Annals of Mathematics Studies
Number 211

A Course on Surgery Theory

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PRINCETON UNIVERSITY PRESS
PRINCETON AND OXFORD
2021

Published by Princeton University Press,
41 William Street, Princeton, New Jersey 08540
6 Oxford Street, Woodstock, Oxfordshire OX20 1TR

press.princeton.edu

Library of Congress Control Number: 2020944360

ISBN 9780691160481
ISBN (pbk.) 9780691160498
ISBN (ebook) 9780691200354

British Library Cataloging-in-Publication Data is available

Editorial: Susannah Shoemaker and Kristen Hop
Production Editorial: Nathan Carr
Production: Brigid Ackerman
Copyeditor: Bhisham Bherwani

The publisher would like to acknowledge the authors of this volume for providing the camera-ready copy from which this book was printed.

This book has been composed in LaTeX.

Printed on acid-free paper. ∞

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

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Preface

Surgery theory is primarily the study of the classification of manifolds. It derives its name from its primary technique, the process called surgery, in which a manifold is modified by excising pieces of it and attaching other pieces in its place. Surgeries occur naturally in cobordisms by the Morse lemma: from the work of Thom, every oriented 3-manifold bounds an oriented 4-manifold, and so it follows that all 3-manifolds can be obtained by surgery on a link in the 3-sphere. Milnor called attention to this construction, and together with Kervaire used it as a powerful tool in the classification of differentiable structures on the n-sphere $\mathbb{S}^n$ for $n \geq 5$. Their work connected this particular classification problem to stable homotopy theory and to an understanding of symmetric and skew-symmetric quadratic forms over the integers $\mathbb{Z}$.

Browder and Novikov extended the work of Milnor and Kervaire to simply connected manifolds, and gave a simple characterization in dimension at least 5 of simply connected closed manifolds that are homotopy equivalent to finitely many others. Sullivan reformulated the theory in a very useful way and discovered that the homotopy theory is much simpler in the piecewise linear category. It is yet simpler in the topological setting due to the work of Kirby-Siebenmann, although Top surgery historically came much later. Wall extended the Browder-Novikov-Sullivan theory to non-simply connected manifolds with the relevant algebra packaged into groups $L_n(\mathbb{Z}[G])$, where $G = \pi_1(M)$ is the fundamental group of the manifold in question.

Browder's book [85] is an excellent introduction to the simply connected theory, while Wall's book [672] is the standard reference for the non-simply connected case. The high points of Wall's book are certainly the classification of manifolds homotopy equivalent to Lens spaces with odd-order fundamental group, and the classification of manifolds homotopy equivalent to the torus, following his own work and that of Hsiang and Shaneson. The former has significant algebraic input, relating to algebraic properties of units and quadratic forms in the integers of cyclotomic fields. One can easily spend decades learning and applying this vein of algebra and number theory, and these ideas are absolutely necessary for understanding manifolds with finite fundamental groups

The main ingredient of the latter classification problem is the Farrell fibering theorem, which describes the conditions for which a manifold is a fiber bundle over the circle. This theorem was interpreted as a calculation of L-groups by Shaneson. This story presages a dichotomy in which results about finite groups are the results of hard algebraic calculations, and results about torsion-free groups mainly stem from geometric considerations. For infinite groups with torsion, one must somehow combine these two ideas together.

This book is a modern course in surgery theory that builds on the insights of several post-Wall generations. Besides increasingly more difficult calculations over the years, there have been new applications and also new conceptual insights. In addition to a deeper understanding of surgery theory itself, the technique of surgery has been extended to homological surgery, local surgery, stratified surgery and many others. Among the main lessons within surgery are the role of functoriality that arises not only in the topological category but also in the context of homology manifolds; the use of controlled methods, i.e. of introducing quantitative estimates into the size of geometric constructions; and the intimate connection between topological surgery and index theory.

Our book is not an independent encyclopedic volume, but an introduction to these later insights. We tried and failed at being comprehensive, saying much less than we wanted about the Borel and Farrell-Jones conjectures, the connection to index theory, the link between 19th century number theory and 20th century quadratic form theory, knot and link theory, stratified spaces, and group actions, among others. Our hope is that, by reluctantly surrendering the goal of comprehensiveness, we have produced a book that will help those who want to gain access to the literature. In particular, we emphasize that there are by now a number of other treatments after the original ones by Browder and Wall, many of which provide points of view that are complementary to ours. We hope that our deficits and theirs occur in different spots, and that by making different choices, the readers can gain a unified picture of the entire subject. Surgery theory is an astonishingly deep and successful mathematical discipline, and we hope to introduce it to a new generation of admirers.

The authors wish to thank the Mathematical Sciences Research Institute, where they were in residence during the fall semester of 2011 for a term-long workshop on quantitative geometry. The first author would like to express his gratitude to DJ Hatfield, Philip Hirschhorn, Hailiang Hu, Andrew Schultz, Hong-Ha Truong, Ismar Volić, and Min Yan for their help during the production of this text, as well as the Brachman-Hoffman fellowship and Jaan Whitehead for their support. The second author would like to thank his thesis advisor, Sylvain Cappell, who first taught him surgery theory in the early 1980s, when the subject seemed much more complex than it does now, and also his many collaborators and students over the years with whom he learned so much of the material explained in this book. The first author would like to thank the second author for the same.

Introduction

The purpose of this book is to explain some of the ideas, techniques, successes, and methods of surgery theory. Surgery is the major tool, at least in high dimensions, for the classification of manifolds. These manifolds come in many types: smooth (Diff) manifolds, piecewise linear (PL) manifolds, and topological (Top) manifolds. In many ways, smooth surgery is the simplest of all the different manifold categories; it was developed first by Kervaire-Milnor [355], Browder [85], and Novikov [484].

Smooth manifolds arise throughout mathematics. They have a well-known and basic theory of tangent bundles. Their bundle theory is understood by maps to Grassmanians, and therefore all questions about bundles are either (a) unstable and related to the topology of Lie groups, or (b) stable and part of K-theory. We frequently understand bundles through characteristic classes. For our purposes, the most important characteristic classes will be the Pontrjagin classes $p_i(\xi) \in H^{4i}(X; \mathbb{Z})$ of oriented vector bundles.

Because of Sard's theorem in the case of smooth manifolds, there is a deep and vital theory of transversality. The notion of transversality gives rise to cobordism theory, as established by Thom [643] (see Stong [621]). As pointed out by Conner and Floyd, bordism theory is a homology theory, and therefore there are powerful tools available such as the Atiyah-Hirzebruch spectral sequence.

The study of critical values, i.e. those points to which a particular $f : M \to \mathbb{R}$ is not transverse, is given by Morse theory, and generic maps f give rise to handlebody decompositions of M. See Milnor [453]. Morse theory ultimately leads to Smale's h-cobordism theorem in the simply connected case, and in the non-simply connected case by Barden, Mazur, and Stallings. These cobordism theorems are essentially involved in almost all proofs of isomorphisms between manifolds, and were immediately used for proofs of the high-dimensional Poincaré conjecture.

The PL category was studied next. While less familiar, it is formally a close cousin of the smooth category. For manifolds with a given triangulation, the regular neighborhoods of simplices provide a natural handlebody structure. Although bundle theory is mostly irrelevant to understanding these manifolds, a good substitute can be found in block bundles. One can also develop an appropriate theory of transversality that is based on linear algebra rather than Sard's theorem. Therefore, in at least one sense, the theory of these manifolds is definitively more elementary than in the smooth case. Two references are Rourke-Sanderson [557–559, 563, 563] and Hudson [325].

The use of block bundles implies that the relevant theory to understand the tangent bundle of a PL manifold is more complicated. Unlike BO with its 8-fold Bott periodicity, the classifying space BPL for PL bundles is much more complicated, and its homotopy groups are coker $(J) \times \mathbb{Z}$ at odd primes every fourth dimension, where J is the Adams J-homomorphism. The term coker (J) is the mysterious part of stable homotopy theory.

Despite this complication in the PL category, it is surprising that surgery on PL manifolds is actually computationally much simpler than on Diff manifolds. In surgery theory of PL manifolds within a fixed homotopy type, the tangent bundles are not arbitrary. Atiyah observed that homotopy equivalent manifolds have fiber homotopy equivalent stable normal bundles. Sullivan emphasized that the collection of degree one normal maps is classified by the classifying space F/PL, where F is the classifying space for spherical fibrations. In fact, the homotopy groups of F are the stable homotopy groups of spheres. A quick look at the homotopy groups for F and PL suggests that we are in a situation of double incomprehensibility but in the most extraordinary way the quotient is completely comprehensible, as if the incomprehensible parts cancel each other out.

A decade or so after these developments, Kirby, assisted by Siebenmann, realized that a good understanding of PL homotopy tori is sufficient to enable a theory of topological manifolds. By making use of Farrell's theory of manifolds that fiber over the circle, itself a cousin of the h-cobordism theorem, Hsiang-Shaneson and Wall quickly handled major parts of the surgery program. The book by Kirby and Siebenmann [361] describes the ensuing theory. Once again there is a bundle theory, for which F/Top is discovered to be slightly cleaner in form than F/PL. In fact, Sullivan describes the homotopy type of F/Top elegantly as the product of Eilenberg-Maclane spaces using localization theory:

$$F/Top_{(2)} \simeq K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_{(2)}, 4) \times K(\mathbb{Z}_2, 6) \times K(\mathbb{Z}_{(2)}, 8) \times \cdots,$$

while $F/Top[{}^1\!/\!{}_2] \simeq BO[{}^1\!/\!{}_2]$, so that $F/Top \simeq \mathbb{Z} \times \Omega^4(F/Top)$. The applicability of Top transversality in all dimensions was only completed by Quinn [526] after Freedman's proof of the four-dimensional Poincaré conjecture. An appropriate handlebody theory was then possible.

Surgery theory demonstrates a periodicity (see Siebenmann's essay in [361]), with a slight imperfection noticed by Nicas [482]. Functoriality in surgery theory was first suggested by Quinn [517] and implemented by Ranicki [536]. The set $\mathcal{S}^{Top}(M)$ of manifold structures on a manifold M becomes part of a sequence of homotopy functors, and their study can now be viewed as the topic of surgery theory, even when divorced from the original geometric context. This manifold structure set is the difference between the theory of local and global quadratic forms. It is an analogue of the Shaferevich group in number theory. One can even move one step farther. Beyond the theory of topological manifolds is the theory of homology manifolds, which exhibits unbroken periodicity and better functoriality, i.e. with respect to more maps. These spaces are studied by Bryant-Ferry-Mio-Weinberger [104], and complete the classification.

Among our goals in this course is to explain the above description of the theory of manifolds. In fact, it can almost all be described by means of one exact sequence:

$$\cdots \to L^w_{n+1}(\mathbb{Z}[G]) \to S^{Top}_n(X,w) \to H_n(X;\mathbb{L}^{\langle 1\rangle}_\bullet(w)) \to L^w_n(\mathbb{Z}[G])$$

where w denotes a map from $\pi = \pi_1(X) \to \mathbb{Z}_2$ associated to orientation, and $\mathbb{L}^{\langle 1\rangle}_\bullet(w)$ is a spectrum related to quadratic form theory over $\mathbb{Z}$. The L-groups are related to Witt groups of w-symmetric quadratic forms over the group ring $\mathbb{Z}[G]$. The symbol $\mathbb{L}^{\langle 1\rangle}_\bullet(w)$ indicates some kind of twisting associated to the orientation on X. The sequence is exceedingly flexible, and can algebraicized and generalized in many different ways. In particular, there is an extension to stratified spaces, where the analogue of $\mathbb{L}^{\langle 1\rangle}_\bullet(w)$ reflects the local structure, and the orientation w twists the object in much more complicated ways, e.g. measured by monodromies.

Despite the evident conciseness of the exact sequence, a surgeon needs to know more in practice. The groups $L_*(\mathbb{Z}[G], w)$, which depend only on the mod 4 dimension of the manifold, its orientation character, and its fundamental group, are notoriously difficult to compute. The L-groups are algebraically defined and can be studied completely algebraically. This study is particularly successful for finite groups, where the problems ultimately relate to issues about representation theory and quadratic forms over rings. While there is no simple general algorithm to compute L-groups for finite groups, there are good methods.

A classical theorem of Browder and Novikov states that the smooth structures for simply connected manifolds is detected up to finite ambiguity by Pontrjagin classes. However, for manifolds with finite fundamental group, there is an additional invariant called the ρ-invariant, a cousin of the Atiyah-Patodi-Singer η-invariant for the signature operator, that enters the analogous classification theorem.

For torsion-free groups, the most beautiful answer is predicted by the Borel conjecture. It asserts that aspherical manifolds M, i.e. those with contractible universal covers, are topologically rigid. Using some functoriality, we obtain an excellent conjectural view of these L-groups, and can develop machinery to extend from aspherical to torsion-free.

General groups are a mixture of these cases. One could combine these two cases by considering orbifolds and stratified spaces, but unfortunately because of a phenomenon called UNil, discovered and developed by Cappell, this idea is incorrect at the prime 2. The apparently correct statement is the Farrell-Jones conjecture that mixes torsion-free and virtually cyclic groups together in an amazing way. The interplay is very natural from the point of view of dynamics, wherein the subgroups of the groups in question are associated to periodic orbits of flows. Our discussion of the Farrell-Jones conjecture in Section 6.8 gives a quick introduction. Lück is currently writing a book covering this subject, and we refer our readers there.

Throughout the book we give many examples that either illustrate these computational techniques or give partial evidence for these conjectures. Of course, just as number theory does not end with the classification of the integers, topology does not end with classification. Frequently one tries to construct some kind of deeper structure on a manifold, and instead one constructs it on something else. Surgery theory will then intervene to show that one actually succeeded in building the structure on the original manifold in question. We will therefore also use the theory for the purposes of constructing embeddings, group actions, and homeomorphisms.

The ideas of surgery theory can be extended to many other theories where the basic ideas apply, but do not have all of the same structure that manifold classification up to homeomorphism does. These theories include the classification up to homology h-cobordism and even aspects of the classification of manifolds of positive scalar curvature. We discuss some of these topics in the last sections of the book.

0.0.1 A summary of chapters

Chapter 1 begins with a review of surgery and describes results that involve no difficult calculations. Many of these results are consequences of the π-π theorem, which describes the situation in which surgery is unobstructed. Besides explicit geometric theorems, such as the Browder-Wall splitting theorem and extension across homology collars, we also give Wall's geometric definition of L-groups and Ranicki's analogue, the algebraic theory of surgery. These theories presage the ideas of spacification in Chapter 4.

Chapter 2 is devoted to the determination of L-groups. We explain some of the mathematics that surround the calculation of L-groups, and see connections to Witt theory, algebraic K-theory and representation theory. The connection to representation theory motivates Dress induction, a key tool in both the space form problem in Chapter 6 and the Borel conjecture for flat manifolds in Chapter 7. We also explain Shaneson's thesis which shows that the Farrell fibering theorem is equivalent to a calculation of L-groups.

Chapter 3 is the other ingredient in the analysis of the surgery exact sequence: the homotopy type of the relevant classifying spaces. The homotopy types of F/PL and F/Top are completely determined; the most important part is the calculation of the set of PL homotopy tori and the Borel conjecture for $\mathbb{Z}^n$, i.e. that any homotopy torus is homeomorphic to the standard torus. We explain the Kirby-Siebenmann triangulation obstruction, and also to show that the most naive noncompact analogue of the Borel conjecture is systematically false.

Chapter 4 is devoted to the reformulation of surgery theory in terms of spaces which in the topological case are close to their own 4-fold loopspace. The topological structure sets can be endowed with a group structure; we show however that a group structure is impossible in the smooth case. The topological structure theory is also the theory of the assembly map, whose importance to the subject is hard to exaggerate. Chapter 5 describes how the assembly map can be exploited to give results. We also give interpretations of the Novikov and Borel conjectures, and enable a connection to index theory.

In Chapter 6, we perform some interesting classifications. For finite fundamental groups, excellent examples are provided by the space form problem, the classification of homotopy lens spaces, and homology propagation. The most important example for infinite fundamental group arising here is Cappell's work on connected sums of projective spaces, later completed through the work of Connolly-Davis, Banagl-Ranicki, and Brookman-Davis-Khan, which gives to this classification theory an extra level of complexity that does not occur in the index-theoretic setting. It provides the platform to discuss the Farrell-Jones conjecture, which is the correct generalization of the Borel conjecture to groups with torsion, or to group rings $R\Gamma$ where R is not regular, even if Γ is torsion-free.

Proofs of the Borel or Farrell-Jones conjecture are nowadays almost always done geometrically. Chapter 7 explains the work of Chapman-Ferry on the α-approximation theorem, which gives a metric condition for a map to be near a homeomorphism, and the work of Farrell-Hsiang that proves the Borel conjecture for a class of manifolds, including flat manifolds for which controlled topology handles the work that algebraic ideas like Dress induction and the theory of Laurent extensions cannot.

