Part III Differential Geometry Lecture Notes

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Contents

1 Introduction

These notes accompany my Michaelmas 2012 Cambridge Part III course on Differential geometry. The purpose of the course is to cover the basics of differential manifolds and elementary Riemannian geometry, up to and including some easy comparison theorems. Time permitting, Penrose's incompleteness theorems of general relativity will also be discussed.

We will give the formal definition of manifold in Section 2. In the rest of this introduction, we first discuss informally how the manifold concept naturally arises from abstracting precisely that structure on smooth surfaces in Euclidean space that allows us to define consistently smooth functions. We will then give a preliminary sketch of the notion of Riemannian metric first in two then in higher dimensions and give a brief overview of some of the main themes of Riemannian geometry to follow later in the course.

These notes are still very much "under construction". Moreover, they are on the whole pretty informal and meant as a companion but not a substitute for a careful and detailed textbook treatment of the material–for the latter, the reader should consult the references described in Section 16.

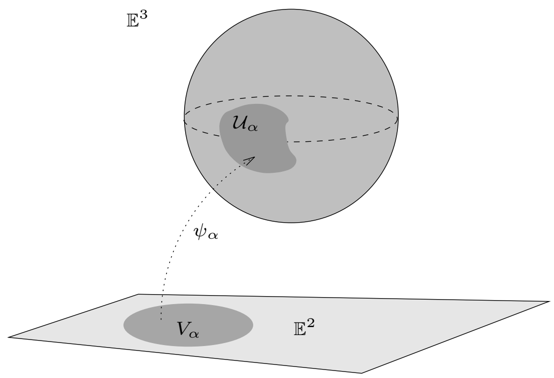
1.1 From smooth surfaces to smooth manifolds

The simplest way that the objects of the form we call smooth surfaces $\mathcal{S} \subset \mathbb{E}^3$ arise are as level sets of a smooth function, say $f(x, y, z) = c$, at a non-critical[1] value c. Example: $\mathbb{S}^2$ as $x^2 + y^2 + z^2 = 1$. It is the implicit function theorem[2] that says that these objects are, in some sense, two dimensional, i.e. that $\mathcal{S}$ can be expressed as the union of the images of a collection of maps $\psi_\alpha : V_\alpha \to \mathbb{E}^3$, $V_\alpha \subset \mathbb{E}^2$, such that ψ_α is smooth, $D\psi_\alpha$ is one-to-one, and denoting $\psi_\alpha(V_\alpha)$ as

[1] i.e. a value $c \in \mathbb{R}$ such that $df(p)$ is surjective for all $p \in f^{-1}(c)$

[2] The reader is assumed familiar with standard results in multivariable analysis.

$\mathcal{U}_\alpha$, ψ_α is a homeomorphism[3] $\psi_\alpha : V_\alpha \to \mathcal{U}_\alpha$.[4]



Let us denote the inverse of the ψ_α's by $\phi_\alpha : \mathcal{U}_\alpha \to V_\alpha$. The collection $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ is known as an *atlas* of $\mathcal{S}$. Each $\mathcal{U}_\alpha, \phi_\alpha$ is called a *chart*, or alternatively, a *system of local coordinates*[5].

The word "differential" in the title of this course indicates that we should be able to do calculus. The point about local coordinates is that it allows us to do calculus *on the surface.*

The first issue:

> *How can we even define what it means for a function on the surface (i.e. a function $f : S \to \mathbb{R}$) to be differentiable?*

Answer:

Definition 1.1. *We say that $f : \mathcal{S} \to \mathbb{R}$ is C^∞ at a point p if $f \circ \phi_\alpha^{-1} : V_\alpha \to \mathbb{R}$ is C^∞ for some α.*

For this to be a good definition, it should not depend on the chart. Let ϕ_α, ϕ_β be different charts containing p. We have

$$f \circ \phi_\alpha^{-1} = f \circ \phi_\beta^{-1} \circ (\phi_\beta \circ \phi_\alpha^{-1})$$

where this is defined.

Proposition 1.1. *$\phi_\beta \circ \phi_\alpha^{-1}$ is C^∞ on the domain where it is defined.*

Proof. Exercise. □

Thus, the definition holds for any compatible chart. The maps $\phi_\beta \circ \phi_\alpha^{-1}$ are sometimes known as *transition functions.*

Now let us forget for a minute that $\mathcal{S} \subset \mathbb{E}^3$. Just think of our surface as the topological space $\mathcal{S}$, and suppose we have been given a collection of homeomorphisms $\phi_\alpha : \mathcal{U}_\alpha \to V_\alpha$, *without knowing that these are $\phi_\alpha = \psi_{-\alpha}^{-1}$ for smooth $\mathbb{E}^3$-valued maps ψ.* Given just this information, suppose we ask:

[3] Here we are taking $\mathcal{S}$ to have the induced topology from $\mathbb{E}^3$. We assume that the reader is familiar with basic notions of point set topology.

[4] For Cambridge readers only: This is precisely the "Part II" definition of a manifold.

[5] Actually, more correctly, one says that the system of local coordinates are the projections $x^i \circ \phi_\alpha$ to the standard coordinates on $\mathbb{R}^2$.

What is the least amount of structure necessary to define consistently the notion that a function $f : \mathcal{S} \to \mathbb{R}$ is smooth?

We easily see that the definition provided by Definition 1.1 is a good definition *provided that the result of Proposition 1.1 happens to hold.* For it is precisely the statement of the latter proposition which shows that if $\phi \circ \phi_\alpha^{-1}$ is smooth at p for *some* α where U_α contains p, then it is smooth for *all* charts.

We now apply one of the oldest tricks of mathematical abstraction. We make a proposition into a definition. The notion of an *abstract smooth surface* distills the property embodied by Proposition 1.1 from that of a surface in $\mathbb{E}^3$, and builds it into the definition.

Definition 1.2. *An* abstract smooth surface *is a topological space $\mathcal{S}$ together with an open cover $\mathcal{U}_\alpha$ and homeomorphisms $\phi_\alpha : \mathcal{U}_\alpha \to V_\alpha$, with V_α open subsets of $\mathbb{R}^2$, such that $\phi_\beta \circ \phi_\alpha^{-1}$, where defined, are C^∞.*

The notion of a smooth n-dimensional manifold $\mathcal{M}$ is defined now precisely as above, where $\mathbb{R}^2$ is replaced by $\mathbb{R}^n$.[6]

Definition 1.3. *A map $f : \mathcal{M} \to \tilde{\mathcal{M}}$ is smooth if $\tilde{\phi}_\beta \circ f \circ \phi_\alpha^{-1}$ is smooth for some α, β.*

Check that this is a good definition (i.e. "for some" implies "for all").

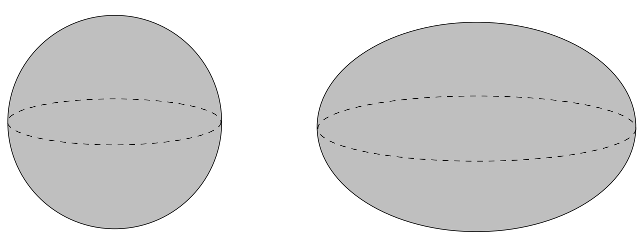
Definition 1.4. *$\mathcal{M}$ and $\tilde{\mathcal{M}}$ are said to be* diffeomorphic *if there exists an $f : \mathcal{M} \to \tilde{\mathcal{M}}$ such that f and f^{-1} are both smooth.*

Exercise: The dimension n of a manifold is uniquely defined and a diffeomorphism invariant.

Examples: $\mathbb{E}^n$, $\mathbb{S}^n$, products, quotients, twisted products (fiber bundles, etc.), connected sums, configuration spaces from classical mechanics. The point is that manifolds are a very flexible category and there is the usual economy provided by a good definition. We will discuss all this soon enough in the course.

1.2 What defines geometry?

The study of smooth manifolds and the smooth maps between them is what is known as *differential topology.* From the point of view of the smooth structure, the sphere $\mathbb{S}^n$ and the set $\frac{x_1^2}{a_1^2} + \cdots \frac{x_{n+1}^2}{a_{n+1}^2} = 1$ are diffeomorphic as manifolds.



To speak about geometry, we must define additional structure. To speak about "differential" geometry, this structure should be defined via the calculus. Without a doubt, the most important such structure is that of a *Riemannian* (or more generally semi-Riemannian) metric.

[6] The actual definition, to be given in the next section, will be enriched by several topological assumptions–so let us not state anything formal here.

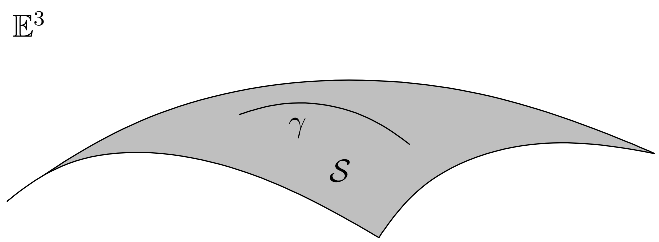
This concept again arises from distilling from the theory of surfaces in $\mathbb{E}^3$ a piece of structure: A surface $\mathcal{S} \subset \mathbb{E}^3$ comes with a notion of how to measure the lengths of curves. This notion can be characterized at the differential level. Formally, we may write

$$dx^2 + dy^2 + dz^2 = E(u,v)du^2 + 2F(u,v)dudv + G(u,v)dv^2, \tag{1}$$

where

$$\begin{aligned} E &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ F &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \\ G &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 . \end{aligned}$$

This is motivated by the chain-rule à la Leibniz. The expression on the right hand side of (1) is called the *first fundamental form*. What does this actually mean? Say that a smooth curve $\gamma : I \to \mathcal{S}$ is given by $(x(t), y(t), z(t)) = (u(t), v(t))$.



Then we can compute its length L in the standard way:

$$L = \int \sqrt{x'^2 + y'^2 + z'^2}dt,$$

and, by the chain rule, we obtain

$$L = \int \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt \tag{2}$$

in our local coordinates on $\mathcal{S}$. It turns out that if $(\tilde{u}, \tilde{v})$ is another coordinate system, then writing $dx^2+dy^2+dz^2 = \tilde{E}d\tilde{u}^2+2\tilde{F}d\tilde{u}d\tilde{v}+\tilde{G}d\tilde{v}^2$, we can compute the relation between E and $\tilde{E}$:

$$\tilde{E} = E\frac{\partial u}{\partial \tilde{u}}\frac{\partial u}{\partial \tilde{u}} + 2F\frac{\partial u}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{u}} + G\frac{\partial v}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{u}}. \tag{3}$$

Now we ask, let us again forget about $\mathbb{E}^3$. Question: What was it about $\mathcal{S}$ that allowed us to unambiguously define lengths of curves? Answer: A set of functions E, F, G defined for each chart, transforming via (3). We distill from the above the following:

Definition 1.5. *A* Riemannian metric *on an abstract* 2*-surface is a collection of smooth functions* $\{E_\alpha\}$*,* $\{F_\alpha\}$*,* $\{G_\alpha\}$ *on an atlas* $\{\mathcal{U}_\alpha\}$*, transforming like in* (3)*, satisfying in addition*

$$E_\alpha > 0, \qquad G_\alpha > 0, \qquad E_\alpha G_\alpha - F_\alpha^2 > 0. \tag{4}$$

In particular, the formula (2) now allows one to define consistently the notion of the length of a smooth curve $\phi : I \to \mathcal{S}$.[7] The condition (4) ensures that our notion of length is positive.[8]

The expression on the right hand side can be generalized to n dimensions, and this defines the notion of a Riemannian metric on a smooth manifold.[9] A couple

$$(\mathcal{M}, g)$$

consisting of an n-dimensional manifold $\mathcal{M}$, together with a Riemannian metric g defined on $\mathcal{M}$, is known as a *Riemannian manifold*. *Riemannian geometry* is the study of Riemannian manifolds.

The reader familiar with the geometry of surfaces has no doubt encountered the so-called *Theorema Egregium* of Gauss. This says that the curvature, originally, defined using the so-called second fundamental form[10], in fact can be expressed as a complicated expression in local coordinates involving up to second derivatives of the compontents E, F, G of the first fundamental form. That is to say, it could have been defined in the first place as said expression. In particular, the notion of curvature can thus be defined for abstract surfaces. One main difficulty in Riemannian geometry in higher dimensions is the algebraic complexity of the analogue of this curvature curvature, which is no-longer a scalar, but a so-called "tensor".

1.3 Geometry, curvature, topology

The following remarks are meant to give a taste of the kinds of results one wants to prove in geometry. Some familiarity with curvature of surfaces will be useful for getting a sense of what these statements mean.

The common thread in these examples is that they relate completeness, curvature and global behaviour (e.g. topology):

Theorem 1.1 (Hadamard–Cartan)**.** *Let $(\mathcal{M}, g)$ be a simply-connected*[11] *n-dimensional complete Riemannian manifold with nonpositive "sectional curvature". Then $(\mathcal{M}, g)$ is diffeomorphic to $\mathbb{R}^n$.*

Theorem 1.2 (Synge)**.** *Let $(\mathcal{M}, g)$ be complete, orientable, even-dimensional and of positive "sectional curvature". Then $(\mathcal{M}, g)$ is simply connected.*

Theorem 1.3 (Bonnet–Myers)**.** *Let $(\mathcal{M}, g)$ be a complete n-dimensional, $n \geq 2$, manifold whose "Ricci curvature" satisfies*

$$\mathrm{Ric} \geq (n-1)kg$$

for some $k > 0$. Then the diameter of $\mathcal{M}$ satisfies

$$\mathrm{diam}(\mathcal{M}) \leq \pi/\sqrt{k}.$$

[7] How to define a smooth curve?

[8] The semi-Riemannian case replaces (4) with the assumption that this determinant is non-zero. See Section 1.4.

[9] As we have tentatively defined them, not all manifolds admit Riemannian metrics. But Hausdorff paracompact ones do. . .

[10] For those who know about the geometry of curves and surfaces. . .

[11] We will often make reference to basic notions of algebraic topology.

1.3.1 Aside: Hyperbolic space and non-euclidean geometry

The set $\mathbb{H}^2$ can be covered by one chart $\{(u,v) : v > 0\}$, and the Riemannian metric is given by

$$\frac{1}{v^2}(du^2 + dv^2). \tag{5}$$

Later on we will recognize $\mathbb{H}^2$ as a complete space form with the topology of $\mathbb{R}^2$ and with constant curvature -1. This defines a so-called non-Euclidean geometry, a geometry satisfying all the axioms of Euclid with the exception of the so-called fifth postulate. In particular, the existence of the Riemannian geometry (5) shows the necessity of the Euclidean fifth postulate to determine Euclidean geometry.

The enigma of why it took so long for this to be understood is in part explained by the following global theorem:

Theorem 1.4. *Let $(\mathcal{S}, g)$ be an abstract surface with Riemannian metric. If $\mathcal{S}$ is complete with constant negative curvature, then $\mathcal{S}$ cannot arise as a subset of $\mathbb{E}^3$ so that g is induced as in* (1) *(in fact, not even as an immersed surface.)*

Compare this with the case of the sphere.

1.4 General relativity

A subject with great formal similarity, but a somewhat diverging epistomological basis, with Riemannian geometry is "general relativity". The basis for this theory is a four dimensional manifold: $\mathcal{M}$, called *spacetime*, together with a so-called *Lorentzian* metric, i.e. a smooth quadratic form $\sum g_{ij}dx^i dx^j$ such that the signature of g is $(-,+,+,+)$. (In two dimensions, Lorentzian vs. Riemannian would just mean that the sign of (4) is flipped.) Pure Lorentzian geometry in full generality is more complicated and less studied than pure Riemannian geometry. What sets general relativity apart from pure geometry, is that in this theory, the Lorentzian metric must satisfy a set of partial differential equations, the so-called Einstein equations. These equations constitute a relation between a geometric quantity, the Einstein tensor[12], and the energy-momentum content of matter. In the case where there is no matter present, these equations take the form

$$\mathrm{Ric} = 0$$

The central questions in general relativity are questions of the dynamics of this system. It is thus a much more rigid subject.

The above comments notwithstanding, there are (quite surprisingly!) some spectacular theorems in general relativity which can be proven via pure geometry. The reason: When the dynamics of matter is not specified, the Einstein equations still yield *inequalities* for this curvature tensor, analogous in many ways to the inequalities in the statement of the previous theorems. This allows one to prove so-called singularity theorems–better termed, the incompleteness theorems, the most important of which is the following result of Penrose

Theorem 1.5 (Penrose, 1965)**.** *Let $(\mathcal{M}, g)$ be a globally hyperbolic Lorentzian 4-manifold with non-compact Cauchy hyper surface satisfying*

$$\mathrm{Ric}(V,V) \geq 0$$

[12]This is an expression derived from the Ricci and scalar curvatures.

for all null vectors V. Suppose moreover that $(\mathcal{M}, g)$ contains a closed trapped 2-surface. Then $(\mathcal{M}, g)$ is future-causally geodesically complete.

This theorem can be directly compared to the Bonnet–Myers theorem referred to before. The elements entering into the proof are actually more or less the same, but the traditional logical sequence of their statements different. Whereas Bonnet–Myers is phrased

$$\begin{array}{c} \text{completeness} \\ \text{mild topological assumption,} \\ \text{Ricci curvature sign} \end{array} \Rightarrow \text{diameter bound}$$

Penrose's theorem is traditionally phrased:

$$\begin{array}{c} \text{mild geometric/topological assumption,} \\ \text{Ricci curvature sign,} \\ \exists \text{trapped surface} \end{array} \Rightarrow \text{incompleteness}$$

The similarity to Bonnet–Myers is more clear if we phrase Penrose's theorem equivalently

$$\begin{array}{c} \text{completeness} \\ \text{lessmild geometric/topological assumption,} \\ \text{Ricci curvature sign,} \\ \exists \text{trapped surface} \end{array} \Rightarrow \text{Cauchy hypersurface is compact}$$

Time permitting, we will discuss these later...

2 Manifolds

2.1 Basic definitions

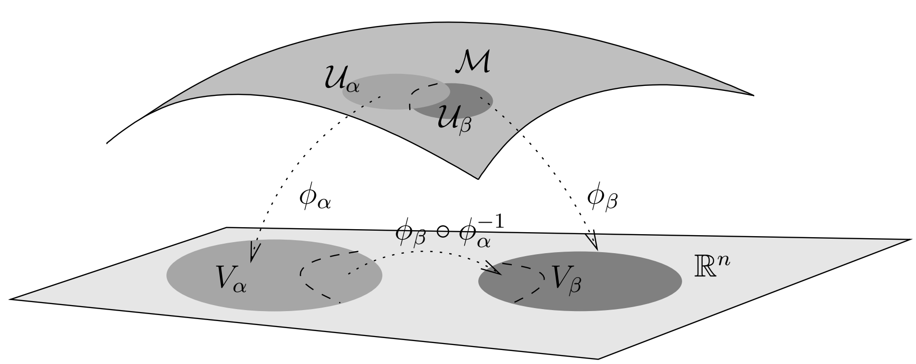
2.1.1 Charts and atlases

Definition 2.1. *Let X be a topological space. A* smooth n-dimensional atlas *on X is a collection $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$, where $\mathcal{U}_\alpha$ are an open cover of X and*

$$\phi_\alpha : \mathcal{U}_\alpha \to V_\alpha,$$

where $V_\alpha \subset \mathbb{R}^n$ are open, such that $\phi_\alpha \circ \phi_\beta^{-1}$ is C^∞ where defined (i.e. on $\phi_\beta(\mathcal{U}_\beta \cap \mathcal{U}_\alpha)$). Each $(\mathcal{U}_\alpha, \phi_\alpha)$ is known as a chart.

