x^c y^d, y^e \rangle,$$

where $T_* = \langle x^c y^d, y^e \rangle$ and we have $0 \leqslant d < e$ and $0 < e$. We must force T_* to be normal in E and ensure that $(rx^a y^b)^4 \in T_*$. Also, since we are only concerned with $r^1 x^a y^b$ modulo T_*, we may take a, b in the ranges $0 \leqslant a < c$ and $0 \leqslant b < e$. The matrix of exponents is

$$\begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & 0 & e \end{pmatrix}.$$

Just as for the previous example, we expand out these words and then attempt to rewrite them as words in the generators of T_*:

$$\mathfrak{C}_2 = \{-d = c\alpha_1, c = d\alpha_1 + e\beta_1, -e = c\alpha_2, 0 = d\alpha_2 + e\beta_2, 0 \leqslant a < c, 0 \leqslant b, d < e\}.$$

We can rewrite the second and fourth of the equations in $\mathfrak{C}_2$ using the identities from the first and third. We get these two new equations:

$$c = -c\alpha_1\alpha_1 - c\alpha_2\beta_1 \quad \text{and} \quad 0 = -c\alpha_1\alpha_2 - c\alpha_2\beta_2.$$

The last condition simply determines $\beta_2 = -\alpha_1$. The previous condition tells us that α_1^2 is congruent to -1 modulo α_2. For -1 to be a square modulo n, a natural number, either n must be a product of prime powers where each prime is congruent to 1 modulo 4 or, $\frac{1}{2}n$ must be of this form. Furthermore, if n (or $\frac{1}{2}n$) is of this form, then there are 2^m choices for the 'square root' of -1, where m is the number of distinct prime factors of n (or $\frac{1}{2}n$ respectively).

The matrix of exponents now becomes

$$\begin{pmatrix} 1 & a & b \\ 0 & c & -c\alpha_1 \\ 0 & 0 & -c\alpha_2 \end{pmatrix}.$$

We see that, given c and $-\alpha_2$, there are c choices for a, $-c\alpha_2$ choices for b, and 2^m choices for $-c\alpha_1$, where m depends on α_2 as above. Summing over all possible values for a, b, c, α_1 and α_2, we get

$$\sum_{c \in \mathbb{N}, \alpha_2 \in \mathbb{S}} c^{-s} c^{-s} c c \alpha_2^{-s} \alpha_2 \psi(\alpha_2) + \sum_{c \in \mathbb{N}, \alpha_2 \in 2\mathbb{S}} c^{-s} c^{-s} c c \alpha_2^{-s} \alpha_2 \psi(\tfrac{1}{2}\alpha_2).$$

Here ψ and $\mathbb{S}$ are defined as follows:

$$\psi(n) = 2^m, \quad \text{where } m \text{ is the number of distinct prime divisors of } n;$$

$$\mathbb{S} = \{n \mid n \in \mathbb{N} \text{ and for all primes } p \text{ such that } p \mid n, p \equiv 1 \bmod 4\}.$$

If we set $\zeta^*(s) = \sum_{n \in \mathbb{S}} \psi(n) n^{-s}$, then the zeta function for E_1 is

$$\begin{aligned}
&(1 + 2^{-s+1})\,\zeta(2s-2)\,\zeta^*(s-1) \\
&\quad = (1 + 2^{-s+1}) \prod_{p \text{ prime}} \frac{1}{(1 - p^{1-s})(1 + p^{1-s})} \\
&\quad\quad \times \prod_{p \equiv 1 \bmod 4} (1 + 2p^{1-s} + 2p^{2-2s} + \ldots).
\end{aligned}$$

Now

$$\prod_{p \text{ prime}} \frac{1}{(1+p^{1-s})} \prod_{p \equiv 1 \bmod 4} (1 + 2p^{1-s} + 2p^{2-2s} + \dots)$$
$$= \frac{1}{(1+2^{-s+1})} \prod_{p \text{ odd prime}} \frac{1}{(1+p^{1-s})} \prod_{p \equiv 1 \bmod 4} (1+p^{1-s})(1+p^{1-s}+p^{2-2s}+\dots)$$
$$= \frac{1}{(1+2^{-s+1})} \prod_{p \equiv 3 \bmod 4} \frac{1}{(1+p^{1-s})} \prod_{p \equiv 1 \bmod 4} \frac{1}{(1-p^{1-s})}$$
$$= \frac{1}{(1+2^{-s+1})} L(s-1, \chi_4),$$

where χ_4 is the extended primitive residue class character $\chi_4\colon \mathbb{Z} \to (\mathbb{Z}/4\mathbb{Z})^*$ with

$$\chi_4(a) = \begin{cases} 1 & \text{if } a \equiv 1 \bmod 4, \\ -1 & \text{if } a \equiv 3 \bmod 4, \\ 0 & \text{otherwise,} \end{cases}$$

and $L(s, \chi)$ denotes the Dirichlet L-function of χ,

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_p \frac{1}{(1-\chi(p)p^{-s})}.$$