We close the body of the book in Chapter 8 with an overview of various extensions of surgery into other directions. We hope that this part sparks the imagination of the readers and encourages them into examine many potential applications of surgery beyond the classification of manifolds. Among the topics discussed are homological surgery, controlled surgery, homology manifolds, and stratified spaces.

The appendices review some topics in algebraic and geometric topology that appear throughout the book, and also provide a summary of some of the conclusions of Kirby-Siebenmann theory. The latter establishes the basic geometric theorems about topological manifolds that are (1) proved using the PL results of surgery, and then (2) used for establishing topological surgery. Interestingly, they form the basis for the study of homology manifolds and stratified spaces. The former theory lacks transversality, but exhibits more functoriality than in Top theory, while the latter lacks the analogues of PL geometric topology.

Notation

We adopt some notation for the book.

1. The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of integers, rationals, real, and complex numbers, respectively. We will use $\mathbb{Z}_n$ to mean the cyclic group of order n. The notation $\mathbb{Z}/n\mathbb{Z}$ is never used in the book. We use the symbol $\mathbb{Z}_{\geq 0}$ to mean all nonnegative integers. The notation $\mathbb{Q}_{>0}$ and $\mathbb{R}_{\geq 1}$ should be clear.
2. Other symbols used in blackboard bold are a general field $\mathbb{F}$, the L-spectrum $\mathbb{L}_\bullet$, a generic spectrum $\mathbb{E}_\bullet$, the sphere $\mathbb{S}^n$, the disk $\mathbb{D}^n$, and the torus $\mathbb{T}^n$. We also use $\mathbb{S}^{Cat}$ and $\mathbb{N}^{Cat}$ to refer to the block structure set and the blocked set of normal invariants. The term *spacification* will be used in this context for reasons that will be clearer below.[1]
3. The monoid of self-homotopy equivalences of the n-sphere is often given by G_n in the literature, but we will instead use F_n in this text. Therefore BF_n denotes this classifying space and F/Cat denotes the fiber of the map $BCat \to BF$. The symbol G is used for a generic group, as well as the symbols π and Γ.
4. If A and B are topological spaces, we use the notation $[A : B]$ to denote the collection of homotopy classes of maps from A to B. The triple (W, M, M') denotes a manifold M with boundary components M and M'; i.e. $\partial W = M \coprod M'$. The pair (X, Y) denotes that Y is the boundary of X or some particular subset of X. The symbol $\mathring{M}$ is the space obtained by deleting a single point from M.
5. If G is a group, then $\mathbb{Z}[G]$ denotes the integer group ring and $\mathbb{R}[G]$ denotes the real group ring.
6. In the literature the symbol $\mathcal{S}^{Cat}(M, \partial M)$ is used at times to mean the manifold structures on M relative to boundary, and at other times not relative to boundary. In this text we use it to mean *not* relative to the boundary. The symbol $\mathcal{S}^{Cat}(M)$ or $\mathcal{S}^{Cat}(M)_{\text{rel}}$ will mean the set of manifold structures on M relative to the boundary.
7. As is customary we use $\times$ to mean the product of two topological spaces or two groups, # to mean the connected sum of two topological manifolds, $\oplus$ to mean the Whitney sum of bundles or fibrations, $\vee$ to mean the one-point union of two spaces, and $*$ to mean the join of spaces, the free product of groups, or the concatenation of paths.
8. The term $L_n^h(\mathbb{Z}[G])$ or the undecorated $L_n(\mathbb{Z}[G])$ indicates the L-groups used for the purpose of determining the structure set up to homotopy equivalence. The decorated $L_n^s(\mathbb{Z}[G])$ is related to simple homotopy equivalences, and $L_n^p(\mathbb{Z}[G])$ denotes the projective L-groups.
9. For the classifying space for real n-dimensional vector bundles, we write BO_n and BO instead of $BDiff_n$ and $BDiff$. We use the notations F/O for homoge-

[1] Not to be confused with the sphere spectrum $\mathbb{S}^0$.

neous classifying spaces and Ω^O (and Ω^{SO}) for smooth (oriented) cobordism. However, we write $\mathcal{S}^{Diff}(M)$ and $\mathcal{N}^{Diff}(M)$ for the structure set and normal invariants set in the Diff category. There should be no confusion within context.

10. For L-groups, we will usually write $L_*(\mathbb{Z}[\pi])$ instead of $L_*(\pi)$. However, when the expression for π is longer, we will omit the $\mathbb{Z}$ from the notation. Therefore we have $L_*(\pi_1(M))$ or $L_*(\pi_1(M) \to \pi_1(N))$.

Chapter One

The characterization of homotopy types

1.1 A REVIEW OF SURGERY

Surgery theory arose from Milnor's discovery that the 7-sphere has more than one differential structure. Kervaire and Milnor's development of surgery, together with Smale's h-cobordism theorem, classified the smooth structures on the sphere. Surgery was invaluable in work that clarified the relationships between the geometric categories of Diff, PL, and Top, and takes a somewhat different form in each of them. This chapter will discuss fundamental ideas and constructions, aspects common to all three categories, that do not depend on any difficult calculations.

Fairly recent treatments for the basic ideas and constructions in surgery theory can be found in Lück [405] and Ranicki [544]. In this opening section, we will restate the major theorems that join together to produce the surgery exact sequence. Some background material about bundles, bordism, and the PL and Top categories can be found in the appendices.

1.1.1 Geometric Poincaré complexes

The decisive step toward surgery theory was the development of simply connected surgery for Diff manifolds by Browder and Novikov. In this section we state without proof the highlights of the surgery program for an arbitrary fundamental group. The primary goal of surgery theory is to understand the conditions under which there is a homotopy equivalence $M \to X$, where M is some Cat manifold and X is a CW complex. Clearly a necessary condition for the existence of such a map is that the target space X satisfy Poincaré duality.

Let A be an abelian group and let $w\colon \pi \to \mathrm{Aut}\,(A)$ be a representation of π. With the action $g.a = w(g)(a)$, the representation endows A with the structure of a left $\mathbb{Z}[\pi]$-module. We write $\mathbb{Z}^w$ for the $\mathbb{Z}[\pi]$-module whose underlying abelian group is infinite cyclic and we write A^w for the $\mathbb{Z}[\pi]$-module $A \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}^w$.

If X is a path-connected and locally path-connected space with fundamental group π, denote by $S_*(\widetilde{X})$ the singular complex of the universal cover. Form the tensor product $S_*(X; A) = S_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} A$. The homology of this chain complex is denoted

$H_*(X; A^w)$, often called the *homology of X twisted by* $w\colon \pi \to \mathrm{Aut}(A)$. Oftentimes the w is dropped from the notation. Similarly the cohomology group $H^*(X; A^w)$ can be constructed by forming the cochain complex $S^*(X; A) = \mathrm{Hom}_{\mathbb{Z}[\pi]}(S_*(\widetilde{X}), A)$ and defining $H^i(X; A^w) = H_i(\mathrm{Hom}_{\mathbb{Z}[\pi]}(S_*(\widetilde{X}), A))$.

Definition 1.1. *An n-dimensional (geometric) Poincaré complex* $(X, w, [X])$ *is a space X which has the homotopy type of a finite CW complex, together with a homomorphism* $w\colon \pi_1(X) \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$ *called the orientation character, and a generator* $[X]$ *in* $H_n(X; \mathbb{Z}^w) \cong \mathbb{Z}$ *called the fundamental class such that the cap product maps*

$$\cap [X]\colon H^*(X; \mathbb{Z}^w) \to H_{n-*}(X; \mathbb{Z}),$$

$$\cap [X]\colon H^*(X; \mathbb{Z}) \to H_{n-*}(X; \mathbb{Z}^w)$$

are isomorphisms. The point of w is that we can consider both orientable and nonorientable spaces. A similar definition can be given for Poincaré pairs.

Remark 1.2. *To perform surgery up to simple homotopy equivalence requires a more refined notion, that of a simple Poincaré complex. See Chapter 2 for a discussion.*

Remark 1.3. *A smooth oriented n-manifold M has the structure of a Poincaré complex since M has a triangulation, and the fundamental class of M is the cycle given by the sum of all n-dimensional simplices suitably oriented. For a closed Top manifold, Poincaré duality is standard. See for example Milnor-Stasheff [461] or Spanier [604]. On the other hand, finiteness is a consequence of West [699] and Kirby-Siebenmann [361].*

1.1.2 Normal data

To decide whether or not a Poincaré complex X is homotopy equivalent to a Cat m-manifold, one must determine, as an initial approximation, whether or not there is a degree one Cat normal map $(f, b)\colon M \to X$ from a Cat m-manifold M to X. Such a map can be used to decide whether there is any potential Cat bundle that could serve as the stable normal bundle to a manifold homotopy equivalent to X. Candidates are called *Cat normal invariants* for X, and exist iff, for the stable normal spherical fibration $\nu_X : X \to BF$, there is a Cat bundle reduction $\widetilde{\nu}_X : X \to BCat$. We explain these ideas in this section, beginning with those associated with spherical fibrations. See Appendix A.2.

Definition 1.4.

1. *The orientation character of a* $(k-1)$*-spherical fibration* $\alpha\colon X \to BF_k$ *over a finite CW complex X is the first Stiefel-Whitney class* $w_1(\alpha) \in H^1(X; \mathbb{Z}_2)$ *considered as a group homomorphism:*

$$w_1(\alpha)\colon \pi_1(X) \to \mathrm{Aut}(\mathbb{Z}) = \mathbb{Z}_2.$$

2. *The Thom space $T(\alpha)$ of the $(k-1)$-spherical fibration $\alpha\colon X \to BF_k$ is the mapping cone of the map $S(\alpha) \to X$ from the total space $S(\alpha)$.*

We distinguish one particular spherical fibration which is of importance in the surgery program.

Definition 1.5. *A k-dimensional Spivak normal structure (α, ρ) for an m-dimensional Poincaré complex X is a $(k-1)$-spherical fibration $\alpha\colon X \to BF_k$ together with a map $\rho\colon \mathbb{S}^{m+k} \to T(\alpha)$ to the Thom space $T(\alpha)$ such that the following compatibility conditions hold:*

1. *the orientation character of X coincides with the orientation character of the spherical fibration α as maps $\pi_1(X) \to \mathbb{Z}_2$;*
2. *if $h_*\colon \pi^S_{m+k}(T(\alpha)) \to \widetilde{H}_{m+k}(T(\alpha);\mathbb{Z})$ is the Hurewicz map and $[\alpha]_{Th}$ is the Thom class of α, then $[\alpha]_{Th} \cap h_*(\rho) = [X]$ in $H_m(X;\mathbb{Z}^w)$. Recall that capping with the Thom class gives the Thom isomorphism $\widetilde{H}_{m+k}(T(\alpha);\mathbb{Z}) \to H_m(X;\mathbb{Z}^w)$.*

The homology groups should be taken with coefficients in $\mathbb{Z}^w$ if the spaces are nonorientable. The map ρ is often called the Spivak class and the stable map $\alpha\colon X \to BF$ is called the Spivak normal fibration.

We spend a moment to discuss the Spivak fibration associated to a finite Poincaré complex X. If M is a smooth manifold, we define its stable normal bundle by embedding M into a Euclidean space of a very high dimension ($\geq 2\dim(M)+4$), which is unique up to isotopy, and we therefore obtain an intrinsically definable normal bundle. The unit disk bundle of this vector bundle is isomorphic to a small tubular neighborhood of M. Now we consider a finite Poincaré complex. If X is finitely dominated, we can replace X by $X \times \mathbb{S}^1$. We take a PL embedding of X in a high-dimensional Euclidean space and consider a regular neighborhood. The PL homeomorphism type of this neighborhood does not change if one does elementary expansions and collapses to X that embed in the Euclidean space, i.e. if they are below half its dimension. The neighborhood of X is an invariant of the simple homotopy type of X. After crossing with a circle, if needed, we can actually obtain a homotopy invariant.

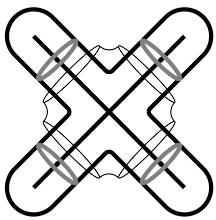


Figure 1.1: The space X with a regular neighborhood

The map of the boundary of the regular neighborhood to X has a spherical homotopy fiber of the dimension predicted by the Poincaré duality dimension of X. This stable spherical fibration is called the *Spivak fibration.* Atiyah had proved earlier that the normal bundles of homotopy equivalent manifolds are equivalent as spherical fibrations; Spivak's construction explains Atiyah's theorem. Spivak also characterized the fibration as the unique stable spherical fibration whose Thom space, i.e. the mapping cone of the projection, has spherical top-dimensional homology class.

We shall see that, for a Poincaré complex, the existence of a normal invariant is essentially equivalent to the existence of a Cat sphere bundle reduction of the Spivak fibration, and that different such reductions correspond to normal cobordism classes of degree one normal invariants.

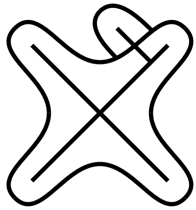


Figure 1.2: The space X after an expansion

If X were a manifold, then the stable normal bundle has the following property: the collapse map from an embedding of X in the sphere to the Thom space of the normal bundle would be the map ρ. If we consider a complex that is homotopy equivalent to the manifold in such a neighborhood, the neighborhood is no longer a bundle neighborhood, but the neighborhood deform retracts to the complex. The restriction to the boundary is still a spherical fibration. Its dimension is determined by the dimension of the Poincaré duality, not the geometric dimension of the complex. These properties hold for all Poincaré complexes and indeed characterize them as asserted in the following theorem of Spivak [605].

Theorem 1.6.

1. *A finite CW complex X is an m-dimensional Poincaré complex iff, for all k and for any closed Cat regular neighborhood $(Y_X, \partial Y_X)$ of a Cat embedding $X \hookrightarrow \mathbb{S}^{m+k}$, the mapping fiber of the inclusion $\partial Y_X \to Y_X$ is a homotopy $(k-1)$-sphere.*
2. *For an m-dimensional Poincaré complex X, a Cat embedding $X \hookrightarrow \mathbb{S}^{m+k}$ with closed Cat regular neighborhood $(Y_X, \partial Y_X)$ determines a k-dimensional Spivak normal structure (α, ρ) with $\mathbb{S}^{k-1} \to S(\alpha) = \partial Y_X \to Y_X \simeq X$ for which $\rho : \mathbb{S}^{m+k} \to Y_X/\partial Y_X = T(\alpha)$ is given by collapse. The image of the fundamental class $[\mathbb{S}^{m+k}] = 1 \in H_{m+k}(\mathbb{S}^{m+k}) \cong \mathbb{Z}$ under the projection ρ is the fundamental class of $(Y_X, \partial Y_X)$ in $\widetilde{H}_{m+k}(Y_X/\partial Y_X) \equiv H_{m+k}(Y_X, \partial Y_X)$.*

The notion of a Spivak normal fibration for a Poincaré complex is completely analogous to the idea of a stable Cat normal bundle. Most familiar are smooth normal bundles, but analogous structures exist in the PL and Top categories. See Haefliger-Wall [281] for PL and Hirsch [311] or Kuiper-Lashof [377] for Top.

Theorem 1.7. *Any two Spivak normal fibrations* (ν, ρ) *and* (ν', ρ') *on an m-dimensional geometric Poincaré complex X are related by a stable fiber homotopy equivalence* $c : \nu \to \nu'$.

Proof. The PL homeomorphism type of a neighborhood does not change if one does elementary expansions and collapses to X that embed in the Euclidean space, i.e. if they are below half its dimension. So the neighborhood of X is an invariant of the simple homotopy type of X and, after crossing with a circle if needed, we can obtain a homotopy invariant. □

When a stable Cat normal bundle is assigned to a Poincaré complex, we have a Cat normal invariant, which we now define.

Definition 1.8. *Let X be an m-dimensional Poincaré complex.*

1. *Suppose we have a Cat bundle* $\eta : X \to BCat_k$ *with orientation character* $w_1(\eta) = w(X) \in H^1(X; \mathbb{Z}_2)$ *and a map* $\rho : \mathbb{S}^{m+k} \to T(\eta)$. *Let*
$$h_* : \pi^S_{m+k}(T(\eta)) \to \widetilde{H}_{m+k}(T(\eta); \mathbb{Z})$$
be the Hurewicz map and U_η *be the Thom class of* η. *Then* (η, ρ) *is a k-dimensional Cat normal invariant for X if* $U_\eta \cap h_*(\rho) = [X] \in H_m(X; \mathbb{Z}^{w(X)})$.
2. *Consider j- and k-dimensional Cat normal invariants* (η_1, ρ_1) *and* (η_2, ρ_2) *of X. They are equivalent if there is a stable bundle isomorphism* $c : \eta_1 \to \eta_2$ *such that the induced map* $T(c)_* : \pi^S_{m+j}(T(\eta_1)) \to \pi^S_{m+k}(T(\eta_2))$ *sends* ρ_1 *to* ρ_2.
3. *The Cat normal structure set* $\mathcal{N}^{Cat}(X)$ *of X is the set of equivalence classes of Cat normal invariants on X.*

In the best of circumstances, we would like the existence of a Cat normal invariant for X to generate for us a Cat manifold M and a homotopy equivalence $M \to X$. Normal invariance, however, is too weak a condition to accomplish the entire job at hand. Instead, it allows us to construct a map $M \to X$ with compatible normal data.