See:



Note that $\phi_\alpha \circ \phi_\beta^{-1}$ is a map from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$, so it makes sense to discuss it's smoothness!

Let X be a topological space and $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ a smooth atlas. Let $(\mathcal{U}, \phi)$ be such that $\mathcal{U} \subset X$ is open, $\phi : \mathcal{U} \to \mathcal{V} \subset \mathbb{R}^n$ a homeomorphism, such that

$$\phi \circ \phi_\alpha^{-1}, \qquad \phi_\alpha \circ \phi^{-1} \qquad \text{are } C^\infty \text{ where defined.} \tag{6}$$

Then $\{(\mathcal{U}_\alpha, \phi_\alpha)\} \cup \{(\mathcal{U}, \phi)\}$ is again an atlas.

Definition 2.2. *Let X be a topological space and $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ a smooth atlas. $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ is* maximal *if for all $(\mathcal{U}, \phi)$ as above satisfying* (6)*, then $(\mathcal{U}, \phi) \in \{(\mathcal{U}_\alpha, \phi_\alpha)\}$.*

One can easily show

Proposition 2.1. *Given an atlas on X, there is a unique maximal atlas containing it.*

Given an atlas $\{(\mathcal{U}_\alpha, \phi_\alpha\}$, the restriction of ϕ_α to all open $\tilde{U} \subset \mathcal{U}_\alpha$ will in particular be in the maximal atlas containing $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$.

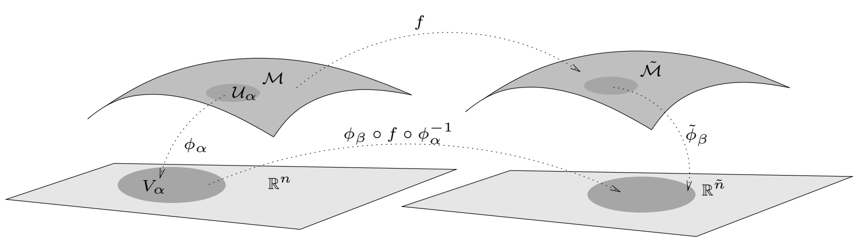
2.1.2 Definition of smooth manifold

Definition 2.3. *A C^∞ manifold of dimension n is a Hausdorff, second countable and paracompact topological space $\mathcal{M}$, together with maximal smooth n-dimensional atlas.*

Given a chart (U_α, ϕ_α), we call $\pi^i \circ \phi_\alpha$ a system of *local coordinates*, where π^i denote the projections to standard coordinates on $\mathbb{R}^n$. Very often in notation we completely suppress ϕ_α and talk about local coordinates $(x^1, \ldots, x^n)$. It is understood that $x^1 = \pi^1 \circ \phi_\alpha$ for a ϕ_α.

2.1.3 Smooth maps of manifolds

Definition 2.4. *A continuous map $f : \mathcal{M} \to \tilde{\mathcal{M}}$ is C^∞ if $\tilde{\phi}_\beta \circ f \circ \phi_\alpha^{-1}$ is C^∞ for all charts where this mapping is defined.*



This definition would be hard to use in practice since maximal atlases are very big! We have however:

Proposition 2.2. *A continuous map $f : \mathcal{M} \to \tilde{\mathcal{M}}$ is smooth iff for all $p \in \mathcal{M}$, there exist charts U_α, $\tilde{U}_\alpha$ around p and $f(p)$, respectively such that $\tilde{\phi}_\beta \circ f \circ \phi_\alpha^{-1}$ is C^∞ where defined.*

This follows immediately from the smoothness of the transition functions. As we said already in the introduction, this is the whole point of the definition of manifolds: it allows us to talk about smooth functions (and more generally smooth maps) by checking smoothness with respect to a particular choice of charts.

If $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are of dimensions m and n respectively we shall often refer to n coordinate components of the map

$$\phi_\alpha^{-1} \circ f \circ \tilde{\phi}_\beta$$

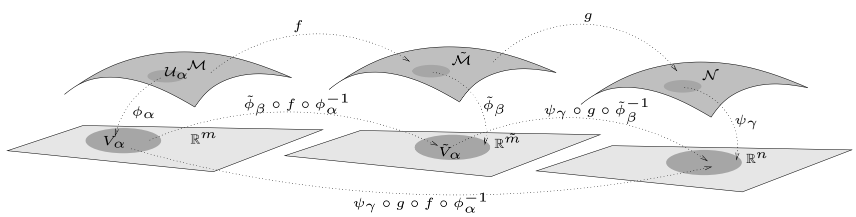
by

$$f^1(x^1, x^2, \ldots, x^m), \ldots, f^n(x^1, \ldots, x^m).$$

With this notation, the map $f : \mathcal{M} \to \tilde{\mathcal{M}}$ is smooth iff the above maps f^i are smooth in some choice of local coordinates around every point.

Proposition 2.3. *If $f : \mathcal{M} \to \tilde{\mathcal{M}}$ is smooth, and $g : \tilde{\mathcal{M}} \to \mathcal{N}$ is smooth, then $g \circ f$ is smooth.*

Proof.



□

Definition 2.5. *$\mathcal{M}$ and $\tilde{\mathcal{M}}$ are said to be* diffeomorphic *if there exists an $f : \mathcal{M} \to \tilde{\mathcal{M}}$ such that f and f^{-1} are both smooth.*

Exercise: The above defines an equivalence relation.

2.1.4 Examples

Example 2.1. *The set $\mathbb{R}^n$ is an n-dimensional manifold defined by (the maximal atlas containing) the atlas consisting of a single chart, the identity map.*

Example 2.2. *$\mathbb{S}^n$, with topology given as the subset*

$$(x^1)^2 + \cdots (x^{n+1})^2 = 1$$

of $\mathbb{R}^{n+1}$, can be given the structure of an n-dimensional smooth manifold with coordinate charts the projections to the coordinate hyperplanes.

To see this, note the transition functions are of the form:

$$(x^1, \ldots, x^{k-1}, x^{k+1}, \ldots x^{n+1}) \mapsto$$
$$(x^1, \ldots, x^{k-1}, \sqrt{1 - \sum_{i \neq k} (x^i)^2}, x^{k+1}, \ldots, x^{h-1}, x^{h+1}, \ldots, x^{n+1})$$

Note. In various dimensions, for instance 7 and (conjecturally) 4, there are differentiable structures inequivalent to the above[13] which live on the same topology. These are called *exotic spheres*.

Example 2.3. *Denote by $\mathbb{R}P^n$ the set of all lines through the origin in $n+1$-dimensional space. This space can be endowed with the structure of an n-dimensional manifold, and is then called* real projective space. *With this structure the map $\pi : \mathbb{S}^n \to \mathbb{R}P^n$ is smooth.*

This is an example of the quotient by a discrete group action. For an extension of this kind of construction, see the first example sheet.

Example 2.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds. Then one can define a natural manifold structure on $\mathcal{M} \times \mathcal{N}$.*

Take $\{(\mathcal{U}_\alpha \times \tilde{\mathcal{U}}_\beta, \phi_\alpha \times \tilde{\phi}_\beta)\}$. Complete the details...

2.2 Tangent vectors

Let $\mathcal{M}$ be a smooth manifold, let $p \in \mathcal{M}$. Let $X(p)$ denote the algebra of locally C^∞ functions at p.[14] Note that if $f \in X(p)$ and $g \in X(p)$ then $fg \in X(p)$, where fg is a locally defined function.

Definition 2.6. *A* derivation *D at p is a mapping $D : X(p) \to \mathbb{R}$ satisfying $D(\lambda f + \mu g) = \lambda Df + \mu Dg$, for λ, μ scalars, and, in addition, $D(fg) = (Df)g(p) + f(p)(Dg)$.*

Proposition 2.4. *The set of derivations at p define a vector space of dimension n, denoted $T_p\mathcal{M}$.*

Proof. The fact that $T_p\mathcal{M}$ is a vector space is clear. Let x^i be a system of local coordinates centred at p. Define a map $\frac{\partial}{\partial x^i}|_p$ by

$$\frac{\partial}{\partial x^i}|_p f = \partial_{x^i} f \circ \phi_\alpha^{-1}|_{\phi_\alpha(p)}$$

where ϕ_α is the name of the chart map defining the coordinates x^i. (Note $\frac{\partial}{\partial x^i}|_p x^j \circ \phi_\alpha = \delta_i^j$. This in particular implies that the $\frac{\partial}{\partial x^i}$ are linearly independent.) Clearly $\frac{\partial}{\partial x^i}|_p$ is a derivation, by the well-known properties of derivatives. We want to show that the $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}$ span $T_p\mathcal{M}$. It suffices to show that if $Dx^i \circ \phi^\alpha = 0$ for all x^i, then $D = 0$. So let D be such a D, and let f be arbitrary. Locally, $f = \alpha_i x^i + g_i x^i$ where $\alpha_i \in \mathbb{R}$ and where g_i are C^∞. Thus, $Df(p) = \alpha_i Dx^i + x_i Dg^i = 0$, since WLOG we can choose p to correspond to the origin of coordinates. □

Note. We have used above the Einstein summation convention, i.e. the convention that whenever we the same index "up" and "down", as in the expression $\alpha_i x^i$, we are to understand $\sum_{i=1}^n \alpha_i x^i$. Note that the index i of $\frac{\partial}{\partial x^i}|_p$ is to be understood as down. Here $n = \dim \mathcal{M}$.

Definition 2.7. *We will call $T_p\mathcal{M}$ the* tangent space *of $\mathcal{M}$ at p, and we will call its elements* tangent vectors.

[13] i.e. such that the resulting manifold is not diffeomeorphic to the above
[14] Exercise: define this space formally in whatever way you choose.

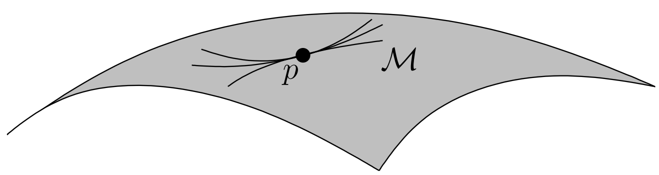
Proposition 2.5. *Let x^i and $\tilde{x}^i$ denote two coordinate systems. Then $\frac{\partial}{\partial \tilde{x}^i}|_p = \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial}{\partial x^j}|_p$, for all p in the common domain of the two coordinate charts.*

Proof. If we apply an arbitrary f to both sides, then by the chain rule, the left and right hand side coincide. Thus, the two expressions correspond to one and the same derivation. $\square$

Notation. In writing $\frac{\partial x^j}{\partial \tilde{x}^i}$ one is to understand $\partial_i(\pi^j \circ \phi \circ \tilde{\phi}^{-1})$, where $\tilde{\phi}$ and ϕ are the two charts corresponding to the local coordinates.[15]

The geometric interpretation of derivations at p: Let γ be a smooth curve through p, i.e. a smooth map $\gamma : (-\epsilon, \epsilon) \to \mathcal{M}$ such that $\gamma(0) = p$. Given f, define a derivation D_γ at p by $D_\gamma f = (f \circ \gamma)'(0)$. All derivations in fact arise in this way. For given $\alpha^i \frac{\partial}{\partial x^i}$, then one can consider the curve $t \mapsto (\alpha^1 t, \ldots, \alpha^n t)$, and it is clear from the definition of partial differentiation in local coordinates that the action of D_γ coincides with that of $\alpha^i \frac{\partial}{\partial x^i}|_p$. We will often denote this tangent vector as γ' or $\dot{\gamma}$.

The curves depicted below, suitably parametrized, all correspond to the same derivation at p.



We thus often visualize tangent vectors as arrows of a given length (related to the above mentioned parametrization) through p in the direction distuingished by these curves. Exercise: Draw on top of this picture such a vector!

2.3 The tangent bundle

From multidimensional calculus, one knows the importance of considering smooth vector fields. We would like a geometric way of describing these in the case of manifolds. It turns out that there is sufficient "economy" in the definition of manifold so as to apply it also to the natural space where these tangent vectors "live". This will allow us to discuss smoothness.

Let $\mathcal{M}$ be an n-dimensional smooth manifold. Define $T\mathcal{M}$ to be the set of tangent vectors in $\mathcal{M}$, i.e.

$$T\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}.$$

Note the natural map

$$\pi : T\mathcal{M} \to \mathcal{M},$$

taking a vector in $T_p\mathcal{M}$ to p. Define an atlas $\{\tilde{\mathcal{U}}_\alpha, \tilde{\phi}_\alpha\}$ as follows:

$$\tilde{\mathcal{U}}_\alpha = \pi^{-1}(\mathcal{U}_\alpha)$$

$$\tilde{\phi}_\alpha : \left\{ \alpha^i \frac{\partial}{\partial x^i}\bigg|_p \right\} \mapsto \phi_\alpha(p) \times (\alpha^1, \ldots, \alpha^n).$$

[15]It is assumed that you know what this means because $\pi^j \circ \phi \circ \tilde{\phi}^{-1} : \mathbb{R}^n \to \mathbb{R}$, so this is partial differentiation from calculus of many variables.

Proposition 2.6. *The above choice of atlas makes $T\mathcal{M}$ into a smooth manifold such that π is smooth.*

Note that, for fixed $x \in \mathcal{M}$, $\tilde{\phi}$ restricted to $T_p\mathcal{M}$ is a linear map.

Definition 2.8. *A* vector field *is just a smooth map $V : \mathcal{M} \to T\mathcal{M}$ such that $\pi \circ V = id$, where id denotes the identity map.*

3 More bundles

3.1 The general definition of vector bundle

The tangent bundle is a special case of the following:

Definition 3.1. *A* smooth vector bundle *of rank n is a map of manifolds $\pi : \mathcal{E} \to \mathcal{M}$, where $\mathcal{M}$ is an m-dimensional manifold for some m, such that, for each p, $\pi^{-1}(p) \doteq \mathcal{E}_p$ is an n dimensional vector space known as the* fibre over p, *and such that there exists an open cover $\tilde{\mathcal{U}}_\alpha$ of $\mathcal{M}$ and smooth maps (so called* local trivialisations

$$\psi_\alpha : \tilde{\mathcal{U}}_\alpha \times \mathbb{R}^n \to E$$

commuting with the two natural projections, i.e. so that$\pi \circ \psi$ is the identity acts as $\mathcal{U}_\alpha \times \mathbb{R}^n \to \tilde{\mathcal{U}} \to \mathcal{M}$, and such that moreover $\psi|_{\{p\}\times\mathbb{R}^n} : \{p\} \times \mathbb{R}^n \to E_p$ are linear isomorphisms.

Let us note that given $\mathcal{E}$ as above, we can construct a special atlas compatible with its smooth structure as follows. Given an atlas $\mathcal{U}_\alpha$ for $\mathcal{M}$ which without loss of generality satisfies $\mathcal{U}_\alpha \subset \tilde{\mathcal{U}}_\alpha$, we may define a map $\tilde{\phi}_\alpha$ by composing $\phi_\alpha \times \mathrm{id} \circ \psi^{-1}$

$$\tilde{\phi}_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{R}^m$$

and this collection yields an atlas for $\mathcal{E}$. Note moreover that the restrictions of the transition functions to the fibres

$$\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}|_{\tilde{\phi}_\alpha(\pi^{-1}(p))} : \{\phi_\alpha(p)\} \times \mathbb{R}^m \to \{\phi_\beta(p)\} \times \mathbb{R}^m \tag{7}$$

are linear maps.

Conversely, given a topological space $\mathcal{E}$, a manifold $\mathcal{M}$ and maps $\tilde{\phi}_\alpha$ satisfying (7), then this induces on $\mathcal{E}$ the structure of a smooth vector bundle of rank n. In particular, the fibres $\mathcal{E}_p = \pi^{-1}$ acquire the structure of a vector space. For defining

$$\lambda v_p + \mu w_p = \tilde{\phi}_\alpha^{-1}(\lambda\tilde{\phi}_\alpha v_p + \mu\tilde{\phi}_\alpha w_p)$$

for some chart, we have by (7) that

$$\tilde{\phi}_\alpha^{-1}(\lambda\tilde{\phi}_\alpha v_p + \mu\tilde{\phi}_\alpha w_p) = \tilde{\phi}_\beta(\lambda\tilde{\phi}_\beta v_p + \mu\tilde{\phi}_\beta w_p),$$

and thus the definition is chart independent.

Definition 3.2. *A* smooth section *of a vector bundle $\mathcal{E}$ is a map $\sigma : \mathcal{M} \to \mathcal{E}$ such that $\pi \circ \sigma = id$.*

Thus, in this language, vector fields are smooth sections of the tangent bundle.

3.2 Dual bundles and the cotangent bundle

First a little linear algebra. Given a finite dimensional vector space V (over $\mathbb{R}$), we can associate the dual space V^* consisting of all linear functionals $f : V \to \mathbb{R}$. This is a vector space of the same dimension as V.

Given now a map $\phi : V \to W$, then there is a natural map $\phi^* : W^* \to V^*$ defined by $\phi^*(g)(v) = g(\phi(v))$. Thus, given an *isomorphism* $\phi : V \to W$, there exists a map $\psi : V^* \to W^*$ defined by $\psi = (\phi^*)^{-1}$.

Now, given a vector bundle $\pi : \mathcal{E} \to \mathcal{M}$, we can define a vector bundle $\mathcal{E}^*$ called the dual bundle, where

$$\mathcal{E}^* = \bigcup_{p \in \mathcal{M}} (\mathcal{E}_p)^*$$

and where the charts $\chi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to V_\alpha \times \mathbb{R}^m$ of $\mathcal{E}^*$, when restricted to the fibres,

$$\chi_\alpha|_{\mathcal{E}^*} = \theta \circ \tilde{\psi}_\alpha|_{\mathcal{E}^*}$$

where $\tilde{\psi}_\alpha|_{\pi^{-1}(p)}$ denotes the map from $\mathcal{E}^*_p \to \mathbb{R}^{m*}$ induced from $\tilde{\phi}_\alpha|_{\mathcal{E}}$, where $\tilde{\phi}$ denote the coordinate charts of $\mathcal{E}$, and θ denotes some fixed[16] linear isomorphism $\theta : \mathbb{R}^{m^*} \to \mathbb{R}^m$.

Definition 3.3. *The dual bundle of the tangent bundle is denoted $T^*\mathcal{M}$ and is called the* cotangent bundle*. Elements of $T^*\mathcal{M}$ are called* covectors*, and sections of $T^*\mathcal{M}$ are called* 1*-forms.*

Let dx^i denote the dual basis[17] to $\frac{\partial}{\partial x^i}$.

Proposition 3.1. *Change of basis: $d\tilde{x}^j = \frac{\partial \tilde{x}^j}{\partial x^i} dx^i$.*

Note if $f \in C^\infty(\mathcal{M}, \mathbb{R})$, then there exists a one one form, which we will denote df, defined by

$$df(X) = X(f).$$

This is called the differential of f. Clearly, in local coordinates,

$$df = \frac{\partial f}{\partial x^i} dx^i$$

We can think of d as a linear operator

$$d : C^\infty(\mathcal{M}, \mathbb{R}) \to \Gamma(T^*M).$$

Much more about this point of view later.

3.3 The pull-back and the push forward

Let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map.