Hence

$$(1+2^{-s+1})\,\zeta(2s-2)\,\zeta^*(s-1) = \zeta(s-1)L(s-1, \chi_4).$$

The group E_2 is isomorphic to **p2**, while E_4 is isomorphic to **p1**, and for the purposes of this demonstration we assume that **p1** and **p2** have already been investigated. We take the contributions from these subgroups to be the appropriate Dirichlet series multiplied by the respective indices on E. We must be careful not to overcount subgroups in the case of E_2, and only count those whose join with T is E_2. The contributions are then summed, to obtain

$$\zeta_{\mathbf{p4}}(s) = \zeta(s-1)L(s-1, \chi_4) + 2^{-s}\zeta(s-1)\,\zeta(s-2) + 2^{-2s}\zeta(s)\,\zeta(s-1)$$

and our demonstration is complete.

The method we use to find a formula for the normal zeta function of a crystallographic group is very similar to the above except that the number of conditions which need to be examined rises sharply [**15**].

6. *Theory versus practice*

In this section we return to the theoretical arguments of § 2 and interpret the zeta functions of the wallpaper groups in the context of zeta functions of orders. We compare the results of § 4 with the formula derived at the end of § 2 for the zeta function at good primes.

Table 1 compares the calculations of the zeta functions counting all subgroups of finite index. For each example we record

