Homework 1

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Lemma 1. *Every open subset of* $\mathbf{R}$ *is homeomorphic to the countable disjoint union of intervals.*

Proof. Let U be an arbitrary non-empty open subset of $\mathbf{R}$. Let $R = U \times U$ be an equivalence relation where, for arbitrary $x, y \in U$, we have $(x, y) \in R$ and $x \sim y$ if and only if $[\min\{x, y\}, \max\{x, y\}] \subseteq U$. The set R is an equivalence relation since:

1. Let $x \in U$ is arbitrary. Since $[x, x] = \{x\} \subset U$, we have that $(x, x) \in R$.
2. Let $x, y \in U$. If $(x, y) \in R$ then $[\min\{x, y\}, \max\{x, y\}] \subseteq U$, and since $[\min\{y, x\}, \max\{y, x\}] = [\min\{x, y\}, \max\{x, y\}] \subseteq U$, we have that $(y, x) \in R$.
3. Let $x, y, z \in U$. Assume, without loss of generality, that $x \leq y \leq z$. If $(x, y) \in R$ and $(y, z) \in R$ then $[x, y] \subseteq U$ and $[y, z] \subseteq U$, so $[x, y] \cup [y, z] = [x, z] \subseteq U$. Therefore, $(x, z) \in R$.

We will write the collection of equivalence classes of R as $\mathcal{R}$. Each $I \in \mathcal{R}$ is a subset of U; equivalence classes partition the set on which the equivalence relation is defined, so each equivalence class is a disjoint subset of U and $\cup_{I \in \mathcal{R}} I = U$. Let $I \in \mathcal{R}$ and $x \in I$ be arbitrary. Since $I \subseteq U$ and U is open, there exists some $\epsilon \in \mathbf{R}^+$ such that $(x - \epsilon, x + \epsilon) \subseteq U$. For all $y \in (x - \epsilon, x + \epsilon) \subseteq U$, we have that $[\min\{x, y\}, \max\{x, y\}] \subseteq (x - \epsilon, x + \epsilon) \subseteq U$, so $y \sim x$. This implies that $(x - \epsilon, x + \epsilon) \subseteq \{y \in U : y\ x\} = I$, and since x was arbitrary, we have that I is open.

Since I is open we can choose some open interval $(a, b) \subseteq I$, and since $\mathbf{Q}$ is dense, the set $I \cap \mathbf{Q}$ is nonempty. The rationals are denumerable, so they are well-ordered, which implies that there exists a least element $\min I \cap \mathbf{Q}$. This gives a function $\pi^{-1} : \mathcal{R} \to \mathbf{Q} : I \mapsto \min I \cap U$. Since all $I \in \mathcal{R}$ are disjoint, $\pi^{-1}(I) = \pi^{-1}(J)$ implies that $I = J$, so π^{-1} is injective, and $\mathcal{R}$ is countable. This gives an injective mapping $\iota : \mathcal{R} \to \mathbf{N}$. Let the image of this mapping be $M = \iota(\mathcal{R})$. The function ι is given by π^{-1} and the function that denumerates $\pi^{-1}(\mathcal{R})$ *consecutively* by applying a total ordering to $\pi^{-1}(\mathcal{R})$; that is, $M = 1, 2, 3, \ldots \subseteq \mathbf{N}$ (this is finite or equal to $\mathbf{N}$).

We can write a function $\phi : U \to (0,1) \times M$ given by

$$x \mapsto (\varphi_J(x), \iota(J)),$$

where $J \in \mathcal{R}$ is the equivalence class of x and φ_J is the homeomorphism from J to $(0,1)$ given by claim 3. We can define $\phi^{-1} : (0,1) \times M \to U : (y,i) \mapsto \varphi^{-1}_{\iota^{-1}(i)}(y)$; this gives that

$$\phi^{-1}(\phi(x)) = \phi^{-1}(\varphi_{[x]}(x), \iota([x])) = \varphi^{-1}_{\iota^{-1}(\iota([x]))}(\varphi_{[x]}(x)) = x,$$

and so ϕ is bijective.

Take M as a metric space with the discrete metric and $(0,1)$ as a metric space with the usual metric. The direct product of two metric spaces is a metric space, so $(0,1) \times M$ is a metric space.

Let $V \times G \subseteq (0,1) \times M$ be open. This implies that $G = \{i\}$ for some $i \in M$, since M is a discrete metric space and G is open. Then, we have that $\phi^{-1}(V \times G) = \varphi^{-1}_{\iota^{-1}(i)}(V)$; this preimage is open since $\varphi^{-1}_{\iota^{-1}(i)}$ is continuous. This implies that ϕ is continuous, so ϕ is a homeomorphism.

Since U was arbitrary, we have that every open subset of $\mathbf{R}$ is homeomorphic to a countable disjoint union of open subsets of $\mathbf{R}$, where the disjoint union is constructed with consecutive natural numbers.

□

Proposition 2. *There are only countably many non-homeomorphic open subsets of* $\mathbf{R}$.