Definition 3.4. *For each p, the differential of F is a map $(F_*)_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$ which takes D to $\tilde{D}$ with $\tilde{D}g = D(g \circ F)$.*

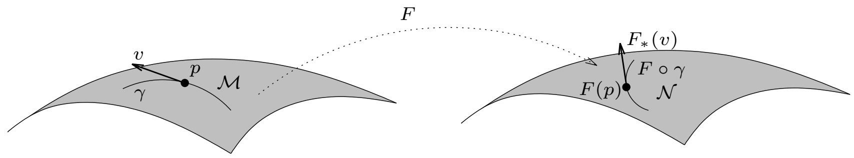
[16] i.e. not depending on α

[17] Recall this notion from linear algebra!

We can also describe the map F_* in terms of the equivalent characterization of tangent vectors as explained at the end of Section 2.2. Let v be a tangent vector and let γ be a curve such that $v = \gamma'$. Then

$$F_*(v) = (F \circ \gamma)'.$$

See below:



We have now

Definition 3.5. *We can define a map* $F^* : \Gamma(\mathcal{T}^*\mathcal{N}) \to \Gamma(\mathcal{T}^*\mathcal{M})$ *by* $F^*(\omega)(X) = \omega(F_*(X))$.

Definition 3.6. *Let* $F : \mathcal{M} \to \mathcal{N}$ *be smooth. We say that* F *is an* immersion *if* $(F_*)_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$ *is injective for all* p. *We say that* f *is an embedding if it is an* immersion *and* F *itself is* 1-1. *In the latter case, if* $\mathcal{M} \subset \mathcal{N}$ *and* F *is the identity, we call* F *a* submanifold.[18]

Example 3.1. *Let* $\mathcal{M}$ *be a manifold and* $\mathcal{U} \subset \mathcal{M}$ *an open set. Then* $\mathcal{U}$ *is a submanifold with the induced maps as charts.*

More interesting:

Proposition 3.2. *Let* $\mathcal{M}$ *be a smooth manifold, and let* $f_1, \dots f_d$ *be smooth functions. Let* $\mathcal{N}$ *denote the common zero set of* f_i *and assume* $df_1, \dots df_m$ *span a subset of dimension* d' *in* $T_p^*\mathcal{M}$, *for all* p, *where* d' *is constant. Then* $\mathcal{N}$ *can be endowed with the structure of a closed submanifold of* $\mathcal{M}$.

See the example sheet!

3.4 Multilinear algebra

The tangent and cotangent bundles are the simplest examples of *tensor bundles*. These are where the objects of interest to us in geometry "live". To understand them, we will need a short diversion into multilinear algebra.

Let U, V be vector spaces. We can define a vector space $U \otimes V$ as the free vector space generated by the symbols $u \otimes v$ as $u \in U$, $v \in V$, modulo the subspace generated by $u \otimes (\alpha v + \beta \tilde{v}) - \alpha u \otimes v + \beta u \otimes \tilde{v}$ and $(\alpha u + \beta \tilde{u}) \otimes v - \alpha u \otimes v + \beta \tilde{u} \otimes v$. This space is indeed a vector space. In fact, if U has dimension n, with basis $e_1, \dots, e_n$, and V has dimension m, with dimension $f_1, \dots, f_n$, then $U \otimes V$ has dimension nm, with basis $\{e_i \otimes f_j\}$.

Proposition 3.3. *We collect some facts about* $U \otimes V$.

1. $U \otimes V$ *has the following universal mapping property. If* $B : U \times V \to W$ *is bilinear then it factors uniquely as* $\tilde{B} \circ h$ *where* $h : U \times V \to U \otimes V$ *is defined by* $h : (u, v) \mapsto u \otimes v$, *and where* $\tilde{B}$ *is linear.*

[18] Note other conventions where F an embedding is required to be a homeomorphism onto its image.

2. $(U \otimes V) \otimes W = U \otimes (V \otimes W)$*. So we can write without fear* $U \otimes V \otimes W$*.*

3. $U \otimes V \cong V \otimes U$*,*

4. $\mathrm{Hom}(U, V) \cong U^* \otimes V$*.*

5. $(U \otimes V)^* \cong U^* \times V^*$

The proof of this proposition is left to the reader. Let us here give only the definition of isomorphism number 4 above, more precisely the map $\leftarrow$ as follows: If $\sum c_{ij} u_i^* \otimes v_j$ is an element of $U^* \otimes V$ we send it to the element of $\mathrm{Hom}(U, V)$ defined by

$$u \mapsto \sum_{ij} c_{ij} u_i^*(u) v_j.$$

(You also have to check that this is well defined...)

Definition 3.7. *Let* $f : U \to \tilde{U}$*,* $g : V \to \tilde{V}$ *be linear. Then we can define a map* $f \otimes g : U \otimes V \to \tilde{U} \otimes \tilde{V}$ *taking* $\sum u_\alpha \otimes v_\alpha \to \sum \tilde{u}_\alpha \otimes \tilde{v}_\beta$*, where* $\tilde{u}_\alpha = f(u_\alpha)$*,* $\tilde{v}_\alpha = g(v_\alpha)$*.*

Definition 3.8. *Define the map* $C : U^* \otimes U \to \mathbb{R}$ *by*

$$C\left(\sum a_\alpha u_\alpha^* \otimes u_\alpha\right) = a_\alpha \sum u_\alpha^*(u_\alpha)$$

Finally, we note that if we compose the map C with the isomorphism from 4 of Proposition 3.3 (with $U = V$), we obtain a map $\mathrm{Hom}(U, U) \to \mathbb{R}$. This map is called the *trace*.

Exercise: Show this map indeed coincides with the trace of an endomorphism as you may have seen it in linear algebra.

3.5 Tensor bundles

Now let $\mathcal{E}$, $\mathcal{E}'$ be vector bundles. We can define $\mathcal{E} \otimes \mathcal{E}'$, etc., in view of Definition 3.7. (This tells us how to make transition functions.) The bundles of the form

$$T\mathcal{M} \otimes \cdots T\mathcal{M} \otimes T^*\mathcal{M} \otimes \cdots T^*\mathcal{M}$$

are known as *tensor bundles*. If there are say d copies of $T\mathcal{M}$, and d' of $T^*\mathcal{M}$, we notate the bundle by $T_d^{d'}\mathcal{M}$, and say the bundle of d-contravariant and d'-covariant tensors. A basis for the fibres over p, in local coordinates, is given by

$$\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_d}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_{d'}}|_p.$$

The transformation law:

$$\begin{aligned} &\frac{\partial}{\partial x^{k_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{k_d}} \otimes dx^{l_1} \otimes \cdots \otimes dx^{l_{d'}}|_p \\ = &\frac{\partial \tilde{x}^{i_1}}{\partial x^{k_1}} \frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}} \frac{\partial}{\partial \tilde{x}^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial \tilde{x}^{i_d}} \otimes d\tilde{x}^{j_1} \otimes \cdots \otimes d\tilde{x}^{j_{d'}}|_p. \end{aligned}$$

Note let $F : \mathcal{M} \to \mathcal{N}$. Then can define

$$F^* : \Gamma\left(\bigotimes_{i=1}^{d'} T^*\mathcal{N}\right) \to \Gamma\left(\bigotimes_{i=1}^{d'} T^*\mathcal{M}\right)$$

How? Exercise.

Get used to the following notation: "Let $A^{i_1\dots i_d}_{j_1\dots j_{d'}}$ be a tensor" meaning: Let $A:\mathcal{M}\to T^{d'}_d\mathcal{M}$ be a smooth section given in local coordinates by

$$A = A^{i_1\dots i_d}_{j_1\dots j_{d'}}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\frac{\partial}{\partial x^{i_d}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_{d'}}.$$

The point of referring to the indices is simply as a convenient way to display the type of the tensor.

With the results of Section 3.4, we can play all sorts of games in the spirit of the above. We can construct the bundle $\mathrm{Hom}(\mathcal{E},\mathcal{E}')$. We can construct a natural isomorphism of bundles $\mathrm{Hom}(\mathcal{E},\mathcal{E}')\cong\mathcal{E}^*\otimes\mathcal{E}'$.

4 Riemannian manifolds

A Riemannian metric is to be an inner product on all the fibres, varying smoothly. The point is, in view of the previous section, we can now define what "varying smoothly" means: Since an inner product is a bilinear map $T_p\mathcal{M}\times T_p\mathcal{M}\to\mathbb{R}$, which is also symmetric and positive definite, then it can be considered an element of $(T_p\mathcal{M}\otimes T_p\mathcal{M})^*$, and thus, $T^*_p\mathcal{M}\otimes T^*_p\mathcal{M}$.[19] We will thus define

Definition 4.1. *A* Riemannian metric *g on a smooth manifold $\mathcal{M}$ is an element $g\in\Gamma(T^*\mathcal{M}\otimes T^*\mathcal{M})$ such that for all $V,W\in T_p\mathcal{M}$, $g(V,W)=g(W,V)$ and $g(V,V)\geq 0$, with $g(V,V)=0$ iff $V=0$. A pair $(\mathcal{M},g)$, where $\mathcal{M}$ is a smooth manifold and g a Riemannian metric on $\mathcal{M}$, is called a* Riemannian manifold.

In local coordinates we have

$$g = g_{ij}dx^i\otimes dx^j.$$

The symmetry condition $g(V,W)=g(W,V)$ gives in local coordinates $g_{ij}=g_{ji}$.

Note: Comparison with the classical notation. In differential geometry of surfaces, one writes classically expressions like the right hand side of (1). In interpreting this notation, you are supposed to remember that this is a symmetric 2-tensor, and thus you are to replace $dudv$ in our present notation by $\frac{1}{2}(du\otimes dv+dv\otimes du)$. On the other hand, one also encounters expressions like $dudv$ in a completely different context, namely in double integrals. Here, one is supposed to interpret $dudv$ as an antisymmetric 2-tensor, a so-called 2-form, and replace it, in more modern notation, by $du\wedge dv$. To avoid confusion, we will never again see in these notes expressions like $dudv$...

Definition 4.2. *Let $\gamma: I\to\mathcal{M}$ be a curve. We define the length of γ as*

$$\int_I\sqrt{g(\gamma',\gamma')}dt.$$

Let γ and $\tilde\gamma$ be curves in $\mathcal{M}$ going through p, such that $\gamma'\neq 0$, $\tilde\gamma'\neq 0$. We define the angle between γ and $\tilde\gamma$ to be

$$\cos^{-1}(g(\gamma',\tilde\gamma')(g(\gamma',\gamma')g(\tilde\gamma',\tilde\gamma'))^{-1/2}).$$

Note. Invariance under reparametrizations.

[19] Exercise: make this identification formal in the language of isomorphisms of bundles described in the end of the previous section.

4.1 Examples

The simplest example of a Riemannian manifold is $\mathbb{R}^n$ with $g = \sum_i dx^i \otimes dx^i$.

From this example we can generate others by the following proposition

Proposition 4.1. *Given a Riemannian metric g on a manifold $\mathcal{N}$, and an immersion $i : \mathcal{M} \to \mathcal{N}$, then i^*g is a Riemannian metric on $\mathcal{M}$.*

Thus, applying the above with $\mathcal{N} = \mathbb{R}^n$ and $g = \sum_i dx^i \otimes dx^i$, we obtain in particular

Example 4.1. *If $\mathcal{M}$ is a submanifold of Euclidean space of any dimension $i : \mathcal{M} \to \mathbb{R}^n$, then $i^*(g)$ is a Riemannian metric on $\mathcal{M}$.*

4.2 Construction of Riemannian metrics

4.2.1 Overkill

Note. We can construct a Riemannian metric by applying the following:

Theorem 4.1. *(Whitney) Let $\mathcal{M}^n$ be second countable[20]. Then there exists an embedding (homeomorphic to its image with subspace topology) $F : \mathcal{M} \to \mathbb{R}^{2n+1}$.*

Actually, it turns out that all Riemannian metrics arise in this way:

Theorem 4.2. *(Nash) Let $(\mathcal{M}, g)$ be a Riemannian manifold. Then there exists an embedding $F : \mathcal{M} \to \mathbb{R}^{(n+2)(n+3)/2}$ such that $g = F^*(e)$ where e denotes the euclidean metric on $\mathbb{R}^{(n+2)(n+3)/2}$.[21]*

Embeddings of the above form are known as *isometric embeddings*.

Although by the above Riemannian geometry is nothing other than the study of submanifolds of $\mathbb{R}^N$ with the induced metric from Euclidean space, the point of view of the above theorem is rarely helpful. We shall not refer to it again in this course.

4.2.2 Construction via partition of unity

We can construct on any manifold a Riemannian metric in a much more straightforward fashion using a so-called *partition of unity*.

It may be useful to recall the definition of paracompact, which is a basic requirement of the underlying topology in our definition of manifold.

Definition 4.3. *A topological space is said to be paracompact if every open cover $\{V_\beta\}$ admits a locally finite, refinement, i.e. a collection of open sets $\{U_\alpha\}$ such that for every p, there exists an open set U_p containing p and only only finitely many U_α such that $U_p \cap U_\alpha \neq \emptyset$.*

Proposition 4.2. *Let $\mathcal{M}$ be a manifold (paracompact by our Definition 2.3). Let $\{U_\alpha\}$ be a locally finite atlas such that $\bar{U}_\alpha$ is compact. Then there exists a collection χ_α of smooth functions $\chi_\alpha : \mathcal{M} \to \mathbb{R}$, compactly supported in U_α, such that $1 \geq \chi_\alpha \geq 0$, $\sum \chi_\alpha = 1$.[22]*

[20] Note that a connected component of a paracompact manifold is second countable.

[21] Note that the original theorem of Nash needed a higher exponent.

[22] Evaluated at any point p, this sum is to be interpreted as a finite sum, over the (by assumption!) finitely many indices α where $\chi_\alpha(p) \neq 0$.

We call the collection χ_α a *partition of unity* subordinate to $\mathcal{U}_\alpha$.

Using this, we can construct a Riemannian metric on any paracompact manifold. First let us note the following fact: If g_1, g_2 are inner products on a vector space, so is $a_1 g_1 + a_2 g_2$ for all $a_1, a_2 > 0$.

From this fact and the definition of partition of unity one easily shows:

Proposition 4.3. *Given a locally finite subatlas $\{(\phi_\alpha, U_\alpha)\}$ and a partition of unity χ_α subordinate to it, $\sum \chi_\alpha \phi_\alpha^* e$ is a Riemannian metric on $\mathcal{M}$, where e denotes the Euclidean metric on $\mathbb{R}^n$.*

Proof. Note that by the paracompactness, in a neighbourhood of any p, $\sum \chi_\alpha \phi_\alpha^* e$ can be written $\sum_{\alpha : U_p \cap U_\alpha \neq \emptyset} \chi_\alpha \phi_\alpha^* e$ from which the smoothness is easily inferred. The symmetry is clear, and the positive definitively follows by our previous remark. □

4.3 The semi-Riemannian case

One can relax the requirement that metrics be positive definite to the requirement that the bilinear map g be *non-degenerate*, i.e. the condition that $g(V, W) = 0$ for all W implies $V = 0$. A $g \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M})$ satisfying $g(X, Y) = g(Y, X)$ and the above non-degeneracy condition is known as a *semi-Riemannian metric*.

By far, the most important case is the so-called Lorentzian case, discussed in Section 14. This is characterised by the property that a basis of the tangent space $E_0, \ldots E_m$, can be found so that $g(E_i, E_j) = 0$ for $i \neq j$, $g(E_0, E_0) = -1$, and $g(E_m, E_m) = 1$. Note that it is traditional in Lorentzian geometry to parameterise the dimension of the manifold by $m + 1$.

At the very formal level, one can discuss semi-Riemannian geometry in a unified way–until the convexity properties of the Riemannian case start being important. On the other hand, a good exercise to see the difference already is to note that in the non-Riemannian case, there are topological obstructions for the existence of a semi-Riemannian metric.

4.4 Topologists vs. geometers

Here we should point out the difference between *geometric topologists* and *Riemannian geometers*.

Geometric topologists study smooth manifolds. In the study of such a manifold, it may be useful for them to define a Riemannian metric on it, and to use this metric to assist them in defining more structures, etc. At the end of the day, however, they are interested in aspects that don't depend on which Riemannian metric they happened to construct. An example of topological invariants constructed with the help of a Riemannian metric are the so-called Donaldson invariants and the Seiberg-Witten invariants. Another more recent triumph is Grisha Perelman's proof [11] of the (3 dimensional case[23] of) Poincaré conjecture using a system of partial differential equations known as *Ricci flow*, completing a programme begun by R. Hamilton:

Theorem 4.3. *Let $\mathcal{M}$ be a simply connected compact manifold. Then $\mathcal{M}$ is homeomorphic to the sphere.*

[23] The $n = 2$ and $n \geq 4$ case having been settled earlier.

For Riemannian geometers, on the other hand, the objects of study are *from the beginning* Riemannian manifolds. You don't get to choose the metric. The metric is given to you, and your task is to understand its properties. The Riemannian geometry of $(\mathcal{M}, g)$ is interesting even if the topology of $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^n$. In fact, for the first half-century of its existence, higher dimensional Riemannian geometry concerned precisely this case.

4.5 Isometry

Every goemetric object comes with its corresponding notion of "sameness". For Riemannian manifolds, this is the notion of isometry.

Definition 4.4. *A diffeomorphism $F : (\mathcal{M}, g) \to (\mathcal{N}, \tilde{g})$ is called an* isometry *if $F^*(\tilde{g}) = g$. The manifolds $\mathcal{M}$ and $\mathcal{N}$ are said to be isometric.*

We can also define a local isometry.

Definition 4.5. *$(\mathcal{M}, g)$ and $(\mathcal{N}, \tilde{g})$ are* locally isometric *at p, q, if there exist neighborhoods $\mathcal{U}$ and $\tilde{\mathcal{U}}$ of p, q, and an isometry $F : \mathcal{U} \to \tilde{\mathcal{U}}$. F is then called a* local isometry.

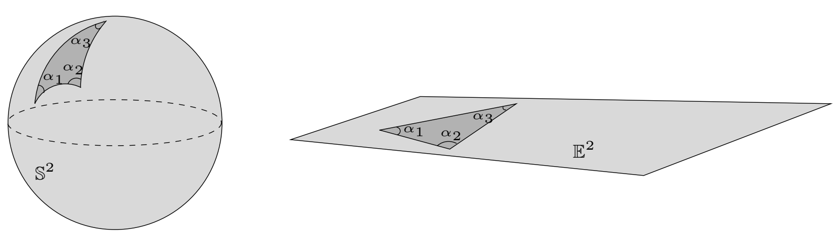
A priori it is not obvious that all two Riemannian manifolds of the same dimension are not always locally isometric. (Actually, they are in dimension 1–exercise!)

In later sections we will develop the notion of curvature precisely to address this issue. Curvature is defined at the infinitesimal level. To get intuition for it, it is easier to think about distinguished "macroscopic" objects. The most important of these is the notion of *geodesic*.

Definition 4.6. *Let $(\mathcal{M}, g)$ be a Riemannian manifold. A curve $\gamma : (a, b) \to \mathcal{M}$ is said to be a geodesic if it locally minimises arc length, i.e. if for every $t \in (a, b)$ there is an interval $[t-\epsilon, t+\epsilon]$ so that $\gamma|_{[t-\epsilon,t+\epsilon]}$ is the shortest curve from $\gamma(t-\epsilon)$ to $\gamma(t + \epsilon)$.*[24]

We will see later that indeed geodesics exist, indeed given any tangent vector V_p at a point p there is a unique geodesic with tangent vector V_p. Moreover, by our definition above it is manifest that geodesics are preserved by isometries.