Definition 1.9. *The degree of a map* $f : M \to X$ *of connected m-dimensional Poincaré complexes is the integer* $\deg(f) \in \mathbb{Z}$ *such that*
$$f_*[M] = \deg(f)[X] \in H_m(X; \mathbb{Z}^w) \cong \mathbb{Z}.$$

Definition 1.10. *Let M be a Cat m-manifold (possibly with boundary) with a stable Cat normal bundle* $\nu_M : M \to BCat$. *In addition let X be a CW complex with a stable*

Cat bundle $\eta\colon X \to BCat$.

1. *An m-dimensional Cat normal map* $(f,b)\colon (M,\nu_M) \to (X,\eta)$ *is a degree one map* $f\colon M \to X$ *equipped with a stable pullback Cat bundle map* $b\colon \nu_M \to \eta$ *covering* f. *In other words, there is a stable trivialization of* $\tau_M \oplus f^*\eta$.
2. *A Cat normal bordism between two m-dimensional Cat normal maps given by* $(f,b)\colon (M,\nu_M) \to (X,\eta)$ *and* $(f',b')\colon (M',\nu_{M'}) \to (X,\eta)$ *is an* $(m+1)$*-dimensional Cat normal map*

$$((F,B),(f,b),(f',b'))\colon (W,M,M') \to X \times (I,\{0\},\{1\})$$

from a cobordism (W,M,M') *to the cylinder* $X \times (I,\{0\},\{1\})$. *In this case, we say that* f *and* f' *are normally bordant and that* M *and* M' *are normally cobordant.*
3. *The Cat normal map* $(f,b)\colon (M,\nu_M) \to (X,\eta)$ *is a Cat normal homotopy equivalence if* $f\colon M \to X$ *is a homotopy equivalence. Note that a Cat normal homotopy equivalence has degree one.*

We will see that normal cobordism classes of degree one normal maps are in bijective correspondence with normal invariants.

If X is a Poincaré space, we can define the *Cat reduction* of a stable spherical fibration $X \to BF$ as a lift $X \to BCat$ fitting into the commutative diagram

$$\begin{array}{ccc} & & BCat \\ & \nearrow & \downarrow \\ X & \longrightarrow & BF \end{array}$$

The relation between Cat normal invariants, degree one Cat normal maps, and Cat reductions is given in the following normal invariants theorem due to Sullivan.

Theorem 1.11. *(Rourke-Sullivan [564]) Consider an m-dimensional Poincaré complex* X. *Let* $\alpha\colon X \to BF$ *be the induced Spivak normal fibration. The following conditions are equivalent:*

1. *The Cat normal structure set* $\mathcal{N}^{Cat}(X)$ *is nonempty.*
2. *There exists a degree one Cat normal map* $(f,b)\colon M \to X$.
3. *The Spivak normal fibration* $\alpha\colon X \to BF$ *admits a Cat bundle reduction* $\eta\colon X \to BCat$.
4. *The composition* $X \xrightarrow{\alpha} BF \to B(F/Cat)$ *is null-homotopic.*

In this case, we say that X *admits a Cat normal invariant*.

Theorem 1.12. *Let X be a Poincaré complex. The set $[X : F/Cat]$ of homotopy classes of maps from X to the classifying space F/Cat is in bijective correspondence with the set of normal invariants of X, if the set is nonempty.*

Theorem 1.13. *Let X be an n-dimensional Poincaré complex. The following two sets are in bijective correspondence:*

1. *the Cat normal structure set $\mathcal{N}^{Cat}(X)$,*
2. *the set of all degree one Cat normal maps $M \to X$ from n-dimensional Cat manifolds M to X up to Cat normal cobordism.*

Proof. In light of the previous theorem, we only need to give the equivalence of (1) and (2). Let ξ^k be a high-dimensional Cat bundle over X^n. Identify ξ^k with a neighborhood N of X in $\mathbb{S}^{n+k}$ as a homotopy type, giving a map $\mathbb{S}^{n+k} \xrightarrow{\alpha} \mathbb{S}^{n+k}/(\mathbb{S}^{n+k}\backslash N) \simeq T(\xi^k)$, i.e. $\alpha \in \pi_{n+k}(T(\xi^k))$. There is a map

$$\mathbb{S}^{n+k} \xrightarrow{\beta} \mathbb{S}^{n+k}/(\mathbb{S}^{n+k}\backslash N_\xi) \simeq T(\xi^k)$$

that can be made transverse to X^n. By transversality, the inverse image $M^n = \beta^{-1}(X^n)$ is a Cat n-manifold, and the restriction of β to a tubular neighborhood ν_M^k of M in $\mathbb{S}^{n+k}$ gives a bundle map $b\colon \nu_M^k \to \xi^k$. Therefore we have constructed a degree one normal map from a lift.

Conversely, let $f : M^n \to X^n$ be a degree one normal map covered by a bundle map $b\colon \nu_M \to \xi$. There is an induced map $Jb\colon J\nu_M \to J\xi$. Let $\eta : X \to BG$ be the spherical fibration over X given by the image of $J\nu_M$ under Jb. Then $b(\nu_M)\colon X \to BO$ is the reduction of this spherical fibration η. Embed M in a large sphere $\mathbb{S}^{n+k}$ with regular neighborhood N_M. The composition $\mathbb{S}^{n+k} \to \mathbb{S}^{n+k}/(\mathbb{S}^{n+k}\backslash N_M) \simeq T(\nu_M) \to T(\zeta)$ gives an element α in $\pi_{n+k}(T(\zeta))$, which is seen to have the correct properties. One can also check that all of these constructions in either direction are well defined up to equivalence, i.e. normal bordism. □

Remark 1.14. *The reader should be familiar with smooth transversality. See the appendices for the other categories.*

Remark 1.15. *The purpose of the bundle data in surgery theory is as follows. Let M^n be a Poincaré complex and N^n be a Cat manifold. If M and N are Cat isomorphic, their normal (tangential) data should be the same. If $(f, b)\colon N^n \to M^n$ is a degree one Cat normal map, we then have a diagram*

$$\begin{array}{ccc} \nu_N & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ N & \xrightarrow[f]{} & M \end{array}$$

where ν_N is the stable Cat normal bundle of N in some large $\mathbb{R}^n$ and $\xi = (f^{-1})^*(\nu_N)$. Besides ξ, there is another stable Cat bundle over M, i.e. the stable normal bundle ν_M in some large $\mathbb{R}^m$. By the uniqueness theorem of Spivak [605], there is only one spherical fibration over the Poincaré duality space M up to stable fiber homotopy equivalence. Hence we can conclude that ξ and ν are stably fiber homotopy equivalent. In other words, the difference $\xi - \nu_M$ is stably fiber homotopically trivial. Such trivial bundles over M are in bijection with $[M : F/Cat]$. Since we are using stable bundles, we can use normal bundles to obtain equivalent expressions of the above result in terms of tangent bundles. So the role of the normal invariants is to characterize the obstruction to a Cat isomorphism in terms of tangential data. Technically, the normal information is used as part of the surgery process.

Remark 1.16. *A reduction is not merely a bundle in $[X : BO]$ but a bundle with a fiber homotopy equivalence to the Spivak fibration. The difference between any two of them is a bundle together with a fiber trivialization. The equivalence is needed to construct a normal invariant.*

Remark 1.17.

1. *The isomorphism class of the stable Cat normal bundle $\nu_M : M \to BCat$ of an m-manifold is a Cat isomorphism invariant but not a homotopy invariant.*
2. *If $g : N^m \to M^m$ is a homotopy equivalence of m-manifolds, then the stable bundles ν_N and $g^*\nu_M : N \to BCat$ are fiber homotopy equivalent. This theorem is due to Atiyah [19].*
3. *In order for a homotopy equivalence of manifolds to be homotopic to a Cat isomorphism, it must preserve the Cat normal bundles. Homotopic maps pull back to isomorphic Cat bundles.*

1.1.3 The structure set

The purpose of the surgery exact sequence for an m-dimensional Poincaré complex X is to determine the equivalence classes of Cat manifold structures of X. We define this notion in this section and discuss its relationship with the h- and s-cobordism theorems. Many of our subsequent chapters are spent in the determination of these equivalence classes.

Let X be a Poincaré space of dimension m. In Appendix 1.2 we review the process of surgery on a degree one normal map $f : M \to X$ from a Cat m-manifold M. There we explicitly describe the effect M' of this k-surgery (short for k-dimensional surgery) and the resulting trace bordism. Here we present a few results that relate homotopy equivalences with degree one normal maps.

Proposition 1.18. *Let X be a m-dimensional Poincaré complex. The following are equivalent:*

1. *The complex X is homotopy equivalent to a closed Cat m-manifold.*
2. *The natural map $X \to B(F/Cat)$ is null-homotopic and some bordism class of degree one Cat normal maps $(f, b) : M \to X$ contains a homotopy equivalence.*
3. *There is a Cat m-manifold M with a degree one Cat normal map $(f, b) : M^m \to X$ and a sequence of Cat surgeries on M such that the trace (W^{m+1}, M, M') is the domain of a degree one Cat normal bordism*

$$((F, B), (f, b), (f', b')) : (W, M, M') \to X \times (I, \{0\}, \{1\}),$$

with $f' : M' \to X$ a homotopy equivalence.

We have analogous results for Cat manifolds M with boundary.

Proposition 1.19. *Let (X, Y) be an m-dimensional Poincaré pair with $Y \subseteq X$.*

1. *The pair (X, Y) is homotopy equivalent to an m-manifold with boundary iff the natural map $X \to B(F/Cat)$ is null-homotopic and some corresponding bordism class of degree one Cat normal maps $(f, b) : (M, \partial M) \to (X, Y)$ contains a homotopy equivalence.*
2. *If ∂X is already an $(m-1)$-manifold, then (X, Y) is homotopy equivalent relative boundary to an m-manifold $(M, \partial M)$ with boundary iff the natural map $X/Y \to B(F/Cat)$ is null-homotopic and some corresponding bordism class relative boundary of degree one normal maps $(f, b) : (M, \partial M) \to (X, Y)$ contains a homotopy equivalence.*

Remark 1.20. *In reference to (2) above, we say that (X, Y) is homotopy equivalent relative boundary to $(M, \partial M)$ if there is a homotopy equivalence $f : (M, \partial M) \to (X, Y)$ of pairs such that $f|_{\partial M}$ is already a Cat isomorphism.*

Our next goal is to collect Cat manifold structures into a set under some equivalence relation. To do so, we require a few notions that we discuss here.

Definition 1.21. *Let (X, Y) be a Poincaré pair with* $\dim X = m$.

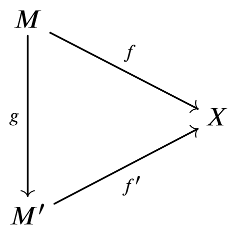
1. *A Cat h-manifold structure on (X, Y) is a homotopy equivalence $g : (M^m, \partial M) \to (X, Y)$ of pairs from a Cat manifold pair $(M, \partial M)$ into (X, Y).*
2. *A Cat s-manifold structure on (X, Y) is a map $g : (M^m, \partial M) \to (X, Y)$ of pairs from a Cat manifold pair $(M, \partial M)$ into (X, Y) such that g and $g|_{\partial M}$ are simple homotopy equivalences.*

Remark 1.22. *For the classification of manifolds up to Cat isomorphism, we choose a finite complex in the homotopy type of the Poincaré complex X, i.e. the choice of a simple homotopy type within the homotopy type (see the appendices). Simple homotopy theory also has an existence analogue, due to Wall [662], that tells us the conditions* *under which a finitely dominated homotopy type contains a finite complex. In the simply connected case, one has a finitely dominated homotopy type on X iff $\bigoplus H_i(X)$ is finitely generated. The Wall finiteness obstruction is an element $w(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$. Not only is this theory analogous to that of Whitehead torsion, but Ferry [237] has reduced it to that theory.*

We come to a few equivalence relations among Cat manifold structures. The following definitions can also be stated for manifolds with boundary if all maps are Cat isomorphisms on the boundary.

Definition 1.23. *Suppose that X is a Poincaré complex.*

1. *We say that two Cat h-manifold structures $f : M \to X$ and $f' : M' \to X$ on X are Cat h-equivalent if there is a Cat h-cobordism W between them with a Cat map $F : W \to X$ that restricts to f and f' on the boundaries.*
2. *We say that two Cat s-manifold structures $f : M \to X$ and $f' : M' \to X$ on X are Cat s-equivalent if there is a Cat s-cobordism W between them with a Cat map $F : W \to X$ that restricts to f and f' on the boundaries.*
3. *We say that two Cat s-manifold structures $f : M \to X$ and $f' : M' \to X$ on X are Cat equivalent if there is a Cat isomorphism $g : M \to M'$ such that the diagram*



is homotopy commutative; i.e. the maps $f' \circ g : M \to X$ and $f : M \to X$ are homotopic.

Remark 1.24. *For high-dimensional manifolds, (2) and (3) are the same.*

If an m-dimensional Poincaré complex X admits a homotopy equivalence $M \to X$ from a Cat manifold M^m, then we can combine all such homotopy equivalences into a set. The following definitions offer a number of ways in which such a set can be defined.

Definition 1.25. *If equivalence is defined by (1), we denote by $\mathcal{S}^{Cat}(X)$ or $\mathcal{S}_h^{Cat}(X)$ the set of equivalence classes of Cat manifold structures, called the Cat structure set of X. If equivalence is defined by (2) or (3), we denote by $\mathcal{S}_s^{Cat}(X)$ the set of equivalence classes of Cat manifold structures, called the simple Cat structure set of X. A typical element of a Cat structure set is denoted by (M, f). We will use the analogous symbol $\mathcal{S}^{Cat}(X)$ when we do not want to distinguish between the two. We can ignore the Cat notation when the results are true in all categories.*

Remark 1.26. *If X itself is a Cat manifold, then there are two reasons why a Cat manifold structure $M \to X$ could be non-trivial in $\mathcal{S}^{Cat}(M)$. First, there may be a manifold M which is homotopy equivalent to X but not Cat isomorphic to it. In this case, we may say that M is exotic. Second, there may be a self-homotopy equivalence $j : X \to X$ that is not homotopic to the identity map. In this case the map j is exotic.*

Definition 1.27. *If M is a Cat n-manifold, and if the Cat structure set $\mathcal{S}^{Cat}(M)$ contains, up to equivalence, the one sole element $id : M \to M$, then we say that M is Cat rigid.*

Much of this book is dedicated to showing the Cat rigidity or nonrigidity of well-known manifolds.

Remark 1.28. *In some classical articles, Cat manifold structures in the PL category were often called homotopy triangulations or h-triangulations. In the Diff category they were often called homotopy smoothings or h-smoothings. Equivalences between manifold structures were called concordances.*

Since a homotopy equivalence is a normal map, and an h-cobordism is a normal cobordism, there is a forgetful map $\mathcal{S}_h^{Cat}(X) \to \mathcal{N}^{Cat}(X)$ or $\mathcal{S}_s^{Cat}(X) \to \mathcal{N}^{Cat}(X)$. The surgery exact sequence explains how different they are.

1.1.4 Wall's L-groups

Let X be an m-dimensional Poincaré complex with fundamental group $\pi = \pi_1(X)$. The Wall L-groups of X are complicated objects which can be described in several different ways. The following is a very brief overview of the most common constructions. Details about L-groups can be found in a variety of texts; see in particular Wall [665], Lück [405], or Ranicki [544]. For simplicity, we shall tacitly assume that all manifolds and Poincaré complexes are oriented, and briefly describe the modifications necessary for the general case.

The L-groups can be described in both a geometric and an algebraic fashion. We start with the geometric description, which is conceptually terrific but calculationally useless. Let X^n be an n-dimensional Poincaré complex. We consider compositions of maps of the form

$$(M^n, \partial M) \xrightarrow{f} (Z^n, \partial Z) \to X,$$

where $(M, \partial M)$ and $(Z, \partial Z)$ are Cat manifolds with boundary and f is a degree one normal map such that the restriction $\partial M \to \partial Z$ is a homotopy equivalence. Two such compositions $(M_i, \partial M_i) \xrightarrow{f_i} (Z_i, \partial Z_i) \to X$ are *equivalent* if there are the following:

1. a cobordism V between M_1 and M_2, which is a manifold with corners,
2. a cobordism W between Z_1 and Z_2, which is also a manifold with corners,

3. a degree one normal map $F: V \to W$ that restricts to f_1 and f_2 on M_1 and M_2.

Definition 1.29. *Let X be a Poincaré complex of dimension n. There is a forgetful map* $\sigma: \mathcal{N}^{Cat}(X) \to L_n(X)$. *The image* $\sigma(f, b)$ *of a degree one Cat normal map* (f, b) *is called the surgery obstruction of* (f, b) *and* σ *itself is called the surgery map.*

Tacit in this definition was the fact that the L-group is independent of Cat. We will review the *meaning* of the surgery obstruction in the upcoming paragraphs.