TABLE 1. *Counting subgroups in the wallpaper groups.*

E	P	A_i	F_i	n_i	e_i	k_i	ε_i	$\zeta_E^{T,\triangleleft}(s)\sim$	$\zeta_E^E(s)\sim$
p1	1	$\mathbb{Q}$	$\mathbb{Q}$	1	1	2	0	$\zeta(s)\,\zeta(s-1)$	$\zeta(s)\,\zeta(s-1)$
p2	$C_2=\langle r\rangle$	$\mathbb{Q}\,\sigma$	$\mathbb{Q}$	1	1	0	0	1	1
		$\mathbb{Q}\,(r-1)$	$\mathbb{Q}$	1	1	2	1	$\zeta(s)\,\zeta(s-1)$	$\zeta(s)\,\zeta(s-1)$
pm, **pg**, **cm**	$C_2=\langle m\rangle$	$\mathbb{Q}\,\sigma$	$\mathbb{Q}$	1	1	1	0	$\zeta(s)$	$\zeta(s)$
		$\mathbb{Q}\,(m-1)$	$\mathbb{Q}$	1	1	1	1	$\zeta(s)$	$\zeta(s-1)$
p2mm, **p2mg**, **p2gg**, **c2mm**	$C_2\times C_2=\langle p\rangle\times\langle q\rangle$	$\mathbb{Q}\,\sigma$	$\mathbb{Q}$	1	1	0	0	1	1
		$\mathbb{Q}\,(pq-1)$	$\mathbb{Q}$	1	1	0	1	1	1
		$\mathbb{Q}\,(p-1)$	$\mathbb{Q}$	1	1	1	1	$\zeta(s)$	$\zeta(s-1)$
		$\mathbb{Q}\,(q-1)$	$\mathbb{Q}$	1	1	1	1	$\zeta(s)$	$\zeta(s-1)$
p4	$C_4=\langle r\rangle$	$\mathbb{Q}\,\sigma$	$\mathbb{Q}$	1	1	0	0	1	1
		$\mathbb{Q}\,(r-1)$	$\mathbb{Q}\,(\omega_4)$	1	1	1	1	$\zeta_{\mathbb{Q}(\omega_4)}(s)$	$\zeta_{\mathbb{Q}(\omega_4)}(s-1)$
p4mm, **p4gm**	$D_8=C_4\cdot C_2=\langle r\rangle\cdot\langle m\rangle$	$\mathbb{Q}\,\sigma$	$\mathbb{Q}$	1	1	0	0	1	1
		$\mathbb{Q}\,(m-1)$	$\mathbb{Q}$	1	1	0	1	1	1
		$\mathbb{Q}\,(r^2-1)$	$\mathbb{Q}$	1	1	0	1	1	1
		$\mathbb{Q}\,(mr^2-1)$	$\mathbb{Q}$	1	1	0	1	1	1
		$\mathrm{M}_2(\mathbb{Q})$	$\mathbb{Q}$	2	1	1	1	$\zeta(2s)$	$\zeta(2(s-1))$
p3	$C_3=\langle r\rangle$	$\mathbb{Q}\,\sigma$	$\mathbb{Q}$	1	1	0	0	1	1
		$\mathbb{Q}\,(r-1)$	$\mathbb{Q}\,(\omega_3)$	1	1	1	1	$\zeta_{\mathbb{Q}(\omega_3)}(s)$	$\zeta_{\mathbb{Q}(\omega_3)}(s-1)$
p31m, **p3m1**	$S_3=C_3\cdot C_2=\langle r\rangle\cdot\langle m\rangle$	$\mathbb{Q}\,\sigma$	$\mathbb{Q}$	1	1	0	0	1	1
		$\mathbb{Q}\,(m-1)$	$\mathbb{Q}$	1	1	0	1	1	1
		$\mathrm{M}_2(\mathbb{Q})$	$\mathbb{Q}$	2	1	1	1	$\zeta(2s)$	$\zeta(2(s-1))$
p6	$C_6=\langle r\rangle$	$\mathbb{Q}\,\sigma$	$\mathbb{Q}$	1	1	0	0	1	1
		$\mathbb{Q}\,(r^2-1)$	$\mathbb{Q}\,(\omega_3)$	1	1	0	1	1	1
		$\mathbb{Q}\,(r^3-1)$	$\mathbb{Q}$	1	1	0	1	1	1
		$\mathbb{Q}\,(r^3-1)\otimes\mathbb{Q}\,(r^2-1)$	$\mathbb{Q}\,(\omega_3)$	1	1	1	1	$\zeta_{\mathbb{Q}(\omega_3)}(s)$	$\zeta_{\mathbb{Q}(\omega_3)}(s-1)$
p6mm	$D_{12}=C_6\cdot C_2=\langle r\rangle\cdot\langle m\rangle$	$\mathbb{Q}\,\sigma$	$\mathbb{Q}$	1	1	0	0	1	1
		$\mathbb{Q}\,(m-1)$	$\mathbb{Q}$	1	1	0	1	1	1
		$\mathbb{Q}\,(r^3-1)$	$\mathbb{Q}$	1	1	0	1	1	1
		$\mathbb{Q}\,(mr^3-1)$	$\mathbb{Q}$	1	1	0	1	1	1
		$\mathrm{M}_2(\mathbb{Q})$	$\mathbb{Q}$	2	1	0	1	1	1
		$\mathrm{M}_2(\mathbb{Q})$	$\mathbb{Q}$	2	1	1	1	$\zeta(2s)$	$\zeta(2(s-1))$