Example 4.2. *The 2-sphere and the plane. The geodesics are great circles, and lines, respectively. In the 2-sphere, the sum of the interior angles of a geodesic triangles is given by $\alpha_1 + \alpha_2 + \alpha_3 = \pi + 2\pi Area$. In the plane $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. Since lengths, areas*[25] *and angles are preserved by local isometries, these spaces can thus not be locally isometric.*



[24] Later, we will find it convenient to define these curves otherwise and infer this property...
[25] as yet undefined...

We will turn in the next section to the developing the tools necessary to consider geodesics in Riemannian geometry. As we shall see, these are governed by ordinary differential equations. So we must first turn to the geometric theory of such equations on manifolds.

5 Vector fields and O.D.E.'s

In this section we will develop the geometric theory of ordinary differential equations, i.e. the theory of integral curves of vector fields on manifolds.

5.1 Existence of integral curves

Back to $\mathbb{R}^n$. I will assume the following fact from the theory of ode's[26].

Theorem 5.1. *Consider the initial value problem*

$$(x^i)' = f^i(x_1, \dots x_n), \tag{8}$$

$$x^i(0) = x_0^i, \tag{9}$$

where f is a Lipschitz function in $\mathcal{U} \subset \mathbb{R}^n$. Then there exists a unique maximal (T_-, T_+), with $-\infty \leq T_- < 0 < T_+$, and a unique continuously differentiable solution

$$x^i : (T_-, T_+) \to \mathbb{R}^n$$

satisfying (8), (9)*. If f is smooth then x is smooth. Moreover, if $T_+ < \infty$, then given any compact subset $K \subset U$, there exists a t_K such that $x(t_K, T_+) \cap K = \emptyset$.*

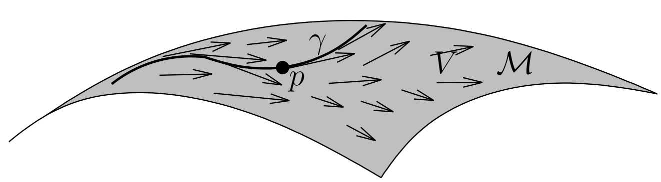
The geometric interpretation of this theorem is:

Theorem 5.2. *Let V be a C^∞ vector field on an open subset $\mathcal{U} \subset \mathbb{R}^n$. Then through any point p in $\mathcal{U}$, there exists a maximal parametrized integral curve γ of V, i.e. a curve $\gamma : (T_-, T_+)$ such that $\gamma' = V$, $\gamma'(0) = p$. Moreover, if $T_+ < \infty$, then given any compact subset $K \subset U$, there exists a t_K such that $\gamma(t_k, T_+) \cap K = \emptyset$.*

The maxaimality statement is simply the following: If $\tilde{\gamma} : (a, b)$ is another parametrized integral curve of V with $\gamma(0) =$, then $T_- \leq a < b \leq T_+$ and $\gamma|_{(a,b)} = \tilde{\gamma}$.

In the example sheet you shall show that this can be extended to manifolds as follows:

Theorem 5.3. *Let $\mathcal{M}$ be a C^∞ manifold, and let V be a C^∞ vector field on $\mathcal{M}$, i.e. $V \in \Gamma(T\mathcal{M})$. Then through any point p in $\mathcal{U}$, there exists a maximal*[27] *parametrized integral curve γ of V, i.e. a curve $\gamma : (T_-, T_+)$ such that $\gamma' = V$, $\gamma'(0) = p$.*



[26] ode's=ordinary differential equations

[27] i.e. a curve not a subset of a larger such curve

Moreover, if $T_+ < \infty$, then given any compact subset $K \subset U$, there exists a t_K such that $\gamma(t_K, T_+) \cap K = \emptyset$. In particular, if $\mathcal{M}$ itself is compact, then γ "exists for all t", i.e. $T_\pm = \pm\infty$.

Definition 5.1. *If for all p, $T_\pm = \pm\infty$, then we call X* complete.

In this language, on a compact manifold $\mathcal{M}$, all vector fields X are complete.

Exercise: Write down a manifold and an incomplete vector field. Write down a non-compact manifold and a complete vector field. Does every non-compact manifold admit an incomplete vector field?

5.2 Smooth dependence on initial data; 1-parameter groups of transformations

Classical O.D.E. theory tells us more than Theorem 5.1. It tells us that solutions depend continuously (in the Lipschitz case) and smoothly (in the smooth case) on initial conditions.

To formulate this in the language of vector fields, let $V \in \Gamma(T\mathcal{M})$.

Proposition 5.1. *For every $p \in \mathcal{M}$ there exists an open set $\mathcal{U}$, a nonempty open interval I and a collection of local transformations*[28]

$$\phi_t : \mathcal{U} \to \mathcal{M},$$

such that $\phi_t(q)$ is the integral curve of V through q given by Proposition 5.1. Moreover $\phi : \mathcal{U} \times I \to \mathcal{M}$ is a smooth map, and

$$\phi_t \circ \phi_s = \phi_{t+s}, \tag{10}$$

on $\mathcal{U} \cap \phi_{-s}(\mathcal{U})$, whenever $t, s, t+s \in I$.

A family of local transformations satisfying (10) is called a 1-parameter local group of transformations. If $I = \mathbb{R}$ then ϕ_t are in fact "global" and (10) defines a group structure on $\{\phi_t\}$.

Note that in particular, the above theorem says that $|T_\pm|$ can be uniformly bounded below in a neighborhood of any point.

It is easy to see using (10) that there is a one to one correspondence between 1-parameter local groups of transformations and vector fields. Check the following: Given such a family ϕ_t, define $X(p)$ to be the tangent vector of $\phi_t(p)$ at $t = 0$. The 1-parameter local group of transformations associated to X is again ϕ_t.

5.3 The Lie bracket

Let $\mathcal{M}$ be a smooth manifold, let X and Y be smooth vector fields: $X, Y \in \Gamma(T\mathcal{M})$.

Definition 5.2. $[X, Y]$ *is the vector field defined by the derivation given by*

$$[X, Y]f = X(Yf) - Y(Xf).$$

[28] i.e. a smooth map $\mathcal{U} \to \mathcal{M}$, where $\mathcal{U} \subset \mathcal{M}$, such that the map is a diffeomorphism to its image

Claim 5.1. *For each p, $[X,Y]|_p$ is indeed a derivation. $[X,Y]$ then defines a smooth vector field.*

Proof. Check the properties of a derivation! Check smoothness! $\square$

Proposition 5.2. *The following hold*

1. $[X,Y] = -[X,Y]$
2. $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$
3. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y]$ *(Jacobi identity)*
4. $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$

The proof of this proposition is a straightforward application of the definition, left to the reader.

So we can say that $\Gamma(T\mathcal{M})$ is a (non-associative) algebra with the bracket operation as multiplication. In general, algebras whose multiplication satisfies 3 above are known as *Lie algebras*. Note finally, that if ϕ is a diffeomorphism then

$$\phi_*[X,Y] = [\phi_* X, \phi_* Y]. \tag{11}$$

We say that ϕ_* is a *Lie algebra isomorphism*.

The above Proposition allows us to easily obtain a formula for $[X,Y]$ in terms of local coordinates. If x^i is a system of local coordinates, first note that

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0.$$

Now using in particular identity 4 of Proposition 5.2, setting $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^i \frac{\partial}{\partial x^i}$, we have

$$[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

Geometric interpretation of $[X,Y]$. Let ϕ_t denote the one-parameter group of transformations corresponding to X

Proposition 5.3.

$$[X,Y]|_p = \lim_{t \to 0} t^{-1}(Y|_p - ((\phi_t)_* Y)_p). \tag{12}$$

Proof. Let f_t denote $f \circ \phi_t$. Claim: $f_t = f + t(Xf) + t^2 h_t$ where h_t is smooth.

Now, we clearly have

$$(\phi_t)_* Y)_p f = Y_{\phi^{-1}(p)}(f \circ \phi_t).$$

Thus the right hand side of (12) applied to f is

$$\begin{aligned}
\lim t^{-1}(Y|_p f - Y_{\phi^{-1}(p)}(f \circ \phi_t)) &= \lim t^{-1}(Y_p f - Y_{\phi^{-1}(p)}(f + t(Xf) + t^2 h_t)) \\
&= \lim t^{-1}(Y_p f - T_{\phi^{-1}(p)} f) - Y_p(Xf) \\
&= X_p(Yf) - Y_p(Xf) \\
&= [X,Y]_p
\end{aligned}$$

as desired. $\square$

Proposition 5.4. *Let ϕ be a diffeomorphism. If ϕ_t generates X, then $\phi^{-1} \circ \phi_t \circ \phi$ generates $\phi_* X$. In particular, for ϕ and ϕ_t to commute, we must have $\phi_* X = X$.*

Proposition 5.5. *$[X, Y] = 0$ if and only if the 1-parameter local groups of transformation commute.*

Proof. Apply (12) to $(\phi_s)_* Y$, use (11) and the relationship $(\phi_s)_* \circ (\phi_t)_* = (\phi_{s+t})_*$. □

5.4 Lie differentiation

The expression (12) looks like differentiation. It is and it motivates a more general definition.

Let
$$\tau \in \Gamma^\infty(T^*\mathcal{M} \otimes \cdots \otimes T^*\mathcal{M} \otimes T\mathcal{M} \otimes \cdots \otimes T\mathcal{M})$$
be a tensor field.

Let $\phi : \mathcal{M} \to \mathcal{M}$ be a diffeomorphism. We may define a tensor field $\widetilde{\phi}\tau$ by the formula
$$\widetilde{\phi}\tau()$$
Exercise. This indeed defines a smooth tensor field of the same type as τ.

Definition 5.3. *Let X be a vector field and let ϕ_t denote the 1-parameter family of local transformations generated by X. Let τ be a tensor field of general type. Then the* Lie derivative of τ by X *is defined to be*
$$\mathcal{L}_X \tau = \lim_{t=0} \frac{1}{t}(\tau - \widetilde{\phi}_t \tau).$$

We collect some properties here:

Proposition 5.6. *We have*

1. $\mathcal{L}_X f = X f$
2. $\mathcal{L}_X Y = [X, Y]$
3. $\mathcal{L}_X(\tau_1 + \tau_2) = \mathcal{L}_X \tau_1 + \mathcal{L}_X \tau_2$
4. $\mathcal{L}_X(\tau_1 \otimes \tau_2) = \mathcal{L}_X \tau_1 \otimes \tau_2 + \tau_1 \otimes \mathcal{L}_X \tau_2$
5. $\mathcal{L}_{fX} g\tau = fg\mathcal{L}_X \tau + f(Xg)\tau$
6. $\mathcal{L}_X C(\tau) = C(\mathcal{L}_X \tau)$.

6 Connections

With our toolbox from the theory of ode's full, let us now return to the study of geometry.

In this section we shall discuss the important notion of connection. To motivate this, let us begin from the study of geodesics in $\mathbb{R}^n$, a.k.a. straight lines. The notion of connection is motivated by the classical interpretation of the geodesic equations in $\mathbb{R}^n$ that geodesics are characterised by the fact that their tangent vector does not change direction.

6.1 Geodesics and parallelism in $\mathbb{R}^n$

We will begin with a discussion of the relevant concepts in the special case of Euclidean space.

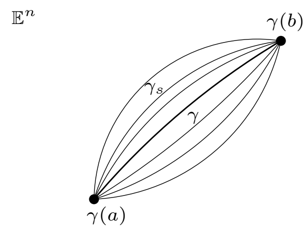
Let us call *geodesics* in $\mathbb{R}^n$ curves

$$\gamma : I \to \mathbb{R}^n$$

which extremize arc length in the following sense: Let $I = [a,b]$, $V = (-\epsilon, \epsilon)$, and consider a smooth[29] map

$$\tilde{\gamma} : I \times V \to \mathbb{R}^n$$

such that $\gamma_s = \tilde{\gamma}|_{I\times\{s\}}$ is a smooth curve in $\mathbb{R}^n$ with $\gamma_0(t) = \gamma(t)$ for all $t \in I$, $\gamma_s(a) = \gamma(a)$, $\gamma_s(b) = \gamma(b)$ for all $s \in J$. We shall call $\tilde{\gamma}$ a smooth variation of γ.



Define $L(s)$ to be the length of the curve γ_s. We would like to derive conditions for $s = 0$ to be a critical point of L for all smooth variations $\tilde{\gamma}$.

Let us for convenience assume γ_0 is parametrized by arc length, i.e. $|\partial_t \gamma_0(t)| = 1$. We compute

$$\begin{aligned}
L'(s)|_{s=0} &= \frac{d}{ds}\int_a^b \sqrt{\partial_t\gamma_s \cdot \partial_t\gamma_s}dt|_{s=0} \\
&= \int_a^b \partial_s \sqrt{\partial_t\gamma_s \cdot \partial_t\gamma_s}dt|_{s=0} \\
&= \int_a^b \partial_t\gamma_s \cdot \partial_s\partial_t\gamma_s dt|_{s=0} \\
&= \int_a^b \partial_t\gamma_s \cdot \partial_t\partial_s\gamma_s dt|_{s=0} \\
&= \int_a^b \partial_t(\partial_t\gamma_s \cdot \partial_s\gamma_s) - \partial_t\partial_t\gamma_s \cdot \partial_s\gamma_s dt|_{s=0} \\
&= \partial_t\gamma_0 \cdot \partial_s\gamma_s|_{s=0}(b) - \partial_t\gamma_0 \cdot \partial_s\gamma_s|_{s=0}(a) - \int_a^b \partial_t\partial_t\gamma_s \cdot \partial_s\gamma_s)dt|_{s=0} \\
&= \int_a^b \frac{d^2}{dt^2}\gamma_0 \cdot \partial_s\gamma_s|_{s=0} dt.
\end{aligned}$$

Now (exercise) it is easy to see that one can construct a variation $\tilde{\gamma}$ of γ such that $\partial_s\gamma_s(t)|_{s=0}$ for $t \in (a,b)$, is an arbitrary smooth vector field along[30] γ,

[29] Exercise: define this in view of the fact that $[a,b]$ is closed.
[30] For a formal definition of this, see Section 7.1.

vanishing at the endpoints. Thus, for the identity $L'(0) = 0$ to hold for all variations $\tilde{\gamma}$,[31] we must have

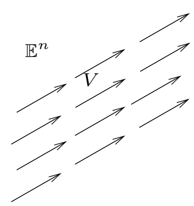
$$\frac{d^2}{dt^2}\gamma_0 = 0. \tag{13}$$

This is the geodesic equation in $\mathbb{R}^n$. It is a second order ode. The general solution is

$$\gamma(t) = (x_0^1 + a^1 t, \ldots, x_0^n + a^n t),$$

i.e. straight lines.

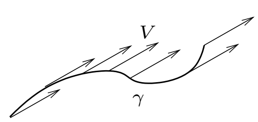
We didn't need any of the so-called qualitative theory of Section 5 to say that there exist solutions to (13), for we could just write them down explicitly! This is related to the following fact: In the case of $\mathbb{R}^n$, it turns out that straight lines are distinguished not only in the variational sense just discussed above but also from the group action point of view. For on $\mathbb{R}^n$ we have the well known translations, which act by isometry. Given a vector V_p at a point p, we can construct a vector field $V : \mathbb{R}^n \to T\mathbb{R}^n$ such that $V(q) = (T^p_q)_*(V_p)$, where T^p_q denotes the translation map $\mathbb{R}^n \to \mathbb{R}^n$ which sends p to q. Geodesics through p tangent to V_p are then integral curves of the vector field V.



Vector fields V constructed as above are known as *parallel*. Geodesics are thus curves whose tangent vector is parallel.

It is somewhat of a miracle that in Euclidean geometry we can identify certain vector fields as parallel so as for this notion to relate to geodesics (defined as length extremizers) in the above sense.

In Riemannian geometry, things are not as simple. An *absolute parallelism* in the sense above does not exist. Nonetheless, one may still define the notion of a vector field being parallel *along a curve*:



More generally, we may still define the notion of the directional derivative of a vector field X in the direction of a vector ξ, to be denoted $\nabla_\xi X$. The vector field X will be called parallel along a curve γ if $\nabla_\xi X = 0$. In particular, we shall be able to define this so that the equation for length extremizing curves is again $\nabla_{\gamma'}\gamma' = 0$, i.e. so that length extremizing curves can again be characterized as those whose tangent is parallel along itself.

At this point one should stop and point out that it is truly remarkable that one can again relate length-extremization and a suitable notion of parallelism, albeit, more restricted than that of an absolute parallelism. The realization

[31] Remember, L depends on the variation $\tilde{\gamma}$, i.e. we should really write $L^{\tilde{\gamma}}$.

that this concept is useful is essentially the contribution of Levi-Civita to the subject of Riemannian geometry.

The task of defining this operation ∇ belongs to the next section.

6.2 Connection in a vector bundle

Our goal in the next section is to relate to a Riemannian manifold $(\mathcal{M}, g)$, an operation ∇ which will allow us to call certain vector fields parallel along curves, and more generally, will allow us to differentiate vector fields along curves, the ones with vanishing derivative called parallel. It turns out, however, that a ∇ operation is a useful concept in more general contexts, independent of Riemannian geometry. Let us start thus in more generality.

Definition 6.1. *Let $\mathcal{M}$ be a smooth manifold, and let $\pi : \mathcal{E} \to \mathcal{M}$ be a vector bundle. A* connection ∇ *on $\mathcal{E}$ is a mapping*

$$\nabla : T\mathcal{M} \times \Gamma(\mathcal{E}) \to \mathcal{E}$$

(we will write $\nabla(\xi, X)$ as $\nabla_\xi X$!) with the following properties:

1. *If $\xi \in T_p\mathcal{M}$ then $\nabla_\xi X \in \mathcal{E}_p$*
2. *$\nabla_{(a\xi+b\tilde{\xi})}X = a\nabla_\xi X + b\nabla_{\tilde{\xi}}X$*
3. *$\nabla_\xi(X + Y) = \nabla_\xi X + \nabla_\xi Y$*
4. *$\nabla_\xi fX = (\xi f)X + f\nabla_\xi X$*[32]
5. *If $Y \in \Gamma(T\mathcal{M})$, then $p \mapsto \nabla_{Y(p)}X$ is an element of $\Gamma(\mathcal{E})$, i.e. is smooth.*

In this class we will be interested in connections on the tangent bundle and related tensor bundles.

Example 6.1. *The flat connection on the tangent bundle of $\mathbb{R}^n$. Let $\frac{\partial}{\partial x^i}$ denote standard coordinates on $\mathbb{R}^n$. (We often call such coordinates Euclidean coordinates.) Let us define $(\nabla_\xi X)^j = \xi^i \partial_i X^j$. Check that this is indeed a connection, and that $\nabla_\xi X = 0$ iff it is parallel in the sense described previously.*

Let ∇ be a connection in the tangent bundle $T\mathcal{M}$, and let x^i be a system of local coordinates. Let us introduce the symbols Γ^i_{jk} by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

Note as always the Einstein summation convention. By the defining properties of connections, the Γ^i_{jk} determine the connection completely by the following formulas: Let $\xi = \xi^i \frac{\partial}{\partial x^i}$, and let $X^i = X^i \frac{\partial}{\partial x^i}$,

$$\begin{aligned} \nabla_\xi X &= \xi^i \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j} + \Gamma^k_{ij} \xi^i X^j \frac{\partial}{\partial x^k} \\ &= \left. \frac{dX^j \circ \gamma(t)}{dt} \right|_{t=0} \frac{\partial}{\partial x^j} + \Gamma^k_{ij} \xi^i X^j \frac{\partial}{\partial x^k}, \end{aligned} \tag{14}$$

[32] Here we are dropping evaluation at a point p from notation. To check the syntax of the formulas, always remember that vectors act on functions and that $\Gamma(\mathcal{E})$ is a module over $C^\infty(\mathcal{M})$.

where $\gamma(t)$ is any curve in $\mathcal{M}$ with $\gamma'(0) = \xi$.