Now we take an algebraic approach and define the L-groups for arbitrary rings Λ with involution $a \mapsto \overline{a}$. For a group π, the group ring $\Lambda = \mathbb{Z}[\pi]$ can be given an involution $\overline{g} = g^{-1}$. Note that, for rings with involution, we have $\overline{ab} = \overline{b}\,\overline{a}$. If n is even, then $L_n^h(\Lambda)$ is defined as follows.

Definition 1.30. *Let* Λ *be ring with involution* $a \mapsto \overline{a}$ *and let* $n = 2k$ *be even. We form the semigroup under orthogonal direct sum of triples* (P, λ, μ) *such that*

1. P *is a free* Λ*-module;*
2. *the map* $\lambda: P \times P \to \Lambda$ *is*
   a) Λ*-linear in the second variable;*
   b) $\lambda(x, y) = (-1)^k \overline{\lambda(y, x)}$ *for all* $x, y \in P$*;*
   c) $\mathrm{ad}(\lambda): P \to \mathrm{Hom}_\Lambda(P, \Lambda)$ *is a homomorphism;*
3. $\mu: P \to Q_k \equiv \Lambda/\langle \nu - (-1)^k \overline{\nu} \rangle$ *satisfies*
   a) $\mu(x) + (-1)^k \overline{\mu(x)} = \lambda(x, x)$ *for all* $x \in P$*;*
   b) $\mu(x + y) - \mu(x) - \mu(y) = [\lambda(x, y)]$ *for all* $x, y \in P$*;*
   c) $\mu(xa) = \overline{a}\mu(x)a$ *for all* $a \in \Lambda$ *and* $x \in P$.

A triple (P, λ, μ) *is a hyperbolic form if there is an isomorphism from* (P, λ, μ) *to a direct sum of copies of*

$$\left(\Lambda \oplus \Lambda,\ \lambda = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix},\ \mu(x) = 0 = \mu(y)\right),$$

where $\{x, y\}$ *is the given basis for* $\Lambda \oplus \Lambda$. *Then the Wall L-group* $L_{2k}^h(\Lambda)$ *or* $L_{2k}(\Lambda)$ *is the Grothendieck group associated to this semigroup modulo one relation: that hyperbolic forms are set to zero.*

Remark 1.31. *The map* λ *is called a* $(-1)^k$*-Hermitian form. The map* μ *is a quadratic form or a quadratic refinement of* λ. *The definitions of isomorphism and other concepts related to forms are given in detail in Section 2.2.*

Definition 1.32. *When* π *is a group and* $\Lambda = \mathbb{Z}[\pi]$, *we can define the simple L-groups* $L_n^s(\mathbb{Z}[\pi])$ *to be the same as the* L^h*-groups except that conditions (1) and (2c) are re-*

placed by the following:

- *(1′) P is a free $\mathbb{Z}[\pi]$-module with a simple equivalence class of bases;*
- *(2c′)* $\mathrm{ad}(\lambda)\colon P \to \mathrm{Hom}_\Lambda(P, \mathbb{Z}[\pi])$ *is a simple homomorphism; i.e. the matrix taking the basis $\{v_1, \dots, v_r\}$ of P to its dual basis in* $\mathrm{Hom}_{\mathbb{Z}[\pi]}(P, \mathbb{Z}[\pi])$ *vanishes in the Whitehead group* $\mathrm{Wh}(\pi)$.

In addition, the isomorphism in the relation must be a simple isomorphism.

Remark 1.33. *If condition (1) is modified so that P is merely a projective Λ-module, then the construction above gives the projective L-groups $L^p_n(\Lambda)$. These groups arise in geometric problems involving noncompact manifolds and are sometimes easier to analyze. Their relationship with L^h can help us study the latter.*

Remark 1.34. *We will refer to h, s, and p in the notation of the L-groups as decorations.*

Notation 1.35. *If the involution is trivial, then the map λ satisfies $\lambda(x, y) = \lambda(y, x)$ if k is even, and $\lambda(x, y) = -\lambda(y, x)$ if k is odd. For general k, we may say that the associated triple (E, λ, μ) is (± 1)-symmetric, or ε-symmetric, where $\varepsilon = (-1)^k$. If the involution is non-trivial, as is the case when complex conjugation is used, the triple (E, λ, μ) is $(-1)^k$-Hermitian or ε-Hermitian. Except in the case of elementary 2-groups, we will always have non-trivial involutions in geometric examples.*

Remark 1.36. *In the case of n even, we arrive at the theory of bilinear and quadratic forms over Λ. The group $L_0(\Lambda)$ is the Witt group of $(+1)$-quadratic forms (symmetric), and $L_2(\Lambda)$ is the Witt group of (-1)-quadratic forms (skew-symmetric). If 2 is invertible in Λ, then condition (3a) gives $\mu(x) = \frac{1}{2}\lambda(x, x)$, so that the quadratic form is determined by the bilinear form. When* $\mathrm{char}(\Lambda) = 2$*, there may be many quadratic forms μ compatible with a given λ. For n odd, the L-groups $L_n(\Lambda)$ and $L^s_n(\Lambda)$ are defined in terms of automorphisms of hyperbolic forms, or equivalently in terms of so-called formations. We will discuss the odd case briefly in Section 2.4, but we do not give the algebraic definition in our review here since it is more complicated.*

Remark 1.37. *At least in the even-dimensional cases discussed above, the L-groups are tautologically 4-periodic. The information used in defining $L_n(\Lambda)$ is the same as the information used in defining $L_{n+4}(\Lambda)$. At the end of this section we give a more geometric interpretation of this isomorphism using $\mathbb{CP}^2$.*

For our purposes we are interested in the integer group ring $\Lambda = \mathbb{Z}[\pi]$ of the fundamental group $\pi = \pi_1(X)$ of a Poincaré complex X. If π is equipped with an orientation homomorphism $w\colon \pi \to \{\pm 1\}$, then we can endow Λ with a canonical anti-involution $a \mapsto \overline{a}$ sending $\sum n_i g_i$ to $\sum w(g_i) n_i g_i^{-1}$. In this case, the Wall groups are denoted by $L_n(\mathbb{Z}[\pi], w)$ and $L^s_n(\mathbb{Z}[\pi], w)$. If the involution w is trivial, we suppress it in the notation for the L-group and just write $L_n(\mathbb{Z}[\pi])$ and $L^s_n(\mathbb{Z}[\pi])$.

Remark 1.38. *If w_1 and w_2 are two non-trivial orientation homomorphisms for an n-*

manifold M with fundamental group π, then certainly the L-groups $L_*(\mathbb{Z}[\pi], w_1)$ and $L_*(\mathbb{Z}[\pi], w_2)$ may not be isomorphic. A simple example is given later in Remark 2.109.

The geometrically defined L-groups $L_*(X)$ only depend on the fundamental group of X, as asserted in the following.

Theorem 1.39. *(Wall [672]) For $n \geq 5$, let $L_n(X)$ be the collection of compositions $(M^n, \partial M) \xrightarrow{f} (Z^n, \partial Z) \to X$ up to the given equivalence. Then $L_n(X)$ is a group under disjoint union and is isomorphic to $L_n(\mathbb{Z}[\pi_1(X)])$.*

Remark 1.40. *A key point in the proof is that Z can be found in any equivalence class such that $\pi_1(Z) \to \pi_1(X)$ is an isomorphism. We will discuss some of the points of this construction in Section 1.5 of this chapter.*

We will examine these L-groups in detail in the next chapter. For now we describe their relationship to the Cat manifold structure set and the Cat normal structure set. The following theorem is due to Wall [672].

Theorem 1.41. *(Wall realization theorem) Let π be a finitely presented group and let $n \geq 5$. Let $x \in L_{n+1}(\mathbb{Z}[\pi])$ and M^n be a closed Cat n-manifold with fundamental group π and orientation character w. Then there is a degree one Cat normal map*

$$(F, B)\colon (W^{n+1}, \partial_0 W, \partial_1 W) \to (M \times I, M \times \{0\}, M \times \{1\})$$

such that

1. *$\sigma(F, B) = x$;*
2. *$\partial_0 W = M$;*
3. *$F|_{\partial_0 W} = id_M$;*
4. *the map $f \equiv F|_{\partial_1 W} : \partial_1 W \to M$ is a homotopy equivalence.*

Here $B\colon \nu_W \to \nu_{M\times I}$ is a bundle map covering F, where the source ν_W is the stable Cat normal bundle of W in Euclidean space, and $\nu_{M\times I}$ is a Cat bundle over $M \times I$. The obvious version holds for L^s as well.

Theorem 1.42. *Let $n \geq 5$. Let X be an n-dimensional simple Poincaré complex and let $\pi = \pi_1(X)$. There is an action $L_{n+1}(\mathbb{Z}[\pi]) \times \mathcal{S}^{Cat}(X) \to \mathcal{S}^{Cat}(X)$ given by the following. Let $x \in L^s_{n+1}(\mathbb{Z}[\pi])$ and $(M, h) \in \mathcal{S}^{Cat}(X)$. If $f : \partial_1 W \to M$ is the homotopy equivalence constructed in (4) of the previous theorem, then the assignment $x.(M, h) = (\partial_1 W, h \circ f)$ gives a well-defined action of $L_{n+1}(\mathbb{Z}[\pi])$ on $\mathcal{S}^{Cat}(X)$.*

Remark 1.43. *One proves that the action is well-defined using surgery in dimension $n + 1$ and the s-cobordism theorem.*

For a Poincaré complex X of dimension m and fundamental group π, the relationship between the structure set $\mathcal{S}^{Cat}(X)$, the normal invariant set $\mathcal{N}^{Cat}(X)$, and the L-group $L_n(\mathbb{Z}[\pi])$ is laid out in the next two theorems.

Theorem 1.44. *(Manifold existence: Wall [672]; see also Browder [85] and Novikov [484]) Let X be an m-dimensional Poincaré complex with $m \geq 5$ and fundamental group π. The following are equivalent:*

1. *The Cat structure set $\mathcal{S}^{Cat}(X)$ is nonempty; i.e. X is homotopy equivalent to an m-dimensional Cat manifold.*
2. *The Spivak normal fibration $\alpha : X \to BF$ has a Cat bundle reduction $\widetilde{\alpha}_X : X \to BCat$ for which the corresponding degree one Cat normal map $(f, b) : M \to X$ has zero surgery obstruction $\sigma(f, b)$ in $L_m(\mathbb{Z}[\pi])$.*
3. *There exists a Cat manifold M^m with a degree one Cat normal map $(f, b) : M \to X$ with a sequence of Cat surgeries on M such that the trace (W, M, M') is the domain of a degree one Cat normal bordism*

$$((F, B), (f, b), (f', b')) : (W, M, M') \to X \times (I, \{0\}, \{1\})$$

with $f' : M' \to X$ a homotopy equivalence.

Theorem 1.45. *(Manifold classification) Let X be an m-dimensional Poincaré complex with $m \geq 5$ and fundamental group π. Assume that $\mathcal{S}^{Cat}(X)$ is nonempty. Let $\eta : \mathcal{S}^{Cat}(X) \to \mathcal{N}^{Cat}(X)$ be as above.*

1. *Let $(f, b) : M \to X$ be a degree one normal map from an m-dimensional Cat manifold M to X. Then $\sigma(f, b) = 0$ in $L_m(\mathbb{Z}[\pi])$ iff (f, b) is bordant to some homotopy equivalence $(f', b') : M' \to X$; i.e. there is a Cat manifold structure $(M', f') \in \mathcal{S}^{Cat}(X)$ such that $\eta(M', f') = (f, b)$ in $\mathcal{N}^{Cat}(X)$.*
2. *Suppose that there is a degree one Cat normal bordism*

$$(F, B) : (W; M_1, M_2) \to X \times (I, \{0\}, \{1\})$$

that restricts to $(f_i, b_i) : M_i \to X$ on M_i. If $\gamma = \sigma(F, B)$ in $L_{m+1}(\mathbb{Z}[\pi])$, then the L-group $L_{m+1}(\mathbb{Z}[\pi])$ acts on $\mathcal{S}^{Cat}(X)$ by $\gamma.(M_1, f_1) = (M_2, f_2)$.

3. *The images under $\eta : \mathcal{S}^{Cat}(X) \to \mathcal{N}^{Cat}(X)$ of two Cat manifold structures (M_1, f_1) and (M_2, f_2) are identified iff there is $\gamma \in L_{m+1}(\mathbb{Z}[\pi])$ such that $\gamma.(M_1, f_1) = (M_2, f_2)$; i.e. there is a degree one Cat normal bordism*

$$(F, B) : (W, M_1, M_2) \to X \times (I, \{0\}, \{1\})$$

such that (F, B) restricts to (f_i, b_i) on M_i and $\gamma = \sigma(F, B)$ in $L_{m+1}(\mathbb{Z}[\pi])$.

The surgery exact sequence puts the statements together.

Theorem 1.46. *(Surgery exact sequence) Let X be an m-dimensional Poincaré com-*

plex with $m \geq 5$ and fundamental group π. Suppose that $\mathcal{S}^{Cat}(X)$ is nonempty. Then $\mathcal{N}^{Cat}(X)$ is nonempty and the surgery sequence of pointed spaces, given as follows, is exact at the two middle terms:

$$\cdots \to L^h_{m+1}(\mathbb{Z}[\pi]) \overset{\alpha}{\dashrightarrow} \mathcal{S}^{Cat}(X) \xrightarrow{\eta} \mathcal{N}^{Cat}(X) \xrightarrow{\sigma} L^h_m(\mathbb{Z}[\pi]).$$

Here the dotted arrow indicates that $L_{m+1}(\mathbb{Z}[\pi])$ acts on $\mathcal{S}^{Cat}(X)$, and exactness at $\mathcal{S}^{Cat}(X)$ means that the point inverses of η are precisely the orbits of the group action on the structure set. Exactness at $\mathcal{N}^{Cat}(X)$ means that the inverse image of $0 \in L_m(\mathbb{Z}[\pi])$ under σ is precisely the image of η.

We now discuss the surgery theory related to manifolds with boundary, i.e. manifold pairs. We will have two sets of definitions.

Definition 1.47. *If (X, Y) is a Cat manifold pair, then a Cat h-manifold structure on (X, Y) is a homotopy equivalence $h\colon (M, \partial M) \to (X, Y)$ from a Cat manifold $(M, \partial M)$ with boundary such that $h|_{\partial M} : \partial M \to Y$ is a homeomorphism. The collection of all equivalence classes of such manifold structures is denoted by $\mathcal{S}^{Cat}(M, \partial M)$.*

Definition 1.48. *A Cat normal invariant on (X, Y) is given by a degree one normal map $f : (M, \partial M) \to (X, Y)$ from a Cat manifold M with boundary such that the restriction $f|_{\partial M} : \partial M \to Y$ is a homotopy equivalence. The collection of all equivalence classes of such manifold structures is denoted by $\mathcal{N}^{Cat}(M, \partial M)$.*

Definition 1.49. *If (X, Y) is a pair, then we consider compositions*

$$(M, \partial^* M, N, \partial N) \xrightarrow{f} (Z, \partial Z^*, \partial Z, \partial^2 Z) \to (X, Y),$$

restricting to maps

1. $M \to Z \to X$,
2. $N \to \partial Z \to Y$,
3. $\partial^* M \to \partial^* Z \to X$,
4. $\partial N \to \partial^2 Z \to Y$,

where

1. $\partial M = N \cup_{\partial N} \partial^* M$;
2. *the map $f : M \to Z$ is a degree one normal map;*
3. *the map $\partial^* M \to \partial^* Z$ is a homotopy equivalence.*

The collection of all equivalence classes of such manifold structures is denoted by $L^h_n(M, \partial M)$.

Remark 1.50. *These sets can certainly be defined in the case of simple homotopy equiv-*

alences.

Theorem 1.51. *(Surgery exact sequence for pairs) If (X, Y) is a Poincaré pair of dimension $n \geq 5$, then there is an exact sequence*

$$L^h_{n+1}(X, Y) \to \mathcal{S}^{Cat}(X, Y) \to \mathcal{N}^{Cat}(X, Y) \to L^h_n(X, Y)$$

of pointed sets, as before.

We can modify all three sets by insisting on the following:

1. The Cat manifold structure $f : (M, \partial M) \to (X, Y)$ restricts to a Cat isomorphism $f|_{\partial M} : \partial M \to Y$. The relative structure set is denoted by $\mathcal{S}^{Cat}(X \text{ rel } Y)$ or $\mathcal{S}^{Cat}(X)_{\text{rel}}$ or simply $\mathcal{S}^{Cat}(X)$.
2. The Cat normal invariant $f : (M, \partial M) \to (X, Y)$ restricts to a Cat isomorphism $f|_{\partial M} : \partial M \to Y$. The relative normal invariant set is denoted $\mathcal{N}^{Cat}(X \text{ rel } Y)$ or $\mathcal{N}^{Cat}(X)_{\text{rel}}$ or simply $\mathcal{N}^{Cat}(X)$.
3. The composition $(M, \partial^* M, N, \partial N) \xrightarrow{f} (Z, \partial^* Z, \partial Z, \partial^2 Z) \to (X, Y)$ restricts to a Cat isomorphism $N \to \partial Z \to Y$. The L-group is denoted $L_n(X \text{ rel } Y)$ or $L_n(X)_{\text{rel}}$ or simply $L_n(X)$.