TABLE 2. *Counting normal subgroups.*

E	P	$1 \leqslant P_* \lhd P$	$[E_{p*}, M_p]$	$\zeta_E^{E_*, \lhd}(s) \sim$
p1	1	1	1	$\zeta(s)\,\zeta(s-1)$
p2	$C_2 = \langle r \rangle$	1	1	$\zeta(s)\,\zeta(s-1)$
		$\langle r \rangle$	M_p	1
pm, **pg**, **cm**	$C_2 = \langle m \rangle$	1	1	$\zeta(s)^2$
		$\langle m \rangle$	$\mathbb{Q}_p y$	$\zeta(s)$
p2mm, **p2mg**, **p2gg**	$C_2 \times C_2 = \langle p \rangle \times \langle q \rangle$	1	1	$\zeta(s)^2$
		$\langle p \rangle$	$\mathbb{Q}_p y$	$\zeta(s)$
		$\langle q \rangle$	$\mathbb{Q}_p x$	$\zeta(s)$
		$\langle pq \rangle$, $\langle p, q \rangle$	M_p	1
c2mm	$C_2 \times C_2 = \langle r \rangle \times \langle m \rangle$	1	1	$\zeta(s)^2$
		$\langle m \rangle$, $\langle rm \rangle$	$\mathbb{Q}_p y$	$\zeta(s)$
		$\langle r \rangle$, $\langle r, m \rangle$	M_p	1
p4	$C_4 = \langle r \rangle$	1	1	$\zeta_{\mathbb{Q}(\omega_4)}(s)$
		$\langle r^2 \rangle$, $\langle r \rangle$	M_p	1
p4mm, **p4gm**	$D_8 = C_4 \cdot C_2 = \langle r \rangle \cdot \langle m \rangle$	1	1	$\zeta(2s)$
		$\langle r \rangle$, $\langle r^2 \rangle$, $\langle r^2, m \rangle$, $\langle r, m \rangle$	M_p	1
p3	$C_3 = \langle r \rangle$	1	1	$\zeta_{\mathbb{Q}(\omega_3)}(s)$
		$\langle r \rangle$	M_p	1
p31m, **p3m1**	$S_3 = C_3 \cdot C_2 = \langle r \rangle \cdot \langle m \rangle$	1	1	$\zeta(2s)$
		$\langle r \rangle$, $\langle r, m \rangle$	M_p	1
p6	$C_6 = \langle r \rangle$	1	1	$\zeta_{\mathbb{Q}(\omega_3)}(s)$
		$\langle r \rangle$, $\langle r^2 \rangle$, $\langle r^3 \rangle$	M_p	1
p6mm	$D_{12} = C_6 \cdot C_2 = \langle r \rangle \cdot \langle m \rangle$	1	1	$\zeta(2s)$
		$\langle r \rangle$, $\langle r^2 \rangle$, $\langle r^2, m \rangle$, $\langle r^3 \rangle$, $\langle r^3, m \rangle$, $\langle r, m \rangle$	M_p	1

(1) the structure of $A = \mathbb{Q}P$ and its decomposition into simple components $A = A_1 \oplus \ldots \oplus A_r$;

(2) F_i, the centre of the simple algebra A_i, and R_i, its ring of integers; A_i is isomorphic to a full ring of matrices of rank m_i over some central F_i-division algebra D_i;

(3) $n_i^2 = \dim_{F_i}(A_i) = m_i^2 e_i^2$ where $e_i^2 = \dim_{F_i}(D_i)$;

(4) the integer k_i defined as follows: if $V = T \otimes \mathbb{Q}$ is the module for A induced by the action of P on T then $V = V_1 \oplus \ldots \oplus V_r$ where $V_i = A_i V = (W_i)^{k_i}$ and W_i is the A_i-module $(D_i)^{m_i}$;

(5) set $\varepsilon_i = 0$ or 1 according to whether $C_{W_i}(P) = C_{W_i}(A)$ is W_i or 0; then

$$\zeta_E^E(s) \sim \prod_{i=1}^{r} \prod_{j=0}^{k_i e_i - 1} \zeta_{R_i}(n_i(s - \varepsilon_i) - j)$$

where $\sim$ will mean up to multiplication by rational functions in p^{-s} for each prime p with $(p, |P|) = 1$ and D_i not split over p.

Table 2 compares the calculations of the normal zeta function. Here we need to record the normal subgroups E_* lying between E and T and the structure of $[E_*, T]$. For $E_* = T$ we will get

$$\zeta_E^{T,\lhd}(s) \sim \prod_{i=1}^{r} \prod_{j=0}^{k_i e_i - 1} \zeta_{R_i}(n_i s - j).$$

In both tables ω_n denotes a primitive nth root of unity, and the element $\sigma = \sum_{g \in P} g$. If $T \gneqq E_* \lhd E$ then we put $P_* = E_*/T$. In Table 1, $\zeta_E^{T,\lhd}(s)$ and $\zeta_E^E(s)$ are each $\sim$-equivalent to the product of the r functions (one corresponding to each simple component of A) listed in the corresponding row.

We refer to [**4**, Example 7.39] for the structure of A in the case that P is a dihedral group. Note that we never get any instance above where a simple component of A involves a division algebra. Hence the only bad primes will be those for which $(p, |P|) = 1$. This tallies when one compares Tables 1 and 2 with the calculations in § 4. (Some care must be taken reading the zeta function of the bad prime from the results in § 4 since some collecting of terms has been done.) If the point group was a quaternion group (not an example which arises in the wallpaper groups) then we would get a division algebra (see [**4**, Example 7.40]).

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