Clearly, connections can be constructed by prescribing arbitrarily the functions Γ_{ij}^k and patching together with partitions of unity.

6.2.1 Γ^i_{jk} is not a tensor!

It cannot be stressed sufficiently that connections are not tensors. We shall see, for instance, that for all $p \in \mathcal{M}$ there always exists a coordinate system such that $\Gamma^i_{jk}(p) = 0$.

The transformation law for Γ^i_{jk} is given by

$$\Gamma^\alpha_{\beta\gamma} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial^2 \tilde{x}^\mu}{\partial x^\beta \partial x^\gamma} + \tilde{\Gamma}^\mu_{\nu\lambda} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} \frac{\partial \tilde{x}^\lambda}{\partial x^\gamma} \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}$$

The difference between two connections $\nabla - \tilde{\nabla}$, however is a tensor.

6.3 The Levi-Civita connection

Let us now return to Riemannian manifolds $(\mathcal{M}, g)$. It turns out that there exists a distinguished connection ∇ that one can relate to g:

Proposition 6.1. *Let $(\mathcal{M}, g)$ be Riemannian. There exists a unique connection ∇ in $T\mathcal{M}$ characterized by the following two properties*

1. *If X, Y are vector fields then, $\nabla_{X(p)}Y - \nabla_{Y(p)}X = [X, Y](p)$.*
2. *If X, Y are vector fields and ξ is a vector then $\nabla_\xi g(X, Y) = g(\nabla_\xi X, Y) + g(X, \nabla_\xi Y)$.*

Proof. Compute $g(\nabla_X Y, Z)$ explicitly using the rules above, and show that this gives a valid connection. □

Since the Levi–Civita connection is determined by the metric one can easily show the following:

Proposition 6.2. *Let $(\mathcal{M}, g)$, $(\tilde{\mathcal{M}}, \tilde{g})$ be Riemannian and suppose that $p \in \mathcal{U} \subset \mathcal{M}$, $q \in \tilde{\mathcal{U}} \subset \tilde{\mathcal{M}}$, and $\phi : \mathcal{U} \to \tilde{\mathcal{V}}$ is an isometry with $\phi(p) = q$. Let γ be a curve in $\mathcal{U}$, and let V be a vector field along γ and let T be a vector tangent to γ. Let ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections of $\mathcal{M}$, $\tilde{\mathcal{M}}$, respectively. Then*

$$\phi_* \nabla_T V = \tilde{\nabla}_{\phi_* T} \phi_* V.$$

In particular, if V is parallel along γ (i.e., $\nabla_T V = 0$), then $\phi_* V$ is parallel along $\phi \circ \gamma$.

6.3.1 The Levi–Civita connection in local coordinates

The Levi-Civita connection in local coordinates. First some notation. We will define the inverse metric g^{ij} as the components of the bundle transformation $T^*\mathcal{M} \to T\mathcal{M}$ inverting the isomorphism $T\mathcal{M} \to T^*\mathcal{M}$ enduced by the Riemannian metric g. More pedestrianly, it is the inverse matrix of g_{ij}, i.e. we have $g^{ij} g_{jk} = \delta^i_k$ where $\delta^i_k = 1$ if $i = k$ and 0 otherwise. Check that

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}).$$

(Check also that the first condition in the definition of the connection is equivalent to the statement $\Gamma^k_{ij} = \Gamma^k_{ji}$.)

6.3.2 Aside: raising and lowering indices with the metric

Since we have just used the so-called inverse metric, we might as well discuss this topic now in more detail. This is the essense of the power of index notation.

The point is that given any tensor field, i.e. a section of

$$T^*\mathcal{M} \otimes \cdots \otimes T^*\mathcal{M} \otimes T\mathcal{M} \otimes \cdots \otimes T\mathcal{M}$$

we can apply the bundle isomorphism defined by the inverse metric on any of the $T^*\mathcal{M}$ factors so as to convert it into a $T\mathcal{M}$. And similarly, we can apply the isomorphism defined by the metric itself to turn any of the factors $T\mathcal{M}$ to convert it to a $T^*\mathcal{M}$.

It is traditional in local coordinates to use the same letter for all the tensors one obtains by applying these isomorphisms to a given tensor. I.e., if $S^{j_1,\dots j_m}_{i_1,\dots i_n}$ is a tensor, then the tensor produced by applying the above isomorphism say to the factor corresponding to the index i_k is given in local coordinates by

$$S^{j_1\dots j_n h}_{i_1\dots \hat{i_k}\dots i_n} = g^{i_k h} S^{j_1\dots j_m}_{i_1\dots i_n}.$$

The hat above denotes that the index is omitted.

This process is known as raising and lowering indices.

Thus, using the metric, we can convert a tensor to one with all indices up or down, or however we like, and we think of these, as in some sense being the "same" tensor.

Finally, this process can be combined with the contraction map. For if say S_{ijkl} is a tensor, then we can raise the index i to obtain S^i_{jkl}, and now apply the contraction map of Definition 3.8 on the factors corresponding to the indices i and j to obtain a tensor S_{kl}. Again, one often uses the same letter to denote this new tensor, as we have just done here, although there is a potential for confusion, as one can define several different contractions, depending on the indices selected.

7 Geodesics and parallel transport

7.1 The definition of geodesic

We may now make the definition

Definition 7.1. *Let $(\mathcal{M}, g)$ be a Riemannian manifold with Levi-Civita connection ∇. A curve $\gamma : I \to \mathcal{M}$ is said to be a geodesic if*

$$\nabla_{\gamma'}\gamma' = 0. \tag{15}$$

Strictly speaking, equation (15) does not make sense, since γ' is a vector field along γ, i.e. it can be though of as a section of the bundle $\gamma^*(T\mathcal{N}) \to I$. Nonetheless, we can use formula (14) to *define* the left hand side of (15). In general, when V is a vector field along a curve γ, and W is a vector tangent to γ, we will use unapologetically the notation $\nabla_W V$. Similarly for vector fields "along" higher dimensional submanifolds, defined in the obvious sense. (Exercise: What is this obvious sense?)

Note that in view of Proposition 6.2, local isometries map geodesics to geodesics. (Show it!)

It turns out that the above notion of geodesic coincides with that of curves locally minimizing arc length:

Theorem 7.1. *Let γ be a geodesic. Then for all $p \in \gamma$, there exists a neighborhood $\mathcal{U}$ of p, so that for all $q, r \in \gamma \cap \mathcal{U}$, denoting by $\gamma_{q,r}$ the piece of γ connecting q and r, we have $d(q,r) = L(\gamma_{q,r})$, where L here denotes length, and moreover, if $\tilde{\gamma}$ is any other piecewise smooth curve in $\mathcal{M}$ connecting q and r, then $L(\tilde{\gamma}) > d(q,r)$.*

The proof of Theorem 7.1 is not immediate, and in fact, reveals various key ideas in the calculus of variations. We will complete the proof several sections later in these notes.

7.2 The first variation formula

For now let us give the following:

Proposition 7.1. *A C^2 curve $\gamma : [a,b] \to \mathcal{M}$ is a geodesic parametrized by a multiple of arc length iff for all C^2 variations $\tilde{\gamma} : [a,b] \times [-\epsilon, \epsilon]$ of γ with $\tilde{\gamma}(a,s) = \gamma(a)$, $\tilde{\gamma}(b,s) = \gamma(b)$, we have*

$$\frac{d}{ds} L\left(\tilde{\gamma}(\cdot, s)\right)\big|_{s=0} = 0.$$

It is this Proposition that relates parellelism with length extremization, i.e. that allows us to recover the analogue in Riemannian geometry of the picture of Section 6.1.

Proof. Let γ be a C^2 curve, and let $\tilde{\gamma}$ be an arbitrary variation of γ.

Let us introduce the notation $N = \tilde{\gamma}_* \frac{\partial}{\partial s}$, $T = \tilde{\gamma}_* \frac{\partial}{\partial t}$, where s is a coordinate in $(-\epsilon, \epsilon)$ and t is a coordinate in $[0, L]$. And define $L(s)$ to be the length of the curve $\tilde{\gamma}(\cdot, s)$.

We have

$$L(s) = \int_a^b \sqrt{g(T_{\tilde{\gamma}(s,t)}, T_{\tilde{\gamma}(s,t)})}\, dt.$$

We now have the technology to mimick the calculation in Section 6.1.

Differentiating L in s, we obtain

$$
\begin{aligned}
L'(s) &= \frac{d}{ds}\int_a^b \sqrt{g(T,T)}dt \\
&= \int_a^b N\sqrt{g(T,T)}dt \\
&= \int_a^b (g(T,T))^{-1/2} g(\nabla_N T, T)dt \\
&= \int_a^b (g(T,T))^{-1/2} g(\nabla_T N, T)dt \\
&= \int_a^b T((g(T,T))^{-1/2} g(N,T)) \\
&\quad - T(g(T,T))^{-1/2})g(N,T) - (g(T,T))^{-1/2} g(N, \nabla_T T)dt \qquad (16) \\
&= g(T,T)^{1/2} g(N,T)]_a^b \\
&\quad - \int_a^b T(g(T,T))^{-1/2} g(N,T) - (g(T,T))^{-1/2} g(N, \nabla_T T)dt \qquad (17) \\
&= \int_a^b -T(g(T,T))^{-1/2} g(N,T) - (g(T,T))^{-1/2} g(N, \nabla_T T)dt \\
&= \int_a^b g(T,T)^{-3/2} g(\nabla_T T, T) g(N,T) - (g(T,T))^{-1/2} g(N, \nabla_T T)dt \qquad (18)
\end{aligned}
$$

Here we have used $[N,T]=0$, $\nabla_N T - \nabla_T N = 0$, and the fact that $N(a,s)=0$, $N(b,s)=0$. Exercise: Why are these statements true?

Now suppose that γ is a geodesic in the sense of Definition 7.1. Since $\nabla_T T|_{\tilde\gamma(0,t)} = 0$, the whole expression on the right hand side of (16) vanishes when evaluated at $s=0$. Since $\tilde\gamma$ is arbitrary, this proves one direction of the equivalence.

To prove the other direction, first we note that given any vector field N along γ, there exists some variation $\tilde\gamma$ such that $N = \tilde\gamma_* \frac{\partial}{\partial s}$. (A nice way to construct such a vector field is via the exponential map discussed in later sections. But this is not necessary.) Thus it suffices to show that if T does not satisfy $\nabla_T T = 0$, then there exists an N such that the expression on the right hand side of (16) is non-zero.

Suppose then that $\nabla_T T(t_0, 0) \neq 0$. There exists a neighborhood (t_1, t_2) of t_0 such that $\nabla_T T(t,0) \neq 0$ for $t \in (t_1, t_2)$. Let N be the vector field along γ such that $N(t) = \nabla_T T(t,0)$. We have

$$
\begin{aligned}
&\int_a^b g(T,T)^{-3/2} g(\nabla_T T, T) g(N,T) - (g(T,T))^{-1/2} g(N, \nabla_T T)dt \\
&= \int_a^b g(T,T)^{-3/2} g(\nabla_T T, T) g(\nabla_T T, T) - g(T,T))^{-1/2} g(\nabla_T T, \nabla_T T)dt \\
&\leq \int_{t_1}^{t_2} g(T,T)^{-3/2} g(\nabla_T T, T) g(\nabla_T T, T) - g(T,T))^{-1/2} g(\nabla_T T, \nabla_T T)dt \\
&< 0.
\end{aligned}
$$

The last inequality follows by noting that the above expression is $-g(T,T)^{-1/2}$

times the norm squared of the projection of $\nabla_T T$ to the orthogonal complement of T, and the latter is certainly nonnegative.

Thus, we must have $\nabla_T T = 0$. $\square$

7.3 Parallel transport

Our goal is to prove the existence of geodesics by reducing to the theory of ordinary differential equations. The geodesic equation in local coordinates is of course a second order equation for γ. It is a first order equation for the tangent vector.

Let us first consider the following simpler situation. Let $\gamma : I \to \mathcal{M}$ be a fixed smooth curve, $I = [a, b]$, denote $\gamma(a)$ as p, $\gamma(b)$ as q, and let T denote the tangent vector γ'. Suppose V is an arbitrary vector at p.

Proposition 7.2. *There exists a unique smooth vector field $\tilde{V}$ along γ such that $\tilde{V}(a) = V$ and*

$$\nabla_T \tilde{V} = 0, \tag{19}$$

i.e. so that $\tilde{V}$ is parallel along γ.

Proof. Writing (19) for the components $\tilde{V}^x$ of $\tilde{V}$ with respect to a local coordinate system, we obtain

$$\frac{d}{dt}\tilde{V}^\alpha = -\Gamma^\alpha_{\beta\gamma} T^\beta(\gamma(t))\tilde{V}^\gamma. \tag{20}$$

Let us consider the manifold $(-\infty, \infty) \times \mathbb{R}^n$, and consider the vector field

$$(s, x) \mapsto (s, -\Gamma^\alpha_{\beta\gamma} T^\beta(\gamma(t))x^\beta),$$

where we set $\Gamma^\alpha_{\beta\gamma} T^\beta(\gamma(s)) = \Gamma^\alpha_{\beta\gamma} T^\beta(\gamma(a))$ for $s < a$, and $\Gamma^\alpha_{\beta\gamma} T^\beta(\gamma(s)) = \Gamma^\alpha_{\beta\gamma} T^\beta(\gamma(b))$ for $s > b$. Then integral curves $(s(t), \tilde{V}(t))$ of this vector field are precisely solutions of (20) for t values in $[a, b]$, after dropping the $s(t)$ which clearly must satisfy $s(t) = t$.[33] Thus, by Theorem 5.3, we have that for the initial value problem at $t = a$, there exists a maximum future time T of existence, and a solution $\tilde{V}(t)$ on $[a, T)$.

On the other hand, since the equation (20) is linear, we know a priori that a solution $\tilde{V}$ is bounded by

$$\sum_\delta |\tilde{V}^\delta(t-a)| \leq \sum_\delta |V^\delta| \exp\left(\left(\sup_{t,\alpha,\gamma} \Gamma^\alpha_{\beta\gamma} T^\beta\right)|t-a|\right)$$

Thus, $(s(t), \tilde{V}(t))$ cannot leave every compact subset of $(-\infty, \infty) \times \mathbb{R}^n$ in finite time, and thus $T = \infty$. In particular, $\tilde{V}$ is defined in all of $[a, b]$ $\square$

We call the vector $W = \tilde{V}(q)$ the *parallel transport* of V to q along γ. One easily sees that parallel transport defines an isometry $T_\gamma : T_p\mathcal{M} \to T_q\mathcal{M}$ of tangent spaces.

[33] We have just here performed a well known standard trick from ode's for turning a so-called non-autonomous system to an autonomous system.

7.4 Existence of geodesics

Now for the existence of geodesics:

Since the geodesic equation is second order, to look for a first order equation, we must go to the tangent bundle.[34] Let x^α be a system of local coordinates on $\mathcal{M}$. Extend this to a system of local coordinates $(x^1, \ldots x^n, p^1, \ldots p^n)$ on $T\mathcal{M}$, where the p^α are defined by

$$V = \sum p^\alpha(V) \frac{\partial}{\partial x^\alpha}$$

for any vector V.

The geodesic equation (15), which in local coordinates can be written in second order form

$$\frac{d^2 x^\alpha}{dt^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt},$$

can now be written as

$$\frac{dx^\alpha}{dt} = p^\alpha \tag{21}$$

$$\frac{dp^\alpha}{dt} = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma. \tag{22}$$

Solutions of the system (21)–(22) are just integral curves on $T\mathcal{M}$ of the vector field

$$p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha}$$

on $\mathcal{TM}$. Remember, the latter is an element of $\Gamma(T(\mathcal{TM}))$. Don't be too confused by this...

We now apply Theorem 5.3, and Proposition 5.1. We call the one-parameter local group of transformations $\phi_t : T\mathcal{M} \to T\mathcal{M}$ generated by this vector field geodesic flow.

Projections $\pi \circ \phi_t$ to $\mathcal{M}$ are then the geodesics we have been wanting to construct. We have thus shown in particular the following:

Proposition 7.3. *Let $V_p \in T\mathcal{M}$. Then there exists a unique maximal arc-length-parametrized geodesic $\gamma : (T_-, T_+) \to \mathcal{M}$ such that $\gamma'(0) = V_0$.*

Thus, we have shown the existence of geodesics.

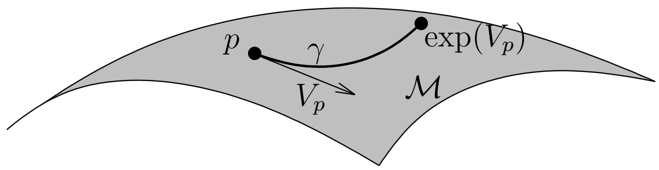
8 The exponential map

Back to ϕ_t. It is easy to see that the domain $\mathcal{U}$ of ϕ_t is star-shaped in the sense that if $V_p \in \mathcal{U}$, for some vector V_p at a point p, then $\lambda V_p \in \mathcal{U}$ for all $0 \leq \lambda \leq 1$. Moreover, (exercise) $\phi_t(V_p) = \phi_{\lambda^{-1}t}(\lambda V_p)$. This implies, that ϕ_1 is defined in a non-empty star-shaped open set.

Definition 8.1. *The map* $\exp : \mathcal{U} \to \mathcal{M}$ *defined by* $\pi \circ \phi_1$*, where* π *denotes the standard projection* $\pi : T\mathcal{M} \to \mathcal{M}$ *and* $\mathcal{U}$ *denotes the domain of* ϕ_1*, is called the* exponential map.

[34] Again, this is just a sophisticated version of the well known trick from ode's of making a second order equation first order.

The map is depicted below:



The curve $\gamma(t)$ is a geodesic tangent to V_p, parametrized by arc length and $\exp(V_p) = \gamma(|V_p|)$. Here $|V_p| = \sqrt{g(V_p, V_p)}$.

As a composition of smooth maps, the exponential map is clearly a smooth map of manifolds. In the next section, we shall compute its differential.

We end this section with a definition:

Definition 8.2. *Let $(\mathcal{M}, g)$ be Riemannian. We say that $(\mathcal{M}, g)$ is* geodesically complete *if the domain $\mathcal{U}$ of the exponential map is $T\mathcal{M}$.*

Equivalently, $(\mathcal{M}, g)$ is geodesically complete if all geodesics can be continued to arbitrary positive and negative values of an arc length parameter.

8.1 The differential of exp

First a computation promised at the end of the last section. What is $\exp_*$?

First, what seems like a slightly simpler situation: for any $p \in \mathcal{M}$ let us denote by $\exp_p$ the restriction of the map exp to $\mathcal{U}_p = \mathcal{U} \cap T_p\mathcal{M}$. This is also clearly a smooth map.