Theorem 1.52. *(Relative surgery exact sequence) If (X, Y) is a Poincaré complex of dimension $n \geq 5$, then there is an exact sequence*

$$L^h_{n+1}(X \text{ rel } Y) \to \mathcal{S}^{Cat}(X \text{ rel } Y) \to \mathcal{N}^{Cat}(X \text{ rel } Y) \to L^h_n(X \text{ rel } Y)$$

of pointed sets. Expressed otherwise, the sequence can be written as follows:

$$L_{n+1}(X) \to \mathcal{S}^{Cat}(X \text{ rel } Y) \to [X/Y : F/Cat] \to L_n(X).$$

Remark 1.53. *The difference in relative and non-relative surgery can be notationally confusing, especially when the relative case is just denoted $\mathcal{S}^{Cat}(X)$ with no mention of Y. The literature is inconsistent in the notation for these sets, so we will have the convention that X refers to the relative case and (X, Y) is non-relative. We may use $\mathcal{S}^{Cat}(X)_{\text{rel}}$ or $\mathcal{S}^{Cat}(X \text{ rel } Y)$ for emphasis. The computations for these sequences are very different. Later we will see how the π-π theorem has great implications on the non-relative surgery exact sequence. In the relative case, only the fundamental group of X matters, so the relative L-group is just denoted $L^h_n(\mathbb{Z}[\pi_1(X)])$. If additionally $\pi_1(Y) = \pi_1(Y)$, then the non-relative L-group can be denoted $L^h_n(\mathbb{Z}[\pi_1(X)], \mathbb{Z}[\pi_1(Y)])$. If a term appears without the rel notation and without a pair, e.g. $\mathcal{S}^{Cat}(X)$, then assume it relative to any boundary pieces.*

Wall proves that the two L-groups $L_n(X)$ and $L_n(\pi_1(X))$ coincide.[1] In a few pages we

[1] We will sometimes omit the $\mathbb{Z}$ when notating the L-group of a group ring, especially if the group in

will give the definition of the L-group $L_n(X, Y)$ of pairs. For now, we use it in stating the following.

Theorem 1.54. *(Wall) If X and Y have fundamental groups $\pi_1(X)$ and $\pi_1(Y)$, respectively, then $L_n(X) \cong L_n(\mathbb{Z}[\pi]_X)$ and $L_n(X, Y) = L_n(\mathbb{Z}[\pi_1(X)], \mathbb{Z}[\pi_1(Y)])$. Moreover, they fit in the exact sequence*

$$\cdots \to L_n(X) \to L_n(Y) \to L_n(X, Y) \to \cdots$$

Theorem 1.55. *The surgery exact sequence can be extended to the left. If X^m is a Poincaré complex with dimension $m \geq 5$ and fundamental group π, then there is an exact sequence*

$$\cdots \to \mathcal{S}^{Cat}(X \times \mathbb{D}^{k+1})_{\text{rel}} \to \mathcal{N}^{Cat}(X \times \mathbb{D}^{k+1})_{\text{rel}} \to L_{m+k}(\mathbb{Z}[\pi]) \dashrightarrow \cdots$$

$$\cdots \to L_{m+2}(\mathbb{Z}[\pi]) \dashrightarrow \mathcal{S}^{Cat}(X \times I)_{\text{rel}} \to \mathcal{N}^{Cat}(X \times I)_{\text{rel}}$$
$$\to L_{m+1}(\mathbb{Z}[\pi]) \dashrightarrow \mathcal{S}^{Cat}(X) \to \mathcal{N}^{Cat}(X) \to L_m(\mathbb{Z}[\pi]).$$

Note that $\mathcal{S}^{Cat}(X \times I)_{\text{rel}}$ and $\mathcal{N}^{Cat}(X \times I)_{\text{rel}}$ can be given a group structure. Indeed, if $W_1 \to X \times I$ and $W_2 \to X \times I$ are maps that restrict to Cat isomorphisms on the boundary, then we can construct the map $W \to X \times I$ where W is the concatenation of W_1 and W_2. A similar construction applies when I is replaced with $\mathbb{D}^k$.

We end this section by mentioning the geometric interpretation of the 4-fold periodicity of Wall's L-groups that is proved in Wall's book.

Theorem 1.56. *Let $m \geq 5$ and let π be a group. Then the map $L_m(\mathbb{Z}[\pi]) \to L_{m+4}(\mathbb{Z}[\pi])$ induced by multiplying by $\mathbb{CP}^2$ is an isomorphism.*

When we cross an element α of L-theory with a manifold of dimension $4k$, we add to γ the symmetric bilinear form induced by cup product on the middle cohomology. In particular, if we cross with any simply connected manifold M^{4k}, then the output is multiplied by the signature of M. Therefore, the complex projective space $\mathbb{CP}^{4k}$ has signature 1 and can be used to induce an isomorphism. See Morgan [472] for the general discussion regarding products with simply connected manifolds of arbitrary dimension, as well as Ranicki [530] for a framework for studying products with non-simply connected manifolds.

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question is notated by a string of more than one character. For example, we will write $L_*(\pi_1(X))$ instead of $L_*(\mathbb{Z}[\pi_1(X)])$. There should be no confusion within the context.

1.2 EXECUTING SURGERY

The method of constructing a homotopy equivalence between a given Poincaré complex X to some Cat manifold M is to begin with a degree one normal map $f : M \to X$ and to modify it by the process of surgery until a desired map $f' : M' \to X$ is reached. The main aim of such modification is to reduce the size of the *relative homotopy group* of the map. If $f : X \to Y$ is a map of topological spaces, we can form the mapping cylinder $M_f = ((X \times I) \coprod Y)/(x \times \{1\} \sim f(x))$. Let the *n-th relative homotopy group* $\pi_n(f)$ be given by $\pi_n(f) = \pi_n(M_f, X)$. If $\pi_k(f) = 0$ for all $k \leq n$, then f is *n-connected.* As these groups fit into a long exact sequence

$$\cdots \to \pi_{n+1}(f) \to \pi_n(X) \to \pi_n(Y) \to \pi_n(f) \to \pi_{n-1}(X) \to \cdots$$

if $\pi_i(f) = 0$ for all i, then X and Y are homotopy equivalent by the Whitehead theorem. Relative groups $H_n(f)$ can similarly be defined for homology. Because there is a relative Hurewicz theorem, one can use homotopy calculations to measure progress. As in the previous section, we only explain the even-dimensional case, referring the reader to Wall [672] for the general case.

Definition 1.57. *Let $n \geq 1$.*

1. *A space X is n-connected if it is connected and $\pi_i(X) = 0$ for all $i \leq n$.*
2. *A map $f : X \to Y$ of connected spaces is n-connected if $f_* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i < n$ and $f_* : \pi_n(X) \to \pi_n(Y)$ is surjective; i.e. if $\pi_i(f) = 0$ for all $i \leq n$.*
3. *A pair of connected spaces (Y, X), where $X \subseteq Y$, is n-connected if the inclusion $f : X \to Y$ is n-connected; i.e. $\pi_i(Y, X) = 0$ for all $i \leq n$.*

We will now discuss the surgery process. Suppose that $f : M \to X$ is a degree one normal map of dimension n for which $\pi_*(f)$ is non-trivial. Our hope is to modify M to arrive at a new degree one normal map $f' : M' \to X$ for which $\pi_*(f')$ is smaller in size than $\pi_*(f)$. By abuse of notation, we often call the new map $f : M \to X$ as well.

We demonstrate the process inductively. Suppose that $f : M \to X$ is a degree one normal map of dimension from a manifold of dimension n to a Poincaré complex of dimension n. Let $k < \left[\frac{n}{2}\right]$ and suppose that $\pi_i(f) = 0$ for all $i \leq k$. Assume that $\alpha \in \pi_{k+1}(f)$ is represented by a diagram

$$\begin{array}{ccc} \mathbb{S}^k & \xrightarrow{j} & \mathbb{D}^{k+1} \\ \phi \downarrow & & \downarrow \Phi \\ M & \xrightarrow[f]{} & X \end{array}$$

such that $\phi\colon \mathbb{S}^k \to M$ is an embedding. The existence of a map $\overline{f}\colon \nu_M \to \xi$ covering f shows that the normal bundle of the embedding ϕ is trivial. Therefore we can extend ϕ to an embedding $\overline{\phi}\colon \mathbb{S}^k \times \mathbb{D}^{n-k} \to M$ that fits the diagram

$$\begin{array}{ccc} \mathbb{S}^k \times \mathbb{D}^{n-k} & \xrightarrow{\bar{j}} & \mathbb{D}^{k+1} \times \mathbb{D}^{n-k} \\ {\scriptstyle\overline{\phi}}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\Phi}} \\ M & \xrightarrow[f]{} & X \end{array}$$

We replace M with a new manifold M' that results from removing the interior of the image of $\mathbb{S}^k \times \mathbb{D}^{n-k}$ in M, then glueing in $\mathbb{D}^{k+1} \times \mathbb{S}^{n-k-1}$ along the boundary, which is the image of $\mathbb{S}^k \times \mathbb{S}^{n-k-1}$. We therefore set

$$M' = (\mathbb{D}^{k+1} \times \mathbb{S}^{n-k-1}) \cup (M \backslash \overline{\phi}(\mathbb{S}^k \times \operatorname{int}(\mathbb{D}^{n-k})))$$

where the union is taken over $\overline{\phi}(\mathbb{S}^k \times \mathbb{S}^{n-k-1})$. Let $f'\colon M' \to X$ be the map defined by f on $M \backslash \overline{\phi}(\mathbb{S}^k \backslash \operatorname{int}(\mathbb{D}^{n-k}))$ and by restriction of $\overline{\Phi}$ on $\mathbb{D}^{k+1} \times \mathbb{S}^{n-k-1}$. It follows that $\pi_i(f') = 0$ for all $i \leq k$, and there is a surjective homomorphism $\pi_{k+1}(f) \to \pi_{k+1}(f')$ whose kernel contains α. This process is called a *k-surgery*. We also say that *the Cat k-surgery kills the homotopy class $[\alpha] \in \pi_k(M)$ of the core Cat k-embedding $g\colon \mathbb{S}^k \times \{0\} \hookrightarrow M$.* More generally, the operation of constructing a new degree one normal map $f'\colon M' \to X$ from the normal data of a given degree one normal map $f\colon M \to X$ is called a *surgery step*. The map $f'\colon M' \to X$ obtained after surgery on M is normally cobordant to $f\colon M \to X$. The cobordism can be constructed from $M \times I$ by glueing $\mathbb{D}^{k+1} \times \mathbb{D}^{n-k}$ along the image of $\overline{\phi}$ to $M \times \{1\}$, i.e.

$$W = (\mathbb{D}^{k+1} \times \mathbb{D}^{n-k}) \cup (M \times I)$$

where the union is taken over $\overline{\phi}(\mathbb{S}^k \times \mathbb{D}^{n-k})$.

Remark 1.58. *When describing the surgery step, we made an assumption that there is an embedding $\phi\colon \mathbb{S}^k \to M$ such that its extension represents a given element $\alpha \in \pi_{k+1}(f)$. By the Whitney embedding theorem [709], this property always holds provided that $k < \left[\frac{n+1}{2}\right]$.*

Theorem 1.59. *Let M be a Cat n-manifold and let $k \leq n$. The following conditions on an element $x \in \pi_k(M)$ are equivalent:*

1. *x can be killed by a Cat k-surgery on M;*
2. *x can be represented by a framed Cat k-embedding $\overline{g}\colon \mathbb{S}^k \times \mathbb{D}^{n-k} \hookrightarrow M$;*
3. *x can be represented by a Cat k-embedding $g\colon \mathbb{S}^k \hookrightarrow M^n$ with trivial Cat normal bundle $\nu_g\colon \mathbb{S}^k \to BCat_{n-k}$.*

Remark 1.60. *The theorem suggests that the possibility of killing an element of $\pi_k(M^n)$*

is entirely determined by the stable normal Cat bundle $\nu_M : M \to BCat$.

We now describe the homotopy effects of a k-surgery in the range $k < \left[\frac{n}{2}\right]$.

Proposition 1.61. *Let* $(F, f, f') : (W^{n+1}, M, M') \to X \times (I, \{0\}, \{1\})$ *be the trace of a k-surgery on a map* $f : M \to X$ *that kills* $x \in \pi_{k+1}(f)$*. Then the relative homotopy groups of F are given by*

$$\pi_i(F) = \begin{cases} \pi_i(f) & \text{if } i \leq k, \\ \pi_{k+1}(f)/\langle x \rangle & \text{if } i = k+1, \end{cases}$$

where $\langle x \rangle \subseteq \pi_{k+1}(f)$ *is the* $\mathbb{Z}[\pi_1(X)]$*-module generated by x. Since F is also the trace of the dual* $(n-k-1)$*-surgery on* $f' : M' \to X$ *killing an element* $x' \in \pi_{n-k}(f')$*, we also have*

$$\pi_j(F) = \begin{cases} \pi_j(f) & \text{if } j \leq n-k-1, \\ \pi_{n-k}(f)/\langle x \rangle & \text{if } j = n-k. \end{cases}$$

Proof. The elements $x = [\phi] = (h, g) \in \pi_{k+1}(f)$ and $x' = [\phi'] = (h', g') \in \pi_{n-k}(f')$ are represented by the core k-embedding and the dual core $(n-k-1)$-embedding

$$\begin{array}{ccc} \mathbb{S}^k & \xrightarrow{\ g\ } & M \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbb{D}^{k+1} & \xrightarrow[\ h\ ]{} & X \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{S}^k & \xrightarrow{\ g'\ } & M' \\ \big\downarrow & & \big\downarrow{\scriptstyle f'} \\ \mathbb{D}^{k+1} & \xrightarrow[\ h'\ ]{} & X \end{array}$$

with $F \simeq f \cup h \simeq f' \cup h' : W \simeq M \cup_g \mathbb{D}^{k+1} \simeq M' \cup_{g'} \mathbb{D}^{n-k} \to X$. □

At the conclusion of a surgery on $(f, b) : (M^m, \nu_M) \to (X, \eta)$ killing $x \in \pi_{n+1}(f)$ (for $n \leq \frac{m-1}{2}$), one arrives at a new map $(f_1, b_1) : (M_1^m, \nu_{M_1}) \to (X, \eta)$ with $\pi_{n+1}(f') = \pi_{n+1}(f)/\langle x \rangle$. This process continues, each time reducing the size of the relative homotopy group.

Corollary 1.62. *For* $k \leq \frac{n-1}{2}$ *a k-connected map* $f : M^n \to X$ *can be made* $(k+1)$*-connected by k-surgeries iff there exists a finite set of* $\mathbb{Z}[\pi_1(X)]$*-module generators* $x_1, \ldots, x_s \in \pi_{k+1}(f)$ *which can be killed by k-surgeries on f.*

Proof. If $k \leq \frac{n-1}{2}$, then the effect of an k-surgery killing an element $x \in \pi_{k+1}(f)$ is a map $f' : M' \to X$ with

$$\pi_i(f') = \pi_i(F) = \begin{cases} \pi_i(f) & \text{if } i \leq k, \\ \pi_{k+1}(f)/\langle x \rangle & \text{if } i = k+1. \end{cases} \qquad \square$$

Our hope is to continue executing surgery until the relative group is trivial. In fact, if we do further surgery to kill the elements of $\pi_*(f)$ without introducing new elements in the previous homotopy group, then Poincaré duality would imply that the resulting map is a homotopy equivalence. Indeed, by Poincaré duality, we only need $\pi_{i+1}(f) = 0$ for all $i \leq \left[\frac{n}{2}\right] - 1$, and we will succeed in building a manifold that is homotopy equivalent to X. However, this process is not always possible.

In the even-dimensional case $n = 2k$, the weak Whitney embedding theorem does not apply. An element $\alpha \in \pi_{k+1}(f)$ is represented by a pair $(\Phi, \phi) : (\mathbb{D}^{k+1}, \mathbb{S}^k) \to (X, M)$ such that $f \circ \phi = \Phi \circ j$:

$$\begin{array}{ccc} \mathbb{S}^k & \xrightarrow{g} & \mathbb{D}^{k+1} \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi} \\ M & \xrightarrow[f]{} & X \end{array}$$

By immersion theory, the map f can be chosen to be an immersion. In fact, the result homotopy type of such a map ϕ depends only on the class α. To perform surgery based at $\alpha \in \pi_{k+1}(f)$ we need ϕ to be regular homotopic to an embedding. It is here that we encounter a restriction in dimensions $k \geq 3$ given by an obstruction called the *self-intersection number* of an immersion. The Whitney trick is required to remove it under an algebraic condition.