Proof. Let $\mathcal{O}$ be the collection of all open subsets of $\mathbf{R}$. Let $R = \mathcal{O} \times \mathcal{O}$, where $(U,V) \in R$ if and only if U is homeomorphic to V. This is an equivalence relation since:

1. Let $U \in \mathcal{O}$ be an arbitrary open subset of $\mathbf{R}$. Since the identity mapping is a homeomorphism, we have that $(U,U) \in R$.

2. Let $U, V \in \mathcal{O}$ be arbitrary. If $(U,V) \in R$, then there exists a homeomorphism $\phi : U \to V$; this function an inverse $\phi^{-1} : V \to U$ by definition, and so $(V,U) \in \mathcal{O}$.

3. Let $U, V, W \in \mathcal{O}$ be arbitrary. If $(U,V) \in R$ and $(V,W) \in R$ then there exist two homeomorphisms $\phi : U \to V$ and $\varphi : V \to W$. The composition of two homeomorphisms is a homeomorphism, so $\varphi \circ \phi : U \to W$ is a homeomorphism, and $(U,W) \in R$.

We can write a function on the set of equivalence classes $\mathfrak{O}$ given by

$$\begin{aligned} f : \mathfrak{O} &\to \mathbf{Z}_{\geq 0} \\ \{U_1, U_2, \ldots\} &\mapsto \xi(\phi(U_1)) \end{aligned}$$

where $\phi : U_1 \to (0,1) \times M$ is the homeomorphism given by lemma 1 on the first open set in the equivalence class. We showed previously that $M \subseteq \mathbf{N}$ containing

$$
\begin{tikzcd}
(0,1)\times M \arrow[rr, "I"] & & (0,1)\times N \\
U_i \arrow[u, "\phi_U"] \arrow[rr, "\phi_U\circ I\circ\phi_V^{-1}"] & & V_j \arrow[u, "\phi_V"]
\end{tikzcd}
$$

Figure 1: This little commutative diagram summarizes what we're doing. The identity homeomorphism in this diagram exists if $M = N$, and ϕ_U and ϕ_J exist by lemma 1.

consecutive numbers. We identify the image of ϕ with a non-negative integer using the function

$$\xi : \{(0,1)\} \times (\{\{1,2,3,\ldots,n\} : n \in \mathbf{Z}^+\} \cup \mathbf{Z}^+) \to \mathbf{Z}_{\geq \mathbf{0}}$$

which maps $((0,1),\{1,2,3,\ldots,n\}) \mapsto n$ and everything else to $((0,1),0)$. We can construct the inverse ξ^{-1} where $\xi^{-1}((0,1),0) = (0,1)\times\mathbf{Z}$ and $\xi^{-1}((0,1),n) = \{1,2,3,\ldots,n\}$ for all $n > 0$, so ξ is injective.

Let $\{U_1,U_2,\ldots\},\{V_1,V_2,\ldots\} \in \mathfrak{O}$ be equivalence classes; if $f(\{U_1,U_2,\ldots\}) = f(\{V_1,V_2,\ldots\})$ then for arbitrary U_i and V_j, the homeomorphisms $\phi_U : U_i \to (0,1)\times M$ and $\phi_V : V_j \to (0,1)\times N$ exist by lemma 1. We know that ξ applied to the images of ϕ_U and ϕ_V is equal, and so the injectivity of ξ implies that $M = N$. This means there is an identity function $I : (0,1)\times M \to (0,1)\times N$, which is a homeomorphism, to construct a homeomorphism $\phi_V \circ I \circ \phi_U^{-1}$. Therefore U_i V_j; since equivalence classes are disjoint, we have that $\{U_1,U_2,\ldots\} = \{V_1,V_2,\ldots\}$. This implies that $\mathfrak{O}$ is countable, so there are countbly many open subsets of $\mathbf{R}$ up to homeomorphism. $\square$

Appendix

Claim 3. *Any open interval $I \subseteq \mathbf{R}$ is homeomorphic to* $(0,1)$.

Proof. Let I be an arbitrary open interval, and define

$$\phi : I \to (0,1).$$

If I is bounded, then let ϕ be given by

$$x \mapsto \frac{x-a}{b-a}.$$

This is clearly injective, since for all $x, y \in I$ we have that $(x-a)(b-a)^{-1} = (y-a)(b-a)^{-1}$ implies $x = y$. The function is also surjective, since for all $z \in (0,1)$ we have that $f(x) = z$ for $x = z(b-a) + a \in (0,1)$.

If I is the ray (a, ∞) then let ϕ be given by

$$x \mapsto a + \tan(x).$$

We can construct the function $\phi^{-1}(x) = \arctan(x-a)$. The function arctan exists since $x \in (0,1)$. We have that $\phi(\phi^{-1}(x)) = a + \tan(\arctan(x-a)) = a + x - a$.

If I is the ray $(-\infty, a)$, then we can use the homeomorphism $x \mapsto -x$ and the argument for the ray (a, ∞).

If I is $\mathbf{R}$, then let ϕ be given by

$$x \mapsto \tan(x).$$

Since $x \in (0,1)$, we can use the inverse $\arctan : \mathbf{R} \to (0,1)$.

In all of the above cases, the function ϕ and its inverse are continuous since they're composed of continuous functions; this gives that ϕ is a homeomorphism. □