We will compute $\left(\exp_{p*}\right)_{0_p}$. It turns out that this is basically a tautology. The only difficulty is in the notation. Remember

$$\left(\exp_{p*}\right)_{0_p} : T_{0_p}(T_p\mathcal{M}) \to T_p\mathcal{M}$$

On the other hand, in view of the obvious[35]

$$T_{0_p}(T_p\mathcal{M}) \cong T_p\mathcal{M}, \tag{23}$$

we can consider the map as a map:

$$\left(\exp_{p*}\right)_{0_p} : T_p\mathcal{M} \to T_p\mathcal{M}.$$

Proposition 8.1. *We have*

$$\left(\exp_{p*}\right)_{0_p} = id \tag{24}$$

Proof. Let $v \in T_p\mathcal{M}$, and consider the curve $t \mapsto tv$. Denote this curve in $T_p\mathcal{M}$ by $\kappa(t)$. This curve is tangent to v. The curve $\exp_p(\kappa(t))$ is found by noting

$$\exp_p(\kappa(t)) = \exp_p(tv) = \phi_t(v) = \gamma(|v|t) \doteq \tilde{\gamma}(t)$$

where ϕ_t denotes geodesic flow, and γ denotes the arc-length geodesic through p tangent to v. By definition of the differential map, we have that

$$(\exp_*)_{0_p} = \tilde{\gamma}'(t) = v,$$

thus, we have obtained (24). ∎

[35] define it!

A little more work (equally tautological as the above) show that

$$(\pi \times \exp_*)_{0_p} : T_p\mathcal{M} \times T_p\mathcal{M} \to T_p\mathcal{M} \times T_p\mathcal{M} \text{ is invertible} \tag{25}$$

as it can be represented as a block matrix consisting of the identity on the diagonal. Here it is understood that

$$\pi \times \exp : T\mathcal{M} \to \mathcal{M} \times \mathcal{M}$$

$\pi \times \exp(v_p) = (p, \exp(v_p))$. Again, the domain and range of the differential map have been identified with $T_p\mathcal{M} \times T_p\mathcal{M}$ by obvious identifications analogous to (23) that the reader is here meant to fill in.

What is the point of all this? We can now apply the following inverse function theorem

Theorem 8.1. *Let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map such that $(F_*)_p : T_p\mathcal{M} \to T_q\mathcal{N}$ is invertible. Then there exists a neighborhood $\mathcal{U}$ of p, such that $F|_\mathcal{U}$ is a diffeomorphism onto its image.*

Proof. Prove this from the inverse function theorem on $\mathbb{R}^n$. □

Applied to the map $\pi \times \exp$, in view of this gives the following:

Proposition 8.2. *Let $p \in (\mathcal{M}, g)$. There exists a neighbhorhood $\tilde{\mathcal{U}} = \mathcal{U} \times \mathcal{U}$ of (p,p) in $\mathcal{M} \times \mathcal{M}$ such that, denoting by $\mathcal{W} = (\pi \times \exp)^{-1}(\tilde{\mathcal{W}})$, we have that $\pi \times \exp|_\mathcal{W} : \mathcal{W} \to \mathcal{U} \times \mathcal{U}$ is invertible.*

That is to say, for any points $q_1, q_2 \in \mathcal{U}$, there exists a $v_{q_1} \in T_{q_1}\mathcal{M}$ such that $\exp(v_{q_1}) = q_2$. (Exercise in tautology: why does this statement follow from the proposition?)

A moment's thought tells us that we can slightly refine the above Proposition. Let us choose $\epsilon > 0$ so that

$$B_{0_q}(\epsilon) \cap \mathcal{W} = \cup q \in \mathcal{U} B_{0_q}(\epsilon) \doteq \tilde{\mathcal{W}}.$$

Again, $\pi \times \exp|_\mathcal{W}$ is a diffeomorphism, and its projection to the first component is $\mathcal{U}$. Let $\mathcal{V} \subset \mathcal{U}$ such that $\mathcal{V} \times \mathcal{V}$ is in the image of this. Let $q_1, q_2 \in \mathcal{V}$. Then there exists a $v_{q_1} \in T_{q_1}\mathcal{M}$ such that $\exp(v_{q_1}) = q_2$, and moreover, such that $|v_{q_1}| < \epsilon$.

Note that the curve $t \to \exp(tv_{q_1}), 0 \leq t \leq 1$ is contained completely in $\mathcal{W}$.

We have thus produced a neighborhood $\mathcal{V}$ with the property that there exists an $\epsilon > 0$ such that any two points q_1 and q_2 of $\mathcal{V}$ can be joined by a geodesic γ of length $< \epsilon$. Moreover, any other *geodesic* joining q_1 and q_2 must have length $\geq \epsilon$. Why?

We can in fact refine this further: We shall prove that γ has length < *any* curve joining q_1 and q_2. Moreover, we shall show that $\mathcal{V}$ can be chosen so that γ is completely contained in $\mathcal{V}$. Such a neighbhorhood is called a *geodesically convex neighborhood*.

8.2 The Gauss lemma

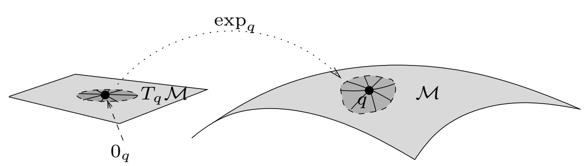
For this, we need a computation originally done in the setting of surfaces in $\mathbb{R}^3$ by Gauss.

First, note the following: By specializing our discussion from before, we have that for every point q, there exists a $B_{0_q}(\epsilon) \subset T_q\mathcal{M}$ such that $\exp_q$ is a diffeomorphism

$$\exp_q : B_{0_q}(\epsilon) \to \mathcal{U}$$

for some $\mathcal{U} \in \mathcal{M}$. In particular, given any coordinate system on $B_{0_q}(\epsilon)$, we obtain a coordinate system on $\mathcal{U}$ by pulling them back via $\exp_q^{-1}$.

For instance, one could choose a system of Cartesian coordinates. The associated coordinates on $\mathcal{M}$ are known as *normal coordinates*. Alternatively, one can choose polar coordinates $r, \theta^1, \theta^{n-1}$, where θ^α are local coordinates on $\mathbb{S}^{n-1}$. (Note of course that these coordinates defined only on a subset of $B_{0_q}(\epsilon)$.) The associated coordinates on $\mathcal{M}$, $r \circ \exp_q^{-1}$, etc., are known as *geodesic polar coordinates*.



The so-called Gauss' lemma is the following proposition:

Proposition 8.3. *On* $\mathcal{M}$,

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1, \qquad g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^\alpha}\right) = 0,$$

i.e., in geodesic polar coordinates, the metric can be written as

$$dr^2 + g_{\alpha\beta} d\theta^\alpha \otimes d\theta^\beta.$$

Proof. Let us prove the first identity first. We can interpret $\frac{\partial}{\partial r}$ as a vector field either on B_{0_q} and on $\mathcal{M}$. On the former, its integral curves are lines through the origin 0_q, parametrized by arc length.[36] By the definition of the map $\exp_q$, it follows that the integral curves of $\frac{\partial}{\partial r}$, interpreted now as a vector field on $\mathcal{M}$, are geodesics parametrized by arc length. Thus, the first identity follows.

For the second, let α be given, let $p \in \mathcal{U}$ and fix a line through the origin 0_q connecting it with the pre-image of p under $\exp_q$. Let us consider the vector fields $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta^\alpha}$ along the image of this line in $\mathcal{M}$, which is a geodesic γ connecting q and p.[37] Let us denote $T = \frac{\partial}{\partial r}$, $N = \frac{\partial}{\partial \theta^\alpha}$.

We are interested in the quantity $g(N, T)$. Since this is differential geometry, let's differentiate and see if we are lucky. We compute

$$Tg(N, T) = g(\nabla_T N, T) + g(N, \nabla_T T).$$

The second term above vanishes in view of the geodesic equation, thus

$$Tg(N, T) = g(\nabla_T N, T).$$

[36] The vector field is of course not defined at the origin. Exercies: deal with this issue.

[37] Again, address for yourself the issue of the fact that *a priori*, these may not be defined everywhere.

On the other hand, since $[N,T]=0$, we have

$$Tg(N,T) = g(\nabla_T N, T) = g(\nabla_N T, T) = \frac{1}{2} N g(T,T) = 0$$

since $g(T,T)=1$ identically. Thus $g(N,T)$ is constant along γ. Since $N=0$ at q, then $g(N,T)=0$ identically. $\square$

The above lemma leads immediately to the following

Proposition 8.4. *Let q, ϵ, $\tilde{\mathcal{U}}$ be as in the above lemma, and let $p \in \tilde{\mathcal{U}}$. Then the radial geodesic γ joining q and p, of length $r(p)$, is length minimizing, i.e. if $\tilde\gamma$ is any other piecewise regular curve joining q and p, then $L(\tilde\gamma) > L(\gamma) = r(p)$.*

The statement is in fact true where piecewise regular is replaced by any rectifiable curve. These are the most general curves for which one can define the notion of length.

Proof. For a piecewise regular curve, we can write the length as

$$L = \int \sqrt{g(\tilde\gamma', \tilde\gamma')} dt.$$

Let us consider separately the case where $\gamma \subset \mathcal{U}$, and when it is not. In the former case, since the curve is contained in our geodesic polar coordinate chart[38], we can write

$$\begin{aligned} L &= \int \sqrt{\left(\frac{dr}{dt}\right)^2 + g_{\alpha\beta}\frac{d\theta^\alpha}{dt}\frac{d\theta^\beta}{dt}} dt \qquad (26) \\ &\geq \int \sqrt{\left(\frac{dr}{dt}\right)^2} dt \\ &\geq \int \frac{dr}{dt} dt \\ &= r(p), \end{aligned}$$

with equality iff[39] $g_{\alpha\beta}\frac{d\theta^\alpha}{dt}\frac{d\theta^\beta}{dt} = 0$, and thus $\theta^\alpha = c$, and $\frac{dr}{dt} \neq 0$. Thus, after reparametrization, $\tilde\gamma$ is the radial geodesic.

In the other case, there is a first time t_0 when $\tilde\gamma$ crosses $r=r(p)$. Redo the above with $\tilde\gamma$ replaced by $\tilde\gamma|_{[0,t_0]}$. $\square$

The above argument actually illustrates a general technique in the calculus of variations for showing that the solution of a variational problem is actually a minimiser. The technique is called: embedding in a field of variations.

8.3 Geodesically convex neighbourhoods

We can now turn to finishing up a task left undone, namely showing the existence of geodesically convex neighbourhoods in the sense described previously.

First a remark. There is something we can say about non-radial geodesics completely contained in a geodesic polar coordinate chart. If κ is such a geodesic

[38] Again, with the usual caveat.

[39] Exercise: Why is this expression positive definite?

then $r \circ \kappa$ cannot have a strict maximum. For suppose t_{max} were such a point. Note that a t_{max}, $\frac{d^2r}{dt^2} \leq 0$, $\frac{dr}{dt} = 0$. But,

$$\begin{aligned} \frac{d^2r}{dt^2}(t_{max}) &= -\Gamma^r_{\beta\gamma}\frac{d\theta^\beta}{dt}\frac{d\theta^\gamma}{dt} \\ &= -\frac{1}{2}g^{rr}(-\partial_r g_{\beta\gamma})\frac{d\theta^\beta}{dt}\frac{d\theta^\gamma}{dt} \\ &= \frac{1}{2}\partial_r g_{\beta\gamma}\frac{d\theta^\beta}{dt}\frac{d\theta^\gamma}{dt} \\ &> 0 \end{aligned}$$

Exercise: why the last strict inequality? This is a contradiction.

We can finally complete our construction of a geodesically convex neighborhood. Let $\mathcal{W}$ be as in Proposition 8.2. We may choose ϵ as in the discussion after that Proposition, and and choose a $\mathcal{V}$ as before, but with $\epsilon/2$ in place of ϵ, and so that addition $\mathcal{V}$ is of the form $\exp_p(B_{0_p}(\epsilon/4)$. We have that $\mathcal{V} \subset \exp_p(B_{0_p}(\epsilon)) \subset \mathcal{W}$. Moreover, we have that any two points q_1, q_2 in $\mathcal{V}$ can be joined by a geodesic in $\mathcal{W}$ of length $< \frac{1}{2}\epsilon$.

Repeating a previous computation (namely, (26), it follows that such a geodesic necessarily must remain in $\exp_p(B_{0_p}(\epsilon))$. Thus, by our result on the absense of maxima of r, it follows that r cannot have a maximum. Since $r(q_1) < \epsilon/4$, $r(q_2) < \epsilon/4$, it follows that $r < \epsilon/4$ throughout the geodesic. That is to say, the geodesic is contained in $\mathcal{V}$. So $\mathcal{V}$ is geodesically convex in the sense claimed.

8.4 Application: length minimizing curves are geodesics

In later sections, we shall give conditions on a Riemannian manifold implying the existence of length minimizing geodesics joining any two given points. The construction of these will involve global considerations.

On the other hand, given a curve which is *not* a geodesic, one can show that it can *not* be length minimizing by completely local considerations. This is in fact yet another application of what we have just done.

Proposition 8.5. *Let $(\mathcal{M}, g)$ be Riemannian, with $p, q \in \mathcal{M}$, and let γ be a piecewise regular curve from p to q. If γ is not a geodesic, then there exists a piecewise regular curve $\tilde{\gamma}$ connecting p to q, such that $L(\tilde{\gamma}) < L(\gamma)$.*

Proof. Suppose γ is not a geodesic. Then there exists a point $\tilde{p} = \gamma(T)$ where γ does not satisfy the geodesic equation. Consider a geodesically convex neighborhood $\mathcal{V}$ centered at $\tilde{p}$. Consider two points $\tilde{q}_1$, $\tilde{q}_2$ on γ, in $\mathcal{V}$, such that say $\gamma(t_1) = \tilde{q}_1$, $\gamma(t_2) = \tilde{q}_2$, with $t_1 < T < t_2$. We know that there exists a geodesic connecting $\tilde{q}_1$ and $\tilde{q}_2$ which is length minimizing. If this geodesic coincides with $\gamma|_{[t_1,t_2]}$, then there is nothing to show. If not, then the curve $\gamma|_{[t_1,t_2]}$ has strictly greater length than this curve, in which case we can replace γ by a shorter curve joining p and q. ∎

9 Geodesic completeness and the Hopf-Rinow theorem

9.1 The metric space structure

An application of the previous is to show that a Riemannian manifold inherits the structure of a metric space related to the Riemannian metric.

Definition 9.1. *Let $(\mathcal{M}, g)$ be a Riemannian manifold. For $x, y \in \mathcal{M}$, let $G_{x,y}$ denote the set of all piecewise smooth curves joining x and y, and for $\gamma \in G_{x,y}$, let $L(\gamma)$ denote the length of the curve γ¿ Define*

$$d(x, y) = \inf_{\gamma \in G_{x,y}} L(\gamma)$$

We have

Proposition 9.1. *The function d defines a distance function on $\mathcal{M}$, i.e. $(\mathcal{M}, d)$ is a metric space.*

Proof. That $d(x, y) = d(y, x)$ is obvious. The triangle inequality is similarly immediate. To spell it out: If $\gamma : [a, b] \to \mathcal{M}$ is a piecewise smooth curve joining x and y, and $\tilde{\gamma} : [\tilde{a}, \tilde{b}]$ is a piecewise curve joining y and z, then $\hat{\gamma} : [a, b+\tilde{b}-\tilde{a}] \to \mathcal{M}$ defined by $\hat{\gamma}(t) = \gamma(t)$ for $t \in [a, b]$, $\hat{\gamma}(t) = \tilde{\gamma}(t)$ for $t \in (b, b + \tilde{b} - \tilde{a}]$ is a piecewise smooth curve joining x and x with $L(\hat{\gamma}) = L(\tilde{\gamma}) + L(\gamma)$. Taking infimums over $G_{x,y}$, $G_{y,z}$, one obtains the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$.

That $d(x, x) \geq 0$ is obvious. If $x \neq y$, then let $\mathcal{U}$ be a geodesically convex neighborhood of x not containing y. In particular, $\mathcal{U}$ contains a geodesic sphere of radius ϵ. Any curve γ joining x and y must cross this geodesic sphere at some point p. Since the radial geodesic from x to p minimizes the length of all curves from x to p, and the length of this curve is ϵ, it follows that $L(\gamma) > \epsilon$. Thus $d(x, y) \geq \epsilon$. □

9.2 Hopf–Rinow theorem

Recall from Section 7.4 the definition of geodesic completeness. In view of the metric space structure, we now have a "competing" notion of completeness, namely, metric completeness. In this section we shall show that this notion is actually equivalent to the notion of geodesic completeness defined earlier. In the process, we shall show that in a geodesically complete Riemannian manifold, it follows that any two points can be connected by a (**not** necessarily unique!) length-minimizing geodesic. This result is essential for global arguments in Riemannian geometry. We shall get a taste of this in the final section.

Theorem 9.1. *Let $(\mathcal{M}, g)$ be a Riemannian manifold, let $x, y \in \mathcal{M}$, and suppose* $\exp_x$ *is defined on all of $T_x\mathcal{M}$. Then there exists a geodesic $\gamma : [0, L] \to \mathcal{M}$ such that*

$$L(\gamma) = d(x, y). \tag{27}$$

Clearly, the assumptions of the above theorem are satisfied if $(\mathcal{M}, g)$ is geodesically complete. Note that in view of the fact that $L(\gamma) \geq d(x, y)$, (27) is equivalent to the statement that for any other curve $\tilde{\gamma}$ joining x and y, then $L(\tilde{\gamma}) \geq L(\gamma)$.

Proof. Let $\mathcal{U}_x$ be a geodesically convex neighborhood of x, and let $\mathcal{S}_\epsilon$ be a geodesic sphere around x of radius $\epsilon > 0$, contained in $\mathcal{U}_x$, such that $\epsilon < d(x,y) = d$. Since $d(\cdot, y)$ is a continuous function on $\mathcal{S}_\epsilon$, and $\mathcal{S}_\epsilon$ is compact, it follows that there exists a $p \in \mathcal{S}_\epsilon$ such that $d(p,y) = \inf_{q\in\mathcal{S}_\epsilon} d(q,y)$. Consider now the radial geodesic γ from x through p, parametrized by arc length. By assumption, $\gamma(t)$ is defined for all values of t. We will show that $\gamma(d) = y$. This will give the result of the theorem.

Note that $\gamma(\epsilon) = p$. Consider the subset $X \subset [\epsilon, d]$ defined by

$$X = \{s \in [\epsilon, d] : d(y, \gamma(s')) = d - s', \forall \epsilon \le s' \le s\}.$$

Clearly, our result follows if we show $d \in X$. Moreover, X is non-empty, because, we have $\epsilon \in X$. To see this note first that every curve κ joining x and y must cross $\mathcal{S}_\epsilon$, points in the connected component of $\mathcal{M}\setminus\mathcal{S}_\epsilon$ containing p have distance $< \epsilon$ away from x. Thus the length of κ is greater than ϵ plus the length of a curve from $\mathcal{S}_\epsilon$ to y. Thus $d(x,y) \ge d(p,y) + \epsilon$. On the other hand the reverse inequality is immediate from the triangle inequality. So indeed $\epsilon \in X$.

Thus, since $[\epsilon, d]$ is connected, it suffices to show that X is open and closed in the topology of $[\epsilon, d]$.

The closedness of X is immediate in view of the continuity of the function $d(y, \cdot)$. To show that X is open, it suffices to show that if $s \in X$ with $s < d$, then $s + \delta \in X$ for δ sufficiently small.