Surgery in the middle dimension is the only obstruction to modifying a given degree one normal map $f : M \to X$ to a homotopy equivalence $f' : M' \to X$. If this obstruction is overcome, i.e. if the surgery obstruction $\sigma(f)$ is zero in the appropriate L-group, then the structure set $\mathcal{S}^{Cat}(X)$ is nonempty.

We would like a way to represent this obstruction. If $f : M^n \to X^n$ is a degree one normal map, denote by $K_n(M)$ the collection of framed immersions $g : \mathbb{S}^n \looparrowright M$, with a choice of path γ_g in $g(\mathbb{S}^n) \subseteq M$ from the basepoint $p \in M$ to $g(1) \in M$, whose image in X under f is nullhomotopic. Any two elements $x, y \in K_n(M)$ can be represented by immersions $g, h : \mathbb{S}^n \looparrowright M$ with transverse self-intersections. Let γ_g and γ_h be the chosen paths from p to $g(1)$ and $h(1)$. We define a pairing $\lambda : K_n(M) \times K_n(M) \to \Lambda$ by geometric intersection numbers. Define

$$D(g, h) = \{(a, b) \in \mathbb{S}^n \times \mathbb{S}^n : g(a) = h(b) \in M\},$$

which is a finite set. For each pair $(a, b) \in D(g, h)$, choose paths δ_g and δ_h in M joining $g(1)$ to $g(a)$ and $h(1)$ to $h(b)$. Define $\gamma(a, b) \in \pi_1(M) = \pi_1(X)$ to be the homotopy class of the loop in M based at p by concatenating the paths $\gamma_g * \delta_g * \delta_h^{-1} * \gamma_h^{-1}$.

Define $\varepsilon(a, b) = \pm 1$ in the following way. Choose an orientation of the tangent plane $T_p(M)$ at the basepoint p of M. Transport it to an orientation for $T_{g(a)}(M) = T_{h(b)}(M)$ by the path $\gamma_g * \delta_g$. If the isomorphism $dg \oplus dh : T_a(\mathbb{S}^n) \oplus T_b(\mathbb{S}^n) \to T_{g(a)}(M)$ is orientation-preserving, let $\varepsilon(a, b) = 1$. Otherwise set $\varepsilon(a, b) = -1$. Now define $I(a, b) \equiv \varepsilon(a, b)\gamma(a, b)$, considered as an element of $\Lambda = \mathbb{Z}[G]$. Define the *geometric* *intersection* of $x, y \in K_n(M)$ by

$$\lambda(x,y) = \sum_{(a,b)\in D(g,h)} I(a,b) \in \Lambda.$$

Some properties include the following:

1. $\varepsilon(b,a) = (-1)^n \varepsilon(a,b)$;
2. $\gamma(b,a) = w(\gamma(a,b))\gamma(a,b)^{-1}$;
3. $I(b,a) = (-1)^n \overline{I(a,b)}$.

Therefore we have the following relationship for all $x, y \in K_n(M)$:

$$\lambda(y,x) = \sum_{(b,a)\in D(h,g)} I(b,a) = \sum_{(a,b)\in D(g,h)} (-1)^n \overline{I(a,b)} = (-1)^n \overline{\lambda(x,y)}.$$

To define the quadratic function $\mu : K_n(M) \to Q_\varepsilon(\Lambda)$, let $x \in K_n(M)$ be given and let $g : \mathbb{S}^n \looparrowright M$ be an immersion with transverse self-intersections representing x. In addition let γ_g be a chosen path in M from p to $g(1)$. Consider the set of double points given by

$$D_2(g) = \{(a,b) \in \mathbb{S}^n \times \mathbb{S}^n : a \neq b \text{ and } g(a) = g(b)\}.$$

Let δ be a path from a to b in $\mathbb{S}^n$. For each $(a,b) \in D_2(g)$, let $\gamma'(a,b)$ be the homotopy class of the loop in M based at p given by $\gamma_g * (g \circ \delta) * \gamma_g^{-1}$. Let $I(a,b) = \varepsilon(a,b)\gamma'(a,b)$, where $\varepsilon \equiv \varepsilon(a,b) = \pm 1$ as defined previously. Then define μ by the equation

$$\mu(x) = \sum_{(a,b)\in D_2(g)/\mathbb{Z}_2} I(a,b) \in Q_\varepsilon(\Lambda).$$

Here the $\mathbb{Z}_2$-action on $D_2(g)$ is given by $(a,b) \mapsto (b,a)$.

In defining λ and μ many choices were made, and one can show that the resulting functions are independent of all these choices. The surgery obstruction as an element of the L-group is also independent of the choices of the preliminary surgeries.

Let $(f,b) : M^{2n} \to X$ be a degree one normal n-connected map with a given ε-quadratic form $(K_n(M), \lambda, \mu)$ over $\Lambda = \mathbb{Z}[G]$ as given above. Let $e_n : \mathbb{S}^{n-1} \times \mathbb{D}^{n+1} \hookrightarrow M$ be an embedding lying in a ball $\mathbb{D}^{2n}$. In this context, the effect of such a surgery is a map $(f_n, b_n) : M_n = M\#(\mathbb{S}^n \times \mathbb{S}^n) \to X$ with kernel Λ-modules given by

$$K_i(M_n) = \begin{cases} K_n(M) \oplus \Lambda \oplus \Lambda^* & \text{if } i = n, \\ 0 & \text{otherwise} \end{cases}$$

and kernel ε-quadratic form given by

$$(K_n(M_n), \lambda_n, \mu_n) = (K_n(M), \lambda, \mu) \oplus \mathcal{H}_\varepsilon(\Lambda),$$

where $\mathcal{H}_\varepsilon(\Lambda)$ is a hyperbolic form.

Theorem 1.63. *Let $n \geq 3$. An n-connected $2n$-dimensional degree one normal map $(f,b): M^{2n} \to X$ is normally bordant to a homotopy equivalence iff there are $k, \ell \in \mathbb{Z}_{\geq 0}$ and an isomorphism of ε-quadratic forms over $\Lambda = \mathbb{Z}[G]$ of the form*

$$(K_n(M), \lambda, \mu) \oplus \mathcal{H}_\varepsilon(\Lambda^k) \cong \mathcal{H}_\varepsilon(\Lambda^\ell),$$

i.e. if it vanishes in the L-theory group $L_{2n}(\mathbb{Z}[G])$.

Geometrically, if $\lambda(x,y) = 0$, then the intersection points between spheres representing x and y can be paired off, so that one can "sweep" the interval in the x-sphere across the interval in the y-sphere, eliminating a pair of intersection points. This procedure is called the *Whitney trick*.

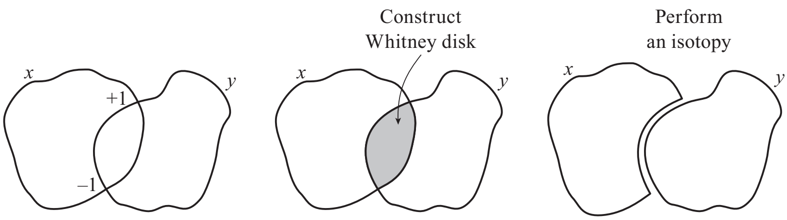


Figure 1.3: Executing the Whitney trick

1.3 THE π-π THEOREM AND ITS APPLICATIONS

This section is devoted to a discussion of the π-π theorem and a few simple applications in the computation of the structure set. We emphasize the formal power of surgery theory, and its capacity to prove results even in the absence of calculations.

The surgery exact sequence for pairs can be cast in the following way.

Theorem 1.64. *Let π and π' be groups.*

1. *For any morphism $\pi \to \pi'$ there are algebraic L-groups $L^h_*(\pi \to \pi')$ fitting into the exact sequence that the notation suggests:*

$$\cdots \to L^h_n(\mathbb{Z}[\pi']) \to L^h_n(\pi \to \pi') \xrightarrow{\partial} L^h_{n-1}(\mathbb{Z}[\pi]) \to L^h_{n-1}(\mathbb{Z}[\pi']) \to \cdots.$$

2. *The L-group of pairs has the property that every degree one Cat normal map $f: (N, \partial N) \to (X, Y)$ from a Cat n-dimensional manifold N^n with boundary to a m-dimensional Poincaré pair (X,Y) with $\pi_X = \pi_1(X)$ and $\pi_Y = \pi_1(Y)$ deter-*

mines a surgery obstruction element $\sigma(f) \in L_n^h(\pi_Y \to \pi_X)$ *such that* $\sigma(f) = 0$ *if* f *is normally bordant not relative to the boundary to a (simple) homotopy equivalence of pairs. When* $n \geq 6$*, the converse also holds.*

Note that there is no requirement on $\phi|_{\partial N}$*; i.e. we reiterate that the surgery problem here is not relative to the boundary. We will keep emphasizing this point because it is different from the terminology of "relative homology" and other conventions that exist in the literature.*

An important special case of the surgery exact sequence is the π-π theorem. Let $n \geq 6$ and consider a (simple) n-dimensional Poincaré pair with $\pi_1(X) \cong \pi_1(Y)$. Then a degree one normal map $f : (N, \partial N) \to (X, Y)$ from an n-dimensional manifold with boundary is normally bordant *not relative to the boundary* to a (simple) homotopy equivalence of pairs. From this point of view, the special case is simply the algebraic result that $L_*^h(\pi \to \pi') = 0$ for an isomorphism $\pi \to \pi'$, which is completely forced by the framework.

We state the theorem in full.

Theorem 1.65. *(π-π theorem: Wall [672]) Let* $n \geq 5$ *and let* M *be an* n*-dimensional Cat manifold with boundary* ∂M*. Suppose that the fundamental groups of* M *and* ∂M *are both isomorphic to a given group* π*. If* $\mathcal{S}^{Cat}(M, \partial M)$ *denotes the structure set of* M *not relative to the boundary, then the map* $\eta : \mathcal{S}^{Cat}(M, \partial M) \to \mathcal{N}^{Cat}(M, \partial M) \cong [M : F/Cat]$ *is a bijection.*

From the point of view of the surgery exact sequence above, in this situation the surgery map σ *is the zero map and* $L_{n+1}^h(\pi \to \pi) = 0$ *acts trivially on* $\mathcal{S}^{Cat}(M, \partial M)$*.*

Remark 1.66. *The* π-π *theorem and Wall realization hold for decorations* p *and* s*, as well as for non-trivial orientations. For the projective case, we have not given the geometric interpretation yet, but it really poses no essential new problems when we do.*

Remark 1.67. *The* π-π *theorem describes an "h-principle" for structure sets, i.e. a complete reduction to homotopy theory, much like for Smale-Hirsch immersion theory and Hirsch-Mazur smoothing theory. In this sense then, surgery theory is a systematic case study of the failure of an h-principle.*

1.3.1 Basic applications

Proposition 1.68. *Let* M^m *be a Cat manifold and* X^m *be a finite Poincaré complex of the same fundamental group. If* $f \times id_{\mathbb{S}^n} : M^m \times \mathbb{S}^n \to X^m \times \mathbb{S}^n$ *is a degree one normal map with* $n \geq 2$ *and* $m + n \geq 5$*, then the surgery obstruction* $\sigma(f \times id_{\mathbb{S}^n})$ *vanishes.*

Proof. The map $f \times id_{\mathbb{S}^n}$ is the restriction to the boundary of the degree one normal map

$$(f \times id_{\mathbb{D}^{n+1}}, f \times id_{\mathbb{S}^n}) : (M^m \times \mathbb{D}^{n+1}, M \times \mathbb{S}^n) \to (X \times \mathbb{D}^{n+1}, X \times \mathbb{S}^n).$$

The π-π theorem implies that $\sigma(f \times id_{\mathbb{D}^{n+1}}) = 0$. There is therefore a homotopy of $f \times id_{\mathbb{D}^{n+1}}$ to a homeomorphism. This homotopy however restricts to a homotopy on the boundary from $f \times id_{\mathbb{S}^n}$ also to a homeomorphism. Therefore the surgery obstruction $\sigma(f \times id_{\mathbb{S}^n})$ of the restriction function is also zero. □

Corollary 1.69. *Let X be a finite Poincaré complex of dimension at least 3. Then the following are equivalent:*

1. *$X \times \mathbb{S}^k$ is homotopy equivalent to a closed Cat manifold for some $k \geq 2$;*
2. *$X \times \mathbb{S}^k$ is homotopy equivalent to a closed Cat manifold for all $k \geq 2$;*
3. *the Spivak fibration of X has a Cat reduction.*

Corollary 1.70. *Let $f : M' \to M$ be a simple homotopy equivalence of Cat manifolds. Let ν be the map $\nu : \mathcal{S}^{Cat}(M) \to \mathcal{N}^{Cat}(M)$. Then the following are equivalent:*

1. *the map $f \times id_{\mathbb{S}^k}$ is homotopic to a Cat isomorphism for some $k \geq 2$;*
2. *the map $f \times id_{\mathbb{S}^k}$ is homotopic to a Cat isomorphism for all $k \geq 2$;*
3. *the normal invariant $\nu(f)$ is zero.*

Proof. The theorem above tells us that $f \times id_{\mathbb{S}^k} : M \times \mathbb{S}^k \to X \times \mathbb{S}^k$ has surgery obstruction zero, so $f \times id_{\mathbb{S}^k}$ is normally cobordant to a homotopy equivalence. □

With the π-π theorem, one can prove the following important theorem, the characterization of closed simply connected Top and PL manifolds.

Proposition 1.71. *(Simply connected characterization) Let Cat = PL or Top and let M^m be a closed Cat m-manifold with $m \geq 6$. If M is simply connected and M_0 is the result of removing an open m-simplex from M, then*

$$\mathcal{S}^{Cat}(M) \cong \mathcal{S}^{Cat}(M_0, \partial M_0) \cong [(M_0, \partial M_0) : (F/Cat, *)].$$

Proof. We demonstrate the first isomorphism by producing maps in either direction. Let $f : N \to M$ be a Cat manifold structure in $\mathcal{S}^{Cat}(M)$. If $v \in M$, then f can be deformed by a homotopy to be transverse to v such that $f^{-1}(v)$ itself is a point. In this case, the inverse image of a sufficiently small ball B around v is a ball B' around $f^{-1}(v)$. Therefore we have a restriction homotopy equivalence $f' : N \backslash B \to M \backslash B'$, i.e. an element of $\mathcal{S}^{Cat}(M_0, \partial M_0)$. Conversely, let $f : (N_0, \partial N_0) \to (M_0, \partial M_0)$ be a homotopy equivalence that restricts to a Cat isomorphism on the boundary. Since ∂M_0 is a sphere, we know that ∂N_0 is a homotopy sphere. By the Poincaré conjecture, it is Cat isomorphic to a standard Cat sphere. Therefore a disk can be glued onto N_0 and M_0 to give a homotopy equivalence $f : N \to M$ of closed Cat manifolds. We therefore demonstrated a well-defined map $\mathcal{S}^{Cat}(M_0, \partial_0 M) \to \mathcal{S}^{Cat}(M)$. These two maps are inverses of each other.

We now prove the second isomorphism in the theorem. If M is simply connected, then M_0 is simply connected and ∂M_0 is a sphere, which is also simply connected. Therefore the π-π theorem holds for $(M_0, \partial M_0)$. So $\mathcal{S}^{Cat}(M_0, \partial M_0)$ is in bijective correspondence with $[(M_0, \partial M_0) : (F/Cat, *)]$. □

Remark 1.72. *In Section 3.4 we will analyze the classifying spaces F/Cat and such calculations will be routine exercises in algebraic topology.*

Remark 1.73. *See Section 5.3 for an explicit rational version of Proposition 1.71 even in the smooth category and a partial extension to all fundamental groups. See also Section 6.5 for the correct rational extension to manifolds with finite fundamental group.*

Example 1.74. *In Section 3.4, we will see that $\pi_6(F/PL) \cong \mathbb{Z}_2$ as a consequence of the 5-dimensional PL Poincaré conjecture. The space $\mathbb{CP}^4$ is simply connected and the space $\mathbb{CP}^4_0$ obtained by deleting a top simplex from $\mathbb{CP}^4$ is homotopy equivalent to $\mathbb{CP}^3$. Let $f : \mathbb{CP}^3 \to \mathbb{S}^6$ be a degree one map, and let $g : \mathbb{S}^6 \to F/PL$ be a generator of $\pi_6(F/PL) \cong \mathbb{Z}_2$. Then the composition*

$$\mathbb{CP}^4_0 \simeq \mathbb{CP}^3 \xrightarrow{f} \mathbb{S}^6 \xrightarrow{g} F/PL$$

is a non-trivial PL manifold structure of $\mathbb{CP}^4$.

In Section 3.3 we will examine the PL manifold structures for $\mathbb{CP}^m$.

It is an algebraic topological theorem of Wall [667] that every Poincaré complex X^n is homotopy equivalent to $X^0 \cup_{\mathbb{S}^{n-1}} e^n$, where X^0 is an $(n-1)$-dimensional CW complex and $(X^0, \mathbb{S}^{n-1})$ is a Poincaré pair. In other words, Poincaré complexes can be punctured. With this result, the reasoning for Proposition 1.71 implies the following theorem.

Theorem 1.75. *A simply connected Poincaré complex of dimension $n \geq 5$ is homotopy equivalent to a PL (or Top) n-manifold iff it has a PL (or Top) normal invariant.*

Remark 1.76. *In Chapter 3 we will remove dependence on Wall's puncturing theorem.*

Our last example is the Browder-Wall codimension one splitting theorem. It applies in all three manifold categories.

Theorem 1.77. *(Browder [80] and Wall [672]) Suppose that M is a compact manifold and $M = M_1 \coprod_V M_2$ along a codimension one submanifold V. Suppose that the map $\pi_1(V) \to \pi_1(M_1)$ induced by inclusion is an isomorphism. Let $f : M' \to M$ be a homotopy equivalence. Then f is homotopic to a split map; i.e. there is a map $g : M' \to M$ transverse to V such that $g|_{g^{-1}(V)} : g^{-1}(V) \to V$ is a homotopy equivalence and $g|_{g^{-1}(M_i)} : g^{-1}(M_i) \to M_i$ is a homotopy equivalence for both $i = 1, 2$.*

Proof. The proof involves multiple application of the π-π theorem. First consider the map $f_\pitchfork : f_\pitchfork{}^{-1}(M_1) \to M_1$ from the transverse inverse image of M_1. The surgery obstruction $\sigma(f_\pitchfork)$ of $f_\pitchfork$ lies in $L_n(\pi_1(M_1), \pi_1(V))$. By the π-π theorem, this group vanishes. So we can find a normal cobordism $Y_\pitchfork$ from $f_\pitchfork$ to a homotopy equivalence $h : X \to M_1$. Consider the concatenated cobordism $(M' \times [0,1]) \cup Y_\pitchfork$ with a map $j : (M' \times [0,1]) \cup Y_\pitchfork \to M \times [0,2]$ as a normal map relative to $(M \times \{0\}) \cup (M_1 \times \{2\})$. Then j is another degree one normal map with surgery obstruction lying in $L_{n+1}(\pi_1(M), \pi_1(M_2))$. Using the van Kampen theorem to check the relevant isomorphism, we note that the L-group vanishes by the π-π theorem,

The result of this surgery is an h-cobordism from M' to a manifold M'' with a map having the desired properties. An additional h-cobordism erected over the inverse image of M_2 turns this h-cobordism into an s-cobordism, and therefore a homotopy of f to a map g defined on M' itself, as desired. □

Remark 1.78. *In the topological category, transversality is far from obvious. One essentially needs all the apparatus from Kirby-Siebenmann [361] to translate the PL version above into a legitimate topological argument.*

1.4 PROPAGATION OF GROUP ACTIONS

The subject of group actions on manifolds is an old and important area within topology. One of its oldest themes, initially developed by P. A. Smith, is the interaction between G-actions and homology. Smith [489] proved remarkable fixed-point theorems for group actions that relate the homological properties of a space X to the fixed set F of a G-action on X, where G is a p-group. Most notably, in this context, if X is mod p acyclic, i.e. $\widetilde{H}^*(X; \mathbb{Z}_p) = 0$, then F is also mod p acyclic. Amazing as this theorem was, it was actually believed at first that stronger results could hold with stronger hypotheses on X. However, Lowell Jones showed in his thesis [337] quite the contrary: every finite mod p-acyclic CW complex is the fixed set of a $\mathbb{Z}_p$-action on a finite contractible space, and therefore is homotopy equivalent to the fixed set of such an action on some disk. His systematic study of converses to Smith theory on the disk appears in a number of very important papers in which he develops ideas, albeit unsystematically, related to Poincaré surgery and blocked surgery with non-manifold base.

When the group order and the characteristic of the homology are relatively prime, there is little connection between F and X. For example, Smith shows that, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is periodic with period $n = p^r$ a prime power, i.e. $f^{(n)} = id_{\mathbb{R}^n}$, then the fixed set F is nonempty, since it is mod p acyclic and the empty set is not. Conversely, Conner-Raymond [134] and Kister [282] show that, whenever the period n of f is not a prime power, there is a periodic smooth transformation on a Euclidean space with no fixed points. These examples show the Jones phenomena: for a fixed-point free $\mathbb{Z}_6$-action on $\mathbb{R}^n$, the fixed set of $\mathbb{Z}_3$ need not be mod 2 acyclic: if it were, then Smith's theorem would imply a fixed point for the involution acting on it, and therefore a fixed point for the original $\mathbb{Z}_6$-action. One can modify these constructions to show that any finite-dimensional homotopy type arises as the fixed set of a $\mathbb{Z}_n$ action on some Euclidean space.

The theory of homology propagation is about systematic converses to the Smith phenomenon in the situation where we do not allow very complicated orbit structures. The question can be cast in the following way. Suppose that F is a PL manifold or smooth manifold subset of $\mathbb{D}^n$ which is p-acyclic. Furthermore, suppose that U is a regular neighborhood of F endowed with a $\mathbb{Z}_p$-action for which U is invariant and F is the fixed set. We ask if it is possible to extend the action to the complement, i.e. to find a manifold M whose boundary ∂M is the orbit space of the boundary ∂U, and whose p-fold cover M_p is exactly $\mathbb{D}^n - F$. If we can find a Poincaré $(X, \partial X)$ that solves this problem, then surgery would be helpful in producing a manifold solution, which is a π-π situation when the codimension is at least 3.

The key outcome of Smith theory is that the map $\partial U \to \mathbb{D}^n \backslash \text{Nbd}(F)$ is a $\mathbb{Z}_p$-homology equivalence. More generally, we can consider the propagation problem for free group actions. Suppose that G is a finite group and $f : M \to N$ is a homology equivalence localized at primes dividing the order of G. We ask whether it is possible to propagate a free G-action from N to M or vice versa, to obtain an action on the other space, so that the map f is homotopic to an equivariant one.

In general it is not possible to propagate this action; in fact, there are even homotopy-theoretic obstructions to this kind of propagation. However, for homologically trivial actions, propagation is frequently possible, giving an approach to converses of Smith theory. Chase [162] however gave examples when the action is not homologically trivial, where there is no homotopical propagation possible. See Section 8.2.

The goal of this section is to explain these ideas and some of their applications. We will mostly follow the work of Cappell-Weinberger [131] and Davis-Weinberger [195]. Important papers in this area include Assadi-Browder [17], Assadi-Vogel [18], Davis-Löffler [190], Jones [338, 339], and Quinn [519]. We will return to this theme in Section 6.9.

1.4.1 Basic definitions

We will frequently need localizations of spaces, both here and in our later discussion of the homotopy theory of F/Top. The fundamentals of localization are covered in Appendix A.4. Localizations will also occur in our discussion of local surgery in Section 8.2.

Let P be a set of primes. Unless otherwise noted, in this section we will always assume that our P-Poincaré complexes are endowed with an orientation.

Definition 1.79. *A P-normal invariant for a P-Poincaré complex X is a quadruple of data (M, f, ν, T) given as follows:*

1. *M is a Cat manifold;*
2. *$f : M \to X$ is a map of degree prime to P;*
3. *ν is a Cat bundle over X;*
4. *T is a stable trivialization of $f^*\nu \oplus \tau_M$.*

If P is empty, then we have a $\mathbb{Q}$-normal invariant. In this case, the only restriction on the degree of the map $f : M \to X$ is that it be nonzero.

Proposition 1.80. *If the P-Poincaré complex X has a Cat P-normal invariant, then the Spivak bundle $X \to BSF_{(P)}$ lifts to $BSCat_{(P)}$.*

Proof. The proof follows that of the classical case. □

Definition 1.81. *Two P-Poincaré complexes X_1 and X_2 with the same fundamental group G are P-locally equivalent if there is a map $f : X_1 \to X_2$ inducing isomorphisms on $\mathbb{Z}_{(P)}[G]$-homology. This f is called a P-local equivalence. If P is empty, we say that f is a rational equivalence.*

Definition 1.82. *Let X be a space and let the group G act on X. Suppose that R is a coefficient group. Define a representation $G \to \operatorname{Aut} H_*(X; R)$ as follows. Let $g \in G$ and $\phi_g : X \to X$ be the action of g on X. If $v \in H_*(X; R)$, then define $g \cdot v = (\phi_g)_*(v)$. We say that the action on X is R-homologically trivial if this representation is trivial.*

Remark 1.83. *Let $n \geq 2$ be an integer and let P be the set of prime divisors of n. Then a P-Poincaré complex X with finite fundamental group is also called a mod n Poincaré complex. The notion of a mod n equivalence between mod n Poincaré complexes should be clear. We will also write $\mathbb{Z}_{(n)}$ for $\mathbb{Z}_{(P)}$ in this case.*

Question 1.84. *The homology propagation problem asks the following. Suppose that G is a finite group of order n. Let $f : M \to N$ be a mod n equivalence between simply connected manifolds. Suppose also that R is a subring of $\mathbb{Q}$ and G acts R-homologically trivially on N. We want to know if there is an action G on M such that*

1. *G acts R-homologically trivially on M, and*
2. *the map f is equivariant under the action, i.e. $f(gx) = f(x)$ for all $x \in M$ and $g \in G$.*

If so, we then say that the action of G on N propagates to M along f (with respect to R). We are particularly interested in the case where $R = \mathbb{Z}[{}^1/_n]$.

Before heading into the next sections, we will briefly discuss the Swan homomorphism in K_0-theory which we will use for computing the Wall finiteness obstructions that arise. This homomorphism also appears in Section 6.6, where we discuss the setting studied by Swan.

Definition 1.85. *Let G be a finite group of order n. The Swan homomorphism is the map $\mathrm{Sw}_G : \mathbb{Z}_n^\times \to \widetilde{K}_0(\mathbb{Z}[G])$ defined in the following way. If $r \in \mathbb{Z}_n^\times$, then let $\mathrm{Sw}_G(r)$ be the ideal I_r generated by the augmentation ideal $I = \ker(\mathbb{Z}[G] \to \mathbb{Z})$ and r. Note that the $\mathbb{Z}[G]$-submodule I_r of $\mathbb{Z}[G]$ generated by r and I is projective and represents an element of $\widetilde{K}_0(\mathbb{Z}[G])$.*

Remark 1.86. *There are at least two other equivalent definitions, differing only by sign.*

1. *If M is a finite module of order r with trivial G-action, then a theorem of Rim [546] asserts that there is a finite projective resolution*

$$0 \to P_k \to P_{k-1} \to \cdots \to P_1 \to M \to 0.$$

The Swan homomorphism can be defined by $\sigma(r) = \sum(-1)^i[P_i] \in K_0(\mathbb{Z}[G])$. The sum is well defined by Schaunel's lemma. See for example Kaplansky [345].

2. *If $I_G = \sum_{g\in G} g$, it can be defined as the boundary $K_1(\mathbb{Z}_n) \to K_0(\mathbb{Z}[G])$ term in the Milnor-Mayer-Vietoris sequence associated to the Cartesian square*

$$\begin{array}{ccc} \mathbb{Z}[G] & \longrightarrow & \mathbb{Z}[G]/I_G \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}_n \end{array}$$

where $n = |G|$. Note that $K_1(\mathbb{Z}_n) \cong \mathbb{Z}_n^\times$. See Milnor [457].

We will be primarily interested in the cases for which the Swan homomorphism is the zero map. The Swan homomorphism is non-trivial for the quaternionic group Q_8, but it is trivial for cyclic groups. In fact, if $n \in \mathbb{Z}_8^\times$, then $\mathrm{Sw}_{Q_8}(n) = 0$ iff $n \equiv \pm 1 \bmod 8$. We will see the proof for the cyclic case in a few pages.

If p is an odd prime, then the Swan homomorphism is non-trivial for $G = \mathbb{Z}_p \times \mathbb{Z}_p$. The group $G = \mathbb{Z}_{23}$ has a nonzero K_0-group, but vanishing Swan homomorphism. (See Ullom [651] for more information.)

1.4.2 Mixing techniques

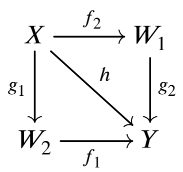
In this section we will set up some machinery of Zabrodsky to show that there are finite simply connected H-spaces that are not homotopy equivalent to Lie groups, the 7-sphere, or products of them. Zabrodsky explained how to mix two different Lie groups with the same rational homotopy type, taking one at the prime 2 and the other at odd primes. The result is an H-space that is not a Lie group.

Let $P \subseteq \mathbb{P}$ be a set of primes and let Q be the complement of P in $\mathbb{P}$. We begin with a few definitions for localizations, which also appear in Appendix A.4.

Definition 1.87. *Let P and Q be complementary nonempty sets of primes, i.e. $P \cup Q = \mathbb{P}$. Let π be a group. Denote by $\langle P \rangle$ the multiplicatively closed set generated by P in $\mathbb{Z}$. We say that π is a P-local group if, for all $n \notin \langle P \rangle$, the map $f_n : \pi \to \pi$ given by $f_n(g) = g^n$ is a bijection. In the literature, the group π is often called $\frac{1}{P}$-local if π is Q-local. Lastly, if P is empty, then a P-local group is also called $\mathbb{Q}$-local.*

Definition 1.88. *Let P be a set of primes and let U and V be spaces. A P-equivalence (or P-local equivalence) between U and V is a map $f : U \to V$ such that $f_* : \pi_1(U) \to \pi_1(V)$ is an isomorphism and $\pi_i(f) : \pi_i(U) \to \pi_i(V)$ is a Q-local group for all i. If $P = 0$, then a P-equivalence is also called a rational equivalence.*

Definition 1.89. *Let P and Q be complementary sets of primes. Then a (P, Q)-square is a homotopy commutative diagram S given by*



where f_1 and f_2 are P-equivalences, g_1 and g_2 are Q-equivalences, and h is a 0-equivalence.

Definition 1.90. *We distinguish between three subdiagrams D of a (P, Q)-square:*

1. *Type I: The pre-pullback diagram consists of just f_1 and g_2.*
2. *Type II: The pre-pushout diagram consists of just f_2 and g_1.*
3. *Type III: The pre-factoring diagram consists of just h.*

In this case, we say that S is a (P, Q)-square extending D.

Definition 1.91. *There is an obvious notion of an equivalence ϕ of diagrams $S^{(1)}$ and $S^{(2)}$, in which all the maps homotopy commute. If there is a subdiagram D as above of two (P, Q)-squares $S^{(1)}$ and $S^{(2)}$, then a map $\phi : S^{(1)} \to S^{(2)}$ is an equivalence rel D if it is an equivalence such that $\phi|_D : D \to D$ is the identity up to homotopy.*

Theorem 1.92. *(Kahn [343]) Let P and Q be complementary sets of primes and let D be a distinguished diagram (Type I, II, or III) consisting of maps of n-dimensional Poincaré complexes. Then there is a (P, Q)-square S that extends D consisting of maps of n-dimensional Poincaré complexes. Moreover, this square S is unique up to equivalence rel D.*

The process of *mixing* is as follows. As an example, we begin with (P, Q)-squares $S^{(1)}$

and $S^{(2)}$ given as follows:

$$\begin{array}{ccc} X^{(1)} & \xrightarrow{f_2^{(1)}} & W_1^{(1)} \\ {\scriptstyle g_1^{(1)}}\big\downarrow & \searrow^{h^{(1)}} & \big\downarrow{\scriptstyle g_2^{(1)}} \\ W_2^{(1)} & \xrightarrow[f_1^{(1)}]{} & Y \end{array} \qquad \begin{array}{ccc} X^{(2)} & \xrightarrow{f_2^{(2)}} & W_1^{(2)} \\ {\scriptstyle g_1^{(2)}}\big\downarrow & \searrow^{h^{(2)}} & \big\downarrow{\scriptstyle g_2^{(2)}} \\ W_2^{(2)} & \xrightarrow[f_1^{(2)}]{} & Y \end{array}$$

Consider then the Type I subdiagram $D^{(1)}$ and $D^{(2)}$ (with $g_2^{(1)}$ and $f_1^{(1)}$ in the first diagram and $g_2^{(2)}$ and $f_1^{(2)}$ in the second diagram). Consider then the mixed Type I subdiagrams $D^{(3)}$ and $D^{(4)}$

$$\begin{array}{ccc} & & W_1^{(2)} \\ & & \big\downarrow{\scriptstyle g_2^{(2)}} \\ W_2^{(1)} & \xrightarrow[f_1^{(1)}]{} & Y \end{array} \qquad \begin{array}{ccc} & & W_1^{(1)} \\ & & \big\downarrow{\scriptstyle g_2^{(1)}} \\ W_2^{(2)} & \xrightarrow[f_1^{(2)}]{} & Y \end{array}$$

and use the above theorem of Kahn to extend these diagrams to unique (P, Q)-squares $S^{(3)}$ and $S^{(4)}$ as follows

$$\begin{array}{ccc} X^{(3)} & \xrightarrow{f_2^{(3)}} & W_1^{(2)} \\ {\scriptstyle g_1^{(3)}}\big\downarrow & \searrow^{h^{(3)}} & \big\downarrow{\scriptstyle g_2^{(2)}} \\ W_2^{(1)} & \xrightarrow[f_1^{(1)}]{} & Y \end{array} \qquad \begin{array}{ccc} X^{(4)} & \xrightarrow{f_2^{(4)}} & W_1^{(1)} \\ {\scriptstyle g_1^{(4)}}\big\downarrow & \searrow^{h^{(4)}} & \big\downarrow{\scriptstyle g_2^{(1)}} \\ W_2^{(2)} & \xrightarrow[f_1^{(2)}]{} & Y \end{array}$$

where the pullbacks $X^{(3)}$ and $X^{(4)}$ are considered the results of *mixing the homotopy types of $X^{(1)}$ and $X^{(2)}$*.