So let $s \in X$, $s < d$. There exists a geodesically convex neighborhood $\tilde{\mathcal{U}}$ of $\gamma(s)$. Let δ be such that the geodesic sphere around $\gamma(s)$ of radius δ, denoted $\tilde{\mathcal{S}}_\delta$, is contained in $\tilde{\mathcal{U}}$, and so that, moreover, $\delta < s < s + \delta < d$. (This will be true for all $\delta \le \delta_0$ for some $\delta_0 > 0$.) As before, let $\tilde{p}$ minimize the distance from $\tilde{\mathcal{S}}_\delta$ to y. Defining $\tilde{\gamma}$ to be the radial geodesic from $\gamma(s)$ to $\tilde{p}$, it follows as before that $d(\tilde{p}, y) = d(\gamma(s), y) + \delta$.

Consider now the distance between $\gamma(s-\delta)$ and $\tilde{p}$. On the one hand, we have, since $s - \delta \in X$, that $d(y, \gamma(s-\delta)) = d - s + \delta$. On the other hand, we know that

$$d(y, \gamma(s-\delta)) \le d(\gamma(s-\delta), \tilde{p}) + d(\tilde{p}, y) = d(\gamma(s-\delta), \tilde{p}) + d - s - \delta.$$

Thus

$$d(\gamma(s-\delta), \tilde{p}) \ge 2\delta.$$

On the other hand, by the triangle inequality, we have

$$d(\gamma(s-\delta), \tilde{p}) \le 2\delta.$$

Thus we have $d(\gamma(s-\delta), \tilde{p}) = 2\delta$. But since γ followed by $\tilde{\gamma}$ is a curve joining $\gamma(s-\delta)$ and $\tilde{p}$ of precisely length 2δ, it follows by the properties of geodesically convex neighborhoods that this curve is a geodesic, i.e. $\tilde{\gamma}$ must coincide with γ. But now the claim follows, since $\tilde{p} = \gamma(s+\delta)$. $\square$

Theorem 9.2. *Let $(\mathcal{M}, g)$ be a Riemannian manifold, and suppose there exists a point x satisfying the assumptions of the previous Theorem. Then $(\mathcal{M}, d)$ is complete as a metric space.*

In particular, the theorem applies in the case $(\mathcal{M}, g)$ is assumed geodesically complete.

Proof. A metric space is complete if every Cauchy sequence converges. Compact spaces are complete, and Cauchy sequences are clearly bounded. Thus it suffices to show that all bounded subsets of $\mathcal{S}$ are contained in compact sets.

Let $\mathcal{B} \subset \mathcal{M}$ be bounded. This means that there exists an $\infty > M > 0$ such that $d(x,y) \leq M$ for all $y \in \mathcal{M}$. By Theorem 9.1, this implies that $\mathcal{B} \subset \exp_x(B_M(0))$ wheere $B_M(0)$ denotes the closed ball of radius M in the $T_x\mathcal{M}$. But $\exp_x : T_x\mathcal{M} \to \mathcal{M}$ is a continuous function and $B_M(0)$ is compact. Thus $\exp_x(B_M(0))$ is compact and our theorem follows.

□

Theorem 9.3. *Let $(\mathcal{M}, g)$ be a Riemannian manifold, and suppose $(\mathcal{M}, d)$ is metrically complete. Then $(\mathcal{M}, g)$ is geodesically complete.*

Proof. Let $\gamma : [0,T) \to \mathcal{M}$ be a geodesic parametrized by arc length, with $T < \infty$. We shall show that γ can be extended to a geodesic $\gamma : [0, T+\delta) \to \mathcal{M}$.

Let $t_i \to T$. Since $d(\gamma(t_i), \gamma(t_j)) \leq |t_i - t_j|$, it follows that $\gamma(t_i)$ is Cauchy. By assumption then, $\gamma(t_i)$ converges to a point y. Let $\mathcal{U}$ denote a geodesically convex neighborhood of y, and let ϵ be such that for every $x_1, x_2 \in \mathcal{U}$, there exists a unique geodesic $\tilde{\gamma}$ of length ϵ in $\mathcal{M}$ starting from x_1, with the property that x_2 is on this geodesic within length ϵ. Choose $\gamma(t_i), \gamma(t_{i+1}) \in \mathcal{U}$, with $T - t_i < \epsilon$, and consider the geodesic $\tilde{\gamma}$ joining $\gamma(t_i)$ and $(\gamma(t_{i+1})$ described above. Since γ is also a geodesic joining $\gamma(t_i)$ and $\gamma(t_{i+1})$ of length less than ϵ, it follows that $\tilde{\gamma}(t - t_i)$ must coincide with $\gamma(t)$ for $t \in [t_i, T)$. Define now $\gamma(t)$ for $t \in [T, t_i + \epsilon)$ by $\gamma(t) = \tilde{\gamma}(t - t_i)$. This is then the required extension. □

10 The second variation

The existence of length minimizing geodesics rests on a *global* argument. On the other hand, a length minimizing geodesic certainly must be length minimizing to *local* deformations. The condition that a geodesic be length minimizing to local deformations is characterized by a differential condition, analogous to the characterization $f'' \leq 0$ for local minima of a function of one variable. It is in this characterization that the so-called curvature tensor first makes its appearance.

Let $\tilde{\gamma} : [0, L] \times (-\epsilon, \epsilon) \to \mathcal{M}$ denote a variation of a smooth curve $\gamma : [0, L] \to \mathcal{M}$, i.e., let $\tilde{\gamma}$ be a smooth map such that $\tilde{\gamma}(\cdot, 0) = \gamma$, and such that $\tilde{\gamma}(\cdot, s)$ is a curve for all $s \in (-\epsilon, \epsilon)$. We have already computed a formula for the first variation in Section 7.1. We now go further and compute a formula for the second variation.

We shall compute this formula in the special case that the original γ is a geodesic parametrized by arc length (so then L is its length), and such that

$$\tilde{\gamma}(0, s) = \gamma(0, s), \tilde{\gamma}(L, s) = \tilde{\gamma}(L, s). \tag{28}$$

It is a good exercise for the reader to write down the general case!

Let us recall the notation $N = \tilde{\gamma}_* \frac{\partial}{\partial s}$, $T = \tilde{\gamma}_* \frac{\partial}{\partial t}$, where s is a coordinate in $(-\epsilon, \epsilon)$ and t is a coordinate in $[0, L]$. Also recall the notation $L(s)$ got the length of the curve $\tilde{\gamma}(\cdot, s)$.

We have (as in Section 7.2) that for all s.

$$L'(s) = \int_0^L (g(T,T))^{-1/2} g(T, \nabla_N T) dt$$

Differentiating in s, and evaluating at 0, we obtain (using that $g(T_{\tilde{\gamma}(t,0)}, T_{\tilde{\gamma}(t,0)}) = 1$

$$\begin{aligned}
L''(0) &= \frac{d}{ds}\int_0^L (g(T,T))^{-1/2} g(T, \nabla_N T)dt|_{s=0} \\
&= \int_0^L -g(T,T)^{-3/2}(g(T,\nabla_N T))^2 + (g(T,T))^{-1/2} N g(T, \nabla_N T)dt|_{s=0} \\
&= \int_0^L g((\nabla_N T, \nabla_N T) - (g(T, \nabla_N T))^2 + g(T, \nabla_N \nabla_N T)dt \\
&= \int_0^L g(\Pi \nabla_N T, \Pi \nabla_N T) + g(T, \nabla_N \nabla_N T)dt \\
&= \int_0^L g(\Pi \nabla_T N, \Pi \nabla_T N) + g(T, \nabla_N \nabla_T N)dt
\end{aligned}$$

where Π denotes the projection to the orthogonal complement of the span of T. We have used in the last line the relation $[N,T] = 0$ and the torsion free property of the connection.

At this point, let us momentarily specialize to the case of $\mathbb{R}^n$ with its Euclidean metric. **In this case, covariant derivatives commute.** (Why?) We may thus write

$$\begin{aligned}
L''(0) &= \int_0^L g(\Pi \nabla_T N, \Pi \nabla_T N) + g(T, \nabla_N \nabla_T N)dt \\
&= \int_0^L g(\Pi \nabla_T N, \Pi \nabla_T N) + g(T, \nabla_T \nabla_N N)dt \\
&= \int_0^L g(\Pi \nabla_T N, \Pi \nabla_T N)dt - g(T, \nabla_N N)]_0^L + \int_0^L g(\nabla_T T, \nabla_N N)dt \\
&= \int_0^L g(\Pi \nabla_T N, \Pi \nabla_T N)dt
\end{aligned}$$

where we have used $\nabla_T T|_{\tilde{\gamma}(t,0)} = 0$ and the boundary condition (28) implying $N(\gamma(0)) = N(\gamma(L)) = 0$.

In particular, we see that in this case $L''(0) \geq 0$, and thus geodesics minimise[40] arc length with respect to near-by variations.

Of course, in the case of Euclidean space, we already knew much more, namely that geodesics globally minimize arc length, i.e. that a geodesic from p to q has the property that its length is strictly less than the length of any other curve from p to q. (In particular, geodesics are unique.) This follows from what we have done already, in view of the existence (for $\mathbb{R}^n$) of **global** geodesic normal coordinates, and Gauss's lemma.

But no matter. It is merely this computation for L'' that we wish to generalize to Riemannian manifolds. **Now, however, covariant derivatives no longer commute.** The analogue of the previous is given as below:

$$\begin{aligned}
L''(0) &= \int_0^L g(\Pi \nabla_T N, \Pi \nabla_T N) + g(\nabla_N \nabla_T N - \nabla_T \nabla_N N, T)dt \\
&= \int_0^L g(\nabla_T N, \nabla_T N) + g(R(T,N)N, T)dt \qquad (29)
\end{aligned}$$

[40] Show that $L''(0) > 0$ for all non-trivial variations satisfying (28).

where

$$R(T,N)N = \nabla_N \nabla_T N - \nabla_T \nabla_N N. \tag{30}$$

The expression $R(T,N)N$ is called the *curvature*. A priori it seems that its value at a point p should depend on the behaviour of the vector fields T and N up to second order. (If this were the case, then the formula (29) would not be particularly useful.) But it fact, $R(T,N)T$ depends only on $T(p)$, $N(p)$, i.e. R is a *a tensor*. (See the next section!) As we shall see, R is an invariant of the metric g; it is the above expression for L'' which gives it perhaps its most natural geometric interpretation in terms of the lengths of nearby curves.

We commence in the following section a systematic discussion of the curvature tensor. We will then return to the second variation formula, and use it to prove theorems relating local assumptions on the behaviour of the curvature tensor, and global geometric properties of the manifold.

11 The curvature tensor

Having got a taste of curvature we now make the general definitions.

Definition 11.1. *Let $(\mathcal{M}, g)$ be a Riemannian manifold, $p \in \mathcal{M}$, and X, Y, and Z be vetor fields in a neighborhood of p. We define*

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z$$

Proposition 11.1. *$R(X,Y)Z$ only depends on $X(p)$, $Y(p)$, and $Z(p)$, i.e. R can be thought of as a section of $T\mathcal{M} \otimes T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes T^*\mathcal{M}$.*

Proof. As usual, an uninspired calculation. ∎

We call R the Riemann curvature tensor. The curvature tensor satisfies the following symmetries

Proposition 11.2.

$$R(X,Y)Z = -R(Y,X)Z$$

$$g(R(X,Y)Z,W) = -g(R(X,Y)W,Z)$$

$$g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$$

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$$

Proof. Again, this is just a computation. The last identiy is known as the first *Bianchi identity.* ∎

Note why the Lie bracket term was absent in (30).

11.1 Ricci and scalar curvature

The algebraic complexity of curvature, and the more subtle relations of this complexity and geometry, is something which we will not in fact really come to terms with in this class. Here, we shall consider algebraically more simple objects that can be obtained from the Riemann curvature.

We can define the so called *Ricci curvature* by applying the contraction C of Definition 3.8. Specifically, considering R as a section of $T\mathcal{M}\otimes T^*\mathcal{M}\otimes T^*\mathcal{M}\otimes T\mathcal{M}$, we may apply the map

$$C : T\mathcal{M}\otimes T^*\mathcal{M}\otimes T^*\mathcal{M}\otimes T^*\mathcal{M}\to T^*\mathcal{M}\otimes T^*\mathcal{M}$$

acting as in Definition 3.8 but only in $T\mathcal{M}$ and the second $T^*\mathcal{M}$ factor, i.e the one "corresponding" to Y in the notation $R(X, Y)Z$. The composition of C and R produces a tensor Ric which is a section of $T^*\mathcal{M}\otimes T^*\mathcal{M}$. This is the so-called Ricci curvature. Alternatively, in view of the relation between contraction and trace of a homomorphism, one easily sees that

$$\mathrm{Ric}(X, Z) = \mathrm{trace} Y \mapsto R(X, Y)Z.$$

From the symmetries of curvature tensor one sees that this is the only non-trivial contraction. Moreover, one sees that the Ricci tensor is symmetric, i.e. that $\mathrm{Ric}(X, Y) = \mathrm{Ric}(Y, X)$.

It is hard to give a sense at this stage for why the Ricci curvature is so important an object in geometry. Written in local coordinates (see next section) it has the same number of components as the metric, and this very pedestrian reason may be why it was first written down. But its true geometric character is much more subtle. In this class, we will only scratch the surface, but we shall see it used in Theorem 12.1 of Section 12.

In general relativity, the importance of Ricci curvature is clear from the beginning. The vacuum Einstein equations are the statement that

$$\mathrm{Ric} = 0$$

A further contraction of Ric (after raising one of the indices!) yields the so-called *scalar curvature*. This is often denoted by R, in the spirit of remarks given in Section 6.3.2. For a formula in terms of indices, see Section 11.3.

11.2 Sectional curvature

The reader familiar with the classical differential geometry of surfaces will have some intuition for the Gauss curvature of a surface, in particular, what it means geometrically for the Gauss curvature to be positive, negative, zero.

It turns out that the curvature tensor defines, for each 2-plane, a notion of curvature that can be thought to correspond to the Gauss curvature. This is called the sectional curvature. It is defined as follows. Given a plane

$$\Pi \subset T_p\mathcal{M}$$

choose vectors X, Y spanning T_p. Define

$$K(\Pi) = \frac{g(R(X, Y)X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}.$$

Note that if X and Y are orthonormal, then

$$K(\Pi) = g(R(X, Y)X, Y)$$

Note that if $\mathcal{M}$ is two dimensional, then K coincides with the usual Gaussian curvature.

It turns out that the condition $K(\Pi) > 0$ for *all* Π is most like the condition of positive Gaussian curvature for surfaces. Similarly, $K(\Pi) < 0$. The subtlety of higher dimensional geometry lies in that $K(\Pi)$ can have different signs for different planes. The condition that Ric is positive (resp. negative) definite can be thought of as a weaker analogue of $K(\Pi) > 0$ for all Π, (resp. $K(\Pi) < 0$ for all Π). We shall get at least some taste of all this in Section 12.

Finally, as an exercise, test your multilinear algebra skills by showing that the collection of all sectional curvatures determines the Riemann curvature tensor.

11.3 Curvature in local coordinates

Since as we saw in the previous section, R can be though of as a section of $T\mathcal{M} \otimes T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes T^*\mathcal{M}$, we can write it as

$$R = R^i_{jkl} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^l$$

where

$$R^i_{jkl} \frac{\partial}{\partial x^i} = R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) \frac{\partial}{\partial x^j}$$

Note: There is never universal agreement in the literature as to the order of the indices, or even the sign convention in the definition of the Riemann curvature tensor. Sectional and Ricci curvature are always defined so as to be positive on, say $\mathbb{S}^n$.

One can derive from the definition of curvature the formulas

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^m_{jk}\Gamma^i_{ml} - \Gamma^m_{jl}\Gamma^i_{mk}$$

$$\mathrm{Ric}_{ij} = R^k_{ijk}$$

$$R = g^{ij} R_{ij}.$$

Thus, if one has explicit formulas for the metric, one can compute the curvature tensor by brute force.

11.4 Curvature as a local isometry invariant

In view of Proposition 6.2, we have immediately that

Proposition 11.3. *Let $(\mathcal{M}, g)$, $(\tilde{\mathcal{M}}, \tilde{g})$ be Riemannian and suppose that $p \in \mathcal{U} \subset \mathcal{M}$, $q \in \tilde{\mathcal{U}} \subset \tilde{\mathcal{M}}$, and $\phi : \mathcal{U} \to \tilde{\mathcal{V}}$ is an isometry with $\phi(p) = q$. Let R, $\tilde{R}$ denote the Riemann curvature tensors of $\mathcal{M}$, $\tilde{\mathcal{M}}$ respectively. Then*

$$\tilde{R}(\phi_* X, \phi_* Y)\phi_* Z = \phi_*(R(X, Y)Z).$$

This is the statement that curvature is a local isometry invariant.

An alternative (and quite cumbersome) way of proving Proposition 11.3 is via the interpretation of curvature in terms of the second variation of arc length. Try it for yourself if you're of that persuasion.

11.5 Spaces of constant curvature

11.5.1 $\mathbb{R}^n$

It should be clear from the definition that in this case we have

$$R^l_{ijk} = 0.$$

In fact, the following statement is true. Let $\mathcal{U}$ be homoeomorphic to the n-disc, and let g be a metric on $\mathcal{U}$ such that R^l_{ijk}. Then there exists a smooth map $\phi : \mathcal{U} \to \mathbb{R}^n$ which is an isometry onto its image. In other words, if $(\mathcal{M}, g)$ has identically vanishing Riemann curvature, then $\mathcal{M}$ is locally isometric to Euclidean space.

The above statement was in fact proven by Riemann in his original lecture which initiated Riemannian geometry. In view also of Propostion 11.3, we see that $R^l_{ijk} = 0$ are necessary and sufficient conditions for a Riemannian manifold to be locally isometric to Euclidean space. This was the original motivation for identifying this expression.

11.5.2 $\mathbb{S}^n$

Write down the metric of $\mathbb{S}^n$ explicitly, and compute the curvature. In particular, show that

$$K(\Pi) = 1$$

for all planes $\Pi \subset T\mathbb{S}^n$. We say that $\mathbb{S}^n$ is a space of constant curvature.

Exercise: How can you cheat in this computation?

Note that, in particular, the curvature tensor is not identically 0. In view of Proposition 11.3, this gives a purely infinitessimal proof that the sphere and Euclidean space are not locally isometric. (Compare with Example 4.6, where the "macroscopic" formula for the area of spherical triangles is used. Actually, this formula can be derived by integrating the curvature in the spherical triangle. This is the celebrated Gauss-Bonnet formula.)

11.5.3 $\mathbb{H}^n$

This space, called hyperbolic n-space, is the higher dimensional analogue of the space $\mathbb{H}^2$ discussed in the introduction.

A useful way of representing this space is as the standard open disc in $\mathbb{R}^n$ with metric

$$g = \frac{4}{\left(1 - (x^1)^2 - \cdots - (x^n)^2\right)^2} e$$

where e denotes the standard Euclidean metric.

Compute for yourself the curvature. In particular, show

$$K(\Pi) = -1$$

for all Π.

It turns out that Riemannian manifolds of constant curvature are locally isometric to $\mathbb{H}^n$, $\mathbb{S}^n$ or $\mathbb{R}^n$. This is not so difficult to show, but we will not show it here...

There is no end to the study of $\mathbb{S}^n$, $\mathbb{H}^n$, and their various quotients. A good geometer should be able to navigate these with ease.

12 Simple comparison theorems

12.1 Bonnet–Myers theorem

Theorem 12.1. *(Bonnet-Myers) Let $(\mathcal{M}, g)$ be a complete Riemannian manifold with*

$$Ric(X, X) \geq (n-1)kg(X, X)$$

for all $X \in T\mathcal{M}$, for some $k > 0$. Then the diameter of $\mathcal{M}$ is less than or equal to $\pi/\sqrt{k}$. In particular, $\mathcal{M}$ is compact.

This is a protypical result of its kind in Riemannian geometry. A weak global assumption (completeness), together with a curvature assumption, give a strong geometric statement (diameter bounded) and more information on the topology (compactness).[41]

Note that the diameter is defined as $\sup_{x,y\in\mathcal{M}} d(x, y)$.

Proof. Let $x, y \in \mathcal{M}$. By Theorem 9.1, there exists a geodesic $\gamma : [0, d] \to \mathcal{M}$, parametrized by arc length, with $\gamma(0) = x$, $\gamma(d) = y$, and $L(\gamma) = d = d(x, y)$. It suffices to show that $d \leq \pi/\sqrt{k}$.

Since γ is length minimizing, then if $\tilde{\gamma}$ is any variation with fixed endpoints, we must have, for the corresponding function $L(s)$, $L''(s) \geq 0$.

Now given a vector field along γ, vanishing at $t = 0$ and $t = d$, we can define a variation $\tilde{\gamma}$ as above. We will define our vector fields (and thus our variations) by making use of an important technique in differential geometry, namely, the choice of a well adapted frame, i.e. a well adapted set of orthogonal vectors spanning the tangent at each point along γ.

In our case, a convenient such frame can be constructed as follows. Let $E_1, \ldots E_{n-1}$, denote vectors at $\gamma(0)$ such that the collection $T, E_1, \ldots E_{n-1}$ comprises a set of orthonormal vectors. We may now parallel transport these vectors along γ to obtain a set of vector fields $T, E_1, \ldots E_{n-1}$. These vector fields remain orthonormal and linearly independent. (Why?)

We now define $n = 1$ vector fields V of the form

$$V_i = f(t)E_i,$$

with $f(t) = 0$, $f(t) = d$. For each we may define a variation ${}^i\tilde{\gamma}$ giving rise to V_i as ${}^i\tilde{\gamma}_* \frac{\partial}{\partial s}$ along γ. By the second variation formula (29) we can compute:

$$\begin{aligned}
0 \leq {}^iL''(s) &= \int_0^d g(\Pi\nabla_T V_i, \Pi\nabla_T V_i) - g(R(T, V_i)T, V_i))dt \\
&= \int_0^d g(\Pi\nabla_T (fE_i), \Pi\nabla_T (fE_i)) - g(R(T, fE_i)T, fE_j)dt \\
&= \int_0^d (f')^2 g(E_i, E_i) + f^2 g(\Pi\nabla_T E_i, \Pi\nabla_T E_i) \\
&\qquad +2ff' g(\Pi\nabla_T (E_i), E_i) - f^2 g(R(T, E_i)T, E_i)dt \\
&= \int_0^d (f')^2 - f^2 g(R(T, E_i)T, E_i)).
\end{aligned}$$

[41]After studying the proof, explore the situation when the completeness assumption is dropped.

The reader should remark why the choice of E_i is so convenient.

Since we have no assumptions about $g(R(T, E_i)T, E_i)$ itself, we cannot infer anything yet. But on summation over i, $g(R(T, E_i)T, E_i)$ yields $Ric(T, T)$, and this, by assumption, we have an inequality. Summing we obtain

$$0 \le \sum_i {}^i L''(s) = (n-1)\int_0^d (f')^2 - \int_0^d f^2 Ric(T,T) \le (n-1)\int_0^d (f')^2 - (n-1)\kappa \int_0^d f^2$$

i.e.

$$\kappa \le \frac{\int_0^d (f')^2}{\int_0^d f^2}$$

The minimum of the expression on the right over all functions vanishing at $t = 0$ and $t = d$ is given by setting $f(t) = \sin \frac{\pi t}{d}$, for which one obtains

$$\kappa \le \frac{\pi^2}{d^2}$$

which implies

$$d \le \pi/\sqrt{k}$$

as deseired. □

12.2 Synge's theorem

First let us define an orientation.

Definition 12.1. *An n-dimensional manifold $\mathcal{M}$ is called orientable if there exists a non-vanishing section $\Gamma(\Lambda^n(T\mathcal{M})$ of the bundle of top-order forms, defined in Section 15.*

Alternatively, the above is a section of $\Gamma \otimes_n T^*\mathcal{M}$ which is alternating as a multilinear map. Note that two non-vanishing sections σ_1, σ_2 satisfy $\sigma_1 = \lambda\sigma_2$ for some non-vanishing smooth function λ.

12.3 Cartan–Hadamard theorem

13 Jacobi fields

We commenced in Sections 7.1 and 10 the study of the set of all piecewise curves from points p to q, via the techniques of the variation calculus. We have derived the first and second variation formulas and have already proven a number of theorems using this.

To go further, we shall need to understand better the structure of the function L thought of as a function on all curves joining p and q. A major difficulty arises from the fact that this space is infinite dimensional.

It turns out, however, that the behaviour of critical points of the length functional L, i.e. geodesics, can be completely analyzed by passing to a finite dimensional subset, so-called geodesic variations. The vector fields N defined by such variations are known as *Jacobi fields*.

Definition 13.1. *Let $\gamma : [0, L] \to \mathcal{M}$ be a geodesic. A* Jacobi field *along γ is a vector field J satisfying*

$$\nabla_T \nabla_T J + R(T, J)T = 0. \tag{31}$$

Equation (31) is known as the *Jacobi equation.* For reasons that shall soon become clear, we could restrict to fields J such that $g(T, J) = 0$, for fields in the direction of T are in some sense trivial.

Proposition 13.1. *Let $\gamma : [0, L] \to \mathcal{M}$ be a geodesic parametrized by arc length, and let $T = \gamma_* \frac{\partial}{\partial t}$. Then J is a Jacobi field if and only if there exists a geodesic variation $\tilde{\gamma}$ of γ such that $J = \tilde{\gamma}_* \frac{\partial}{\partial s}|_{s=0}$.*

Proof. Suppose $\tilde{\gamma}$ is a geodesic variation, i.e. suppose that $\nabla_T T = 0$ throughout the surface spanned by the variation. Let us denote by J the vector field $\tilde{\gamma}_* \frac{\partial}{\partial s}$. Applying J to $\nabla_T T = 0$, we obtain

$$\nabla_J (\nabla_T T) = 0.$$

But by the definition of curvature and the fact that $[J, T] = 0$, we have

$$\begin{aligned} 0 = \nabla_J \nabla_T T &= \nabla_T \nabla_J T + R(T, J)T \\ &= \nabla_T \nabla_T J + R(T, J)T. \end{aligned}$$

This gives the desired result.

For the other direction take just take $\tilde{\gamma}(s, t) = \exp_{\exp_{\gamma(0)} J(0)s}(T + J's)$. ☐

In view of our local existence theorem for ode's, and the linearity of Jacobi equation, we easily obtain the following

Proposition 13.2. *Let γ be a geodesic as before, with $p \in \gamma$ and let J_0, J_0' be vectors at $p = \gamma(t_0)$. Then there exists a Jacobi field along γ such that $J(t_0) = J_0$, $\nabla_{\frac{\partial}{\partial t}} J = J_0'$.*

Proof. Compare with the Proposition 7.2. ☐

Corollary 13.1. *The space of Jacobi fields along γ constitutes an $2n$-dimensional linear subspace of the (infinite dimensional) set of all piecewise regular vector field along γ.*

13.1 The index form $I(V, W)$

The second variation formula motivates the following definition. Given piecewise regular vector fields V, W along a geodesic γ, with $T = \gamma'$, define

$$I(V, W) = \int (g(\nabla_T V, \nabla_T W) - g(R(T, V)T, W))dt. \tag{32}$$

This is a symmetric bilinear form on the space of piecewise regular vector fields. In view of our computation, if V is a variation vector field for a variation $\tilde{\gamma}$ of γ, with fixed endpoints, then

$$L''(0) = I(V, V).$$

We will be interested in knowing when is $L''(0) \geq 0$ for all variations $\tilde{\gamma}$ of a given curve γ. In view of the above, we have reduced this to the question, when is $I(V,V) \geq 0$ for all piecewise regular vector fields along γ, vanishing at the endpoints.

This is progress, for the space of piecewise regular vector fields vanishing at the endpoints is a linear space, and the mapping I is a bilinear form. This is a lot better than understanding the nonlinear mapping L on a huge space of curves.

But in fact, we can do better, and this is the whole point of considering Jacobi fields.

For it turns out that the question of whether $I(V,V) \geq 0$ for all V can be completely resolved by restricting to the finite dimensional subspace of Jacobi fields.

13.2 Conjugate points and the index form

To state the relation between the index form and Jacobi fields, we will make the following definition.

Definition 13.2. *Let $\gamma : [0,L] \to \mathcal{M}$ be a geodesic parametrized by arc length. Points $p = \gamma(t_0)$, $q = \gamma(t_1)$, $t_0 < t_1$ are said to be conjugate along γ if there exists a nontrivial*[42] *Jacobi field J along γ such that $J(t_0) = 0$, $J(t_1) = 0$.*

The main result of this section is contained in:

Proposition 13.3. *Let $\gamma : [0,L] \to \mathcal{M}$ be a geodesic parametrized by arc length, with $p = \gamma(0)$. Suppose V is a piecewise regular vector field along γ vanishing at the endpoints, such that $I(V,V) \leq 0$. Then there exists a point $q = \gamma(t_1)$ for some $t_1 \in (0,L]$, conjugate to p along γ. Moreover, if $I(V,V) < 0$, then there exists such a point for $t_1 \in (0,L)$. Conversely, given a conjugate point $q = t_1 \in (0,L)$, there exits a V vanishing at the endpoints such that $I(V,V) < 0$.*

Proof. The idea of the proof is to write the vector field V in terms of a frame of Jacobi fields vanishing at p, together with T, and then apply the definition of the index form and just compute!

Fix an arbitrary orthonormal frame T, E_1,$\ldots E_{n-1}$ at p. For E_1, $\ldots E_{n-1}$, we can define Jacobi field J_1, $\ldots J_n$ such that $J'_\alpha(0) = E_\alpha$.

Note that these Jacobi fields are linearly independent in the space of vector fields. If their span (together with T) is not n-dimensional at some point q, then there exists a linear combination of the vector fields that vanishes at q. This linear combination yields our desired Jacobi field.

Otherwise, we may assume that the span of T, J_1, $\ldots J_n$ is n-dimensional for $t > 0$. Given our arbitrary V we may write V as

$$V = V^1 T + V^2 J_2 + \ldots + V^n J_n.$$

[42] i.e. not identically vanishing!

Applying (32) we obtain

$$
\begin{aligned}
I(V,V) &= \int (g(\nabla_T V, \nabla_T V) - g(R(T,V)T,V))dt \\
&= -\int g(V, \nabla_T \nabla_T V) + g(R(T,V)T,V)dt \\
&= -\int V^\alpha V^\beta g(J_\alpha, \nabla_T \nabla_T J_\beta) + 2(V^\alpha)(V^\beta)' g(J_\alpha, \nabla_T J_\beta) \\
&\quad + (V^\alpha)(V^\beta)'' g(J_\alpha, J_\beta) + V^\alpha V^\beta g(R(T, J_\alpha)T, J_\beta)dt \\
&= -\int 2(V^\alpha)(V^\beta)' g(J_\alpha, \nabla_T J_\beta) + (V^\alpha)(V^\beta)'' g(J_\alpha, J_\beta)dt \\
&= \int (V^\alpha)'(V^\beta)' g(J_\alpha, J_\beta)dt \geq 0.
\end{aligned}
$$

Actually, in this computation, we have cheated, or better, we haven't justified all the steps. For we have several times applied integration by parts and thrown away the boundary terms. For this, we need that the V^α are regular at 0. Recall that the J_α and V both vanish at 0. Since J_α is piecewise regular, it must vanish at least linearly in t. On the other hand, one easily sees by the Jacobi equation that J_α vanishes at most linearly. Thus our integration by parts is justified.[43]

The full strength of the proposition now follows from carefully considering the various cases of equality in the above. This is again left as an exercise for the reader. □

13.3 The use of Jacobi fields

The results of the previous section open the door to the study of the local length minimizing properties of geodesics by means of the theory of ordinary differential equations applied to the Jacobi equation. And this equation in turn depends on curvature. A classical corpus of results in the theory of ode's allows one to compare solutions of

$$f'' + Kf = 0$$

and

$$f'' + \bar{K}f = 0$$

when, say $K \geq \bar{K}$. Applied to the Jacobi equation, this theory allows us to infer the vanishing of Jacobi fields on a manifold $\mathcal{M}$ whose curvature satisfies a certain inequality, by "comparing" the solutions of the Jacobi equation, with the solutions on a constant-curvature manifold, which are known explicitly. It is this which gives the name to *comparison theorems* in Riemannian geometry.

We have already proven in simple comparison theorems that did not need the characterisation of the index provided by Proposition 13.3. In those cases, we produced directly variation fields on which the index had a negative sign (cf. Sections 12.1 and 12.2), or showed that for all fields it had a positive sign (cf. Section 12.3). Proposition 13.3 allows for more subtle results like the triangle comparison theorem. See Chapter 7 of Chavel [2].

[43] The reader is encouraged to do this carefully.

14 Lorentzian geometry and Penrose's incompleteness theorem

Penrose's celebrated incompleteness theorem is really the Lorentzian analogue of the Bonnet–Myers theorem. To state it, we first have to make some general definitions in Lorentzian geometry.

Definition 14.1. *Let $\mathcal{M}$ be a smooth manifold. A* Lorentzian metric *is a smooth section $g \in \Gamma^\infty(T^*\mathcal{M} \otimes T\mathcal{M})$ satisfying:*

At the formal level, we can import much of the apparatus of Riemannian geometry *as is* to the Lorentzian (or more general semi-Riemannian) case. In particular, Lorentzian metrics define a unique Levi-Civita connection, and we may define the Riemann curvature tensor as before, which satisfies the same symmetries. The first and second variation formulas hold as before.

On the other hand, the failure of positive definiteness for the metric means that we may no longer define a metric space structure using length. The metric retains however convexity properties when restricted to particular directions. Let us first see how directions are distinguished in Lorentzian geometry.

14.1 Timelike, null, and spacelike

In this section $(\mathcal{M}, g)$ is a Lorentzian manifold.

14.1.1 Vectors and vector fields

Definition 14.2. *A vector v is said to be* timelike *if $g(v,v) < 0$, spacelike if $v = 0$ or $g(v,v) < 0$, and* null *otherwise. If a vector is either timelike or null, it is called* causal.

Vector fields V are said to be timelike, etc., if $V(x)$ is a timelike, etc., vector for all x.

14.1.2 Curves

C^1 parametrized curves inherit these appellations from their tangent vectors.

Definition 14.3. *A curve γ is said to be* timelike*, etc., if γ' is timelike, etc.*

In the physical interpretation of general relativity, massive physical particles are constrained to follow timelike curves. (This is what survives of the special relativistic principle "nothing can travel faster than light".) So called freely falling particles follow timelike geodesics. Freely falling massless particles follow null geodesics.

14.1.3 Submanifolds

Definition 14.4. *A submanifold is said to be* timelike*,* spacelike*,* null*, ...*

14.2 Time orientation and causal structure

Definition 14.5. *A time orientation is a globally defined timelike vector field T.*

Note that such a vector field cannot vanish in view of the definition of timelike.

Definition 14.6. *A causal vector v is future directed if $g(v,T) < 0$, and past-directed if $g(v,T) > 0$.*

14.3 Global hyperbolicity

Definition 14.7. *Let $\mathcal{M}$ be a manifold. A parameterised curve $\gamma : (a,b) \to \mathcal{M}$ is inextendible if it is not the restriction of a parameterised curve $\tilde{\gamma}$ defined on a strictly larger domain.*

Definition 14.8. *Let $(\mathcal{M}, g)$ be a Lorentzian manifold. A Cauchy hypersurface is a spacelike hypersurface Σ such that every inextendilbe parametrized causal curve intersects Σ exactly once.*

Definition 14.9. *A Lorentzian manifold is said to be* globally hyperbolic *if it admits a Cauchy hypersurface.*

The significance of globally hyperbolic spactimes is that solutions of hyperbolic equations whose symbol is related to that of the wave equation

$$\Box_g \psi = 0$$

(in particular, the wave equation itself) are uniquely determined by their Cauchy data on a Cauchy hypersuface. This in particular applies to the Einstein equations themselves.

14.4 Closed trapped 2-surfaces

14.5 Statement of Penrose's theorem

14.6 Sketch of the proof

14.7 Examples

14.7.1 Schwarzschild

Consider the manifold $\mathbb{Q} \times \mathbb{S}^2$ with "doubly warped product metric"

14.7.2 Reissner-Nordström

15 Appendix: Differential forms and Cartan's method

16 Guide to the literature

There is no shortage of good books on differential geometry. The discussion here is not in any sense intended as an exhaustive list.

16.1 Foundations of smooth manifolds, bundles, connections

Chapter 1 of Kobayashi–Nomizu [6] is a concise introduction to differential manifolds, and Chapter 2 has a definitive treatment of the general theory of connections in principal bundles. Connections in vector bundles are then a derived notion.

In our course, we have avoided principal bundles all together. A direct introduction to connections in vector bundles, along the lines of the definition here, is given in.

The book of Lee [9] is a thorough introduction to the foundation of differential manifolds.

This material is also nicely covered in Volume 1 of Spivak [13]. (See also section 16.2.)

For very stimulating Russian points of view on the foundations of differential manifolds, I can recommend the first volume of the Geometry series from the Encylcopedia of Mathematical Studies series [7], published by Springer, and Volume 2 of Dubrovin, Fomenko and Novikov's celebrated text [4].

An outline overview of differential manifolds can also be found in the first chapter of the beautiful book of Gallot, Hulin and Lafontaine [5] on Riemannian geometry (see Section 16.2 below).

Bourbaki did not produce a treatment, but if you want to know what it would have looked like, see:

`http://portail.mathdoc.fr/archives-bourbaki/PDF/167bis_nbr_068.pdf`

16.2 Riemannian geometry

There are many good books centred on Riemannian geometry.

The volumes [13] of Spivak are must: Volume 2 gives historical notes on the work of Gauss and early Riemannian geometry as well as an account of connections in principle bundles. The classical differential geometry of surfaces theory is then covered in volume 3 while higher dimensional Riemannian geometry is convered in volumes 4 and 5.

The text book [2] of Chavel is a very accessible introduction to Riemannian geometry, with a special emphasis on volume comparison theorems and the relation with isoperimetric constants.

The book of O'Niell [10] is unique in its attention payed to the general semi-Riemannian case, and covers the basics of Lorentzian geometry in a form amenable to mathematicians who may be scared off from too much physics (cf. Section 16.3).

The book of Gallot, Hulin and Lafontaine [5] is another beautiful introduction to Riemannian geometry, starting from the basics of differential manifolds, and containing a discussion of the Lorentzian case.

A very rewarding approach to Riemannian geometry, emphasising computational technique (all from the point of view of the structure equations for submanifolds) is that given by Peterson [12].

The book of Berger [1] is an extremely stimulating encyclopaedic reference of the entire subject.

16.3 General relativity

Classic introductory books in general relativity with a mathematical point of view are Hawking and Ellis [8] and Wald [14]. The lecture notes of Christodoulou [3] provide a very nice introduction quickly going from basic Lorentzian geometry to highlights of recent research. The book [10] of O'Niell mentioned above is also a dependable reference for Lorentzian causality theory and the singularity theorems.

As I recently learned in a Cambridge bookshop, a good-condition first edition of Hawking and Ellis [8] goes for £450, so if you have one, treat it with care.

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