Notation 1.93. *If $f : V \to W$ is a map of connected CW complexes, let $\Lambda = \mathbb{Z}[\pi_1(W)]$. If B is a right Λ-module, we use the notation $H_*(f; B)$ and $H^*(f; B)$ to denote the homology and cohomology of f with local coefficient group B. If P and Q are complementary sets of primes, let $\mathbb{Z}_{(P)}$ be the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and the reciprocals of primes in Q. For any abelian group A, let $A_P = A \otimes \mathbb{Z}_{(P)}$. If B is a right Λ-module, then so is B_P. Because $\mathbb{Z}_{(P)}$ is $\mathbb{Z}$-flat, it follows that $H_*(f; B_P) = H_*(f; B)_P$ (cf. Theorem A.51).*

Proposition 1.94. *(Kahn [343], Serre [576]) Suppose that $f : V \to W$ is a map of connected CW complexes that induces a π_1-isomorphism. Let P be a set of primes. The following are equivalent:*

1. *The map f is a P-equivalence.*
2. *The homology groups $H_*(f;\Lambda_P)$ vanish.*
3. *The homology groups $H_*(f;B_P)$ vanish for all right Λ-modules B.*
4. *The cohomology groups $H^*(f;B_P)$ vanish for all right Λ-modules B.*

We remove the requirement that f induce a π_1-isomorphism for the following definition.

Definition 1.95. *A map $f : V \to W$ of connected CW complexes is a homology P-equivalence if $H_*(f;\mathbb{Z})_P = 0$.*

Clearly every P-equivalence is a homology P-equivalence by the proposition above.

Proposition 1.96. *Let $f : V \to W$ be a homology P-equivalence of finitely dominated, connected CW complexes. Let h_* and h^* be any generalized homology and cohomology theories. Then the localizations $h_*(f;\mathbb{Z})_P$ and $h^*(f;\mathbb{Z})_P$ are both zero.*

Lemma 1.97. *Let P be a set of primes. Suppose we have a homotopy pullback diagram of the form*

$$\begin{array}{ccc} A & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & Y \end{array}$$

where all maps induce isomorphism on the fundamental group G. Suppose that $\pi_i(X_1)$ is P-local and $\pi_i(X_2)$ is $1/P$-local for all $i \geq 2$. If X_1 is P-local and X_2 is $1/P$-local and Y is $\mathbb{Q}$-local, then A is a $\mathbb{Z}[G]$-Poincaré complex; i.e. if A is an m-dimensional complex, then capping with the fundamental class $[A]$ produces an isomorphism given by $\cap [A] : H^{m-i}(A;\mathbb{Z}[G]) \to H_i(A;\mathbb{Z}[G])$.

Proof. Let $[A]$ be the fundamental class in $H_m(A;\mathbb{Z})$. It maps to a $1/P$-multiple of $[X_1]$ and a P-multiple of $[X_2]$. □

Remark 1.98. *This result enables us to combine Poincaré complexes at different primes that agree rationally. Since F/Cat is an infinite loopspace, a lift of a map $X \to BF$ to $BCat$ is equivalent to a null-homotopy in $B(F/Cat)$, a condition that can be checked one prime at a time.*

Proposition 1.99. *(Kahn [343]) Suppose that X^n is a simply connected complex with $n \geq 5$, and that $X_{(p)}$ is a closed Cat manifold for all primes p. Then X is homotopy equivalent to a closed Cat manifold.*

Remark 1.100. *The paper of Cappell-Weinberger [132] shows that this statement fails in the non-simply connected case.*

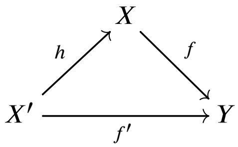
1.4.3 Propagation for CW complexes

We start by showing how to propagate homologically in the category of CW complexes, and then move on to discuss the finiteness obstruction that arises. We start with a simple lemma.

Lemma 1.101. *(Weinberger [686]) Let X be a P-Poincaré complex and suppose that $\pi_1(X)$ acts trivially on $H_*(\widetilde{X};\mathbb{Q})$. Let $\widetilde{X}_{(0)}$ be the rationalization of the universal cover $\widetilde{X}$ of X. Then there is a rational equivalence $X \to \widetilde{X}_{(0)}$; i.e. the map induces an isomorphism on rational homology.*

Proof. Attach a 2-cell for every non-trivial element of $\pi_1(X)$. Elements of the second homology group can be killed rationally by attaching 3-cells to form a space Y. The inclusion $X \hookrightarrow Y$ is a rational equivalence. Let $f : \widetilde{X} \to Y_{(0)}$ be the composite of the maps $\widetilde{X} \to X \to Y \to Y_{(0)}$. Then f is a rational equivalence of simply connected spaces. Since simply connected spaces can be localized, it follows that there is an (integral) equivalence $Y_{(0)} \to \widetilde{X}_{(0)}$. The composite $X \to Y \to Y_{(0)} \to \widetilde{X}_{(0)}$ is then a rational equivalence. □

Proposition 1.102. *(Propagation for CW complexes, Assadi-Browder [17], Cappell-Weinberger [131]) Let G be a finite group with order n. Let $R = \mathbb{Z}$ or $\mathbb{Z}[^1/_n]$ and let $f : X \to Y$ be a $\mathbb{Z}_{(n)}$-homology equivalence between simply connected CW complexes. Suppose that G acts freely and R-homologically trivially on Y. Then there is a CW complex X', not necessarily finite, with a free R-homological G-action such that the diagram*



homotopy commutes, where h is a homotopy equivalence and f' is G-equivariant. Furthermore, if X and Y/G are finitely dominated, then so is X'/G.

Proof. We give a sketch proof. The strategy is to use Y/G at the primes dividing n and X at the remaining primes using localization theory. Examples of such mixing constructions are also seen in Section 1.4. Let W be a CW complex and P be a set of primes. Using the fiberwise localization functor of Bousfield and Kan [73, Ch 5], one can construct a complex $W_{(P)}$ and a map $W \to W_{(P)}$ inducing an isomorphism of fundamental groups and a localization of higher homotopy groups; i.e. $\pi_1(W_{(P)}) = \pi_1(W)$ and $\pi_i(W_{(P)}) = \pi_i(W)_{(P)}$ for all $i \geq 2$. In our original set-up, we know that

Y/G is the homotopy pullback of the diagram

$$\begin{array}{ccc} & & (Y/G)[1/n] \\ & & \downarrow \\ (Y/G)_{(n)} & \longrightarrow & (Y/G)_{(0)} \end{array}$$

By the lemma, since G acts $\mathbb{Z}[1/n]$-homologically trivially on Y, there is a homotopy equivalence $(Y/G)[1/n] \to Y[1/n] \times BG$. Then let Z be the homotopy pullback of

$$\begin{array}{ccc} & & X[1/n] \times BG \\ & & \downarrow \\ (Y/G)_{(n)} & \longrightarrow & X_{(0)} \times BG \end{array}$$

This homotopy pullback Z is finitely dominated because it is finitely dominated at every prime. The desired X' is the G-cover of Z. Since the roles of X and Y can be reversed, the actions propagate in both directions. □

To study the propagation problem for finite complexes, we first present a theorem of Mislin which gives a relationship between the Wall finiteness obstructions of certain finitely dominated CW complexes with the same fundamental group.

Definition 1.103. *Let G be a group with order n. Suppose that $f : Y \to X$ is a $\mathbb{Z}_{(n)}$-local equivalence of finitely dominated mod n Poincaré complexes with fundamental group G. If M_f is the mapping cylinder of f, we define the Wall obstruction of f to be the quantity*

$$w(M_f) = \sum_{i \geq 0} (-1)^i \mathrm{Sw}_G([H_i(\widetilde{M}_f; \mathbb{Z})]) \in \widetilde{K}_0(\mathbb{Z}[G]).$$

Remark 1.104. *We use the notation $H_k(\widetilde{M}_f)$ to indicate the modules of the universal cover. As in the definition of the Swan homomorphism, we can consider $[H_k(\widetilde{M}_f)]$ as an element of $\widetilde{K}_0(\mathbb{Z}[G])$ by taking a finite resolution. With abuse of notation, we called it $\mathrm{Sw}_G([H_k(\widetilde{M}_f)])$. When the action on homology is trivial, then $\mathrm{Sw}_G([H_k(\widetilde{M}_f)]) = \mathrm{Sw}_G(|H_k(M_f)|)$. Note that, if f is a $\mathbb{Z}_{(n)}$-homology equivalence, then $|H_i(M_f; \mathbb{Z})|$ is coprime to n, so that $|H_i(M_f; \mathbb{Z})|$ is genuinely in $\mathbb{Z}_n^{\times}$ for all i. The homology of the mapping cone is the same on the spaces and their covers in the homologically trivial case.*

Proposition 1.105. *(Mislin [465]) Let X and Y be finitely dominated CW complexes with fundamental group G of order n. Let $f : \widetilde{X} \to \widetilde{Y}$ be G-equivariant and a $\mathbb{Z}_{(n)}$-homology equivalence. Then*

$$w(Y) = w(X) + w(M_f) \in \widetilde{K}_0(\mathbb{Z}[G]),$$

where $w(X)$ and $w(Y)$ denote the usual Wall finiteness obstructions of X and Y.

Proof. From the exact sequence

$$0 \to C_*(X) \to C_*(Y) \to C_*(Y, X) \to 0,$$

we have a relationship $w(X) = w(Y) - w(Y, X)$ for the Wall finiteness obstructions for these spaces. Since $H_i(M_f; \mathbb{Z}_{(n)}) = 0$, these homology groups have finite projective length. Therefore

$$w(C_*(Y, X)) = \sum_{i \geq 0} (-1)^i [H_i(M_f; \mathbb{Z})] = \sum_{i \geq 0} (-1)^i \mathrm{Sw}_G(|H_i(M_f; \mathbb{Z})|). \qquad \square$$

Corollary 1.106. *The Swan homomorphism $\mathrm{Sw}_G : \mathbb{Z}^{\times}_{|G|} \to \widetilde{K}_0(\mathbb{Z}[G])$ is the zero map if G is a finite cyclic group.*

Proof. A lens space is the quotient of an odd-dimensional sphere by a free cyclic action. Let $r \in \mathbb{Z}^{\times}_{|G|}$. There is a map $L_1 \to L_2$ of lens spaces that lifts to a degree r map $f_r : \mathbb{S}^{2k-1} \to \mathbb{S}^{2k-1}$. Lens spaces are discussed extensively in Section 6.5. The associated Wall finiteness obstruction is clearly zero, and has two nonzero terms $\mathrm{Sw}_G(|H_0(M_{f_r}; \mathbb{Z}_{(n)})|) - \mathrm{Sw}_G(|H_{2k-1}(M_{f_r}; \mathbb{Z}_{(n)})|)$. Then $0 = \mathrm{Sw}_G(1) - \mathrm{Sw}_G(r)$. But $\mathrm{Sw}_G(1) = 0$ by definition, so $\mathrm{Sw}_G(r) = 0$. $\square$

The reader will find a discussion about group actions on homology spheres in Section 6.9.

Corollary 1.107. *Suppose that Σ is a simply connected $\mathbb{Z}_{|G|}$-homology n-sphere. If a group G acts freely on the standard sphere $\mathbb{S}^n$, then it acts freely on a finite complex that is homotopy equivalent to Σ.*

Proof. Again let G be a finite group of order n. We then arrive at the equation

$$w((\mathbb{S}^m)'/G) = w(\Sigma'/G) - \mathrm{Sw}_G(d) - \sum_{i=1}^{m-1} (-1)^i \mathrm{Sw}_G(|H_i(\Sigma; \mathbb{Z})|)$$

where $d = \deg(f)$. We then use the fact that $\mathrm{Sw}_G(s) + \mathrm{Sw}_G(t) = \mathrm{Sw}_G(st)$ and $\mathrm{Sw}_G(r) = 0$ for all $s, t \geq 2$ and $r \equiv 1 \bmod n$. We can choose the degree of f so that $w((\mathbb{S}^m)'/G) = w(\Sigma'/G)$ and the corollary follows. $\square$

1.4.4 Extension across homology collars

In this section, we apply the π-π theorem of Wall to solve an important manifold propagation problem. This application has direct relevance to the problem of group actions on the disk that motivated our whole discussion.

Theorem 1.108. *(Extension on homology collars, Assadi-Browder [17] and Weinberger [686]) Suppose that* (W, M, N) *is a triad of simply connected manifolds with* $H_*(W, M; \mathbb{Z}_{(n)}) = 0$. *Let* M *admit a free cyclic* $\mathbb{Z}_n$*-action that is* $\mathbb{Z}[^1/_n]$*-homologically trivial. Then the action on* M *extends to such an action on all of* W.

Proof. We propagate in the CW complex category from M to W and then from W to N. Since the Swan homomorphism for $\mathbb{Z}_n$ vanishes, the homotopy quotients are finite. We leave to the reader the verification that we have produced a Poincaré pair (or triad relative to one manifold M/G). The triple $(W'/\mathbb{Z}_n, M'/\mathbb{Z}_n, M/\mathbb{Z}_n)$ has a normal invariant relative to $M/\mathbb{Z}_n$ because the relative $B(F/Cat)$ cohomology group vanishes at $|G|$, and because the projection map of the G-cover is an isomorphism away from $|G|$ by homological triviality.

By the π-π theorem, the map $\mathcal{S}^{Top}(W'/\mathbb{Z}_n, M/\mathbb{Z}_n) \to [(W/M)/\mathbb{Z}_n : F/Top]$ is a bijection, and without loss of generality, we can then assume that $W'/\mathbb{Z}_n$ and $M'/\mathbb{Z}_n$ have manifold structures. With the assumption that $H_*(W, M; \mathbb{Z}_{(n)}) = 0$, it follows that $W/M \to (W/M)/\mathbb{Z}_n$ is a homotopy equivalence and the projection map

$$[(W/M)/\mathbb{Z}_n : F/Top] \to [W/M : F/Top]$$

is an isomorphism. The map induced by taking covers is easily seen to give a commutative diagram

$$\begin{array}{ccc} \mathcal{S}^{Top}(W'/\mathbb{Z}_n, M/\mathbb{Z}_n) & \longrightarrow & [(W/M)/\mathbb{Z}_n : F/Top] \\ \Big\downarrow tr & & pr \Big\downarrow \cong \\ \mathcal{S}^{Top}(W', M) & \longrightarrow & [W/M : F/Top] \end{array}$$

The π-π theorem is used again to show that the top line is an isomorphism, implying that the transfer map $\mathcal{S}^{Top}(W'/\mathbb{Z}_n, M/\mathbb{Z}_n) \to \mathcal{S}^{Top}(W', M)$ is a bijection. □

The following is the extension of the previous theorem to noncyclic groups.

Theorem 1.109. *(Assadi-Browder [17] and Weinberger [686]) Let* $(M^m, \partial_0 M, \partial_1 M)$ *be a simply connected triad of manifolds with* $m \geq 6$. *Let* G *be a finite group of order* n *and suppose that there is a* $\mathbb{Z}[^1/_n]$*-homologically trivial* G*-action on* $\partial_0 M$. *If* $H_*(M, \partial_0 M; \mathbb{Z}_{(n)}) = 0$*, then the* G*-action on* $\partial_0 M$ *extends to such an action on* M *iff*

$$\sum_{i \geq 0} (-1)^i \mathrm{Sw}_G(|H_i(M, \partial_0 M; \mathbb{Z})|) = 0 \in \widetilde{K}_0(\mathbb{Z}[G]).$$

Proof. As before, construct a homotopy quotient of M so that $\partial_0 M \to M$ is equivariant and then again use the homology isomorphism $\partial_1 M \to M$ to do the same for $\partial_1 M$. The finiteness of the triad is from the formula in Section 2. One shows that it has a normal invariant rel $\partial_0 M/G$, which involves homological triviality and localization theory for the space F/Cat. The π-π theorem applies to show that there is a manifold triad homotopic to the original one on which an extension is possible, and finally another surgery argument shows that the extension is possible on the original manifold. □

Remark 1.110. *When we propagate, it is not necessary that the space from which we propagate be simply connected. We only require that the space to which we propagate the action be simply connected, since it is in this space that we use homotopy theory to execute a construction. Therefore simple connectivity can be slightly weakened in all the above theorems.*

Corollary 1.111. *If F is a properly embedded smooth submanifold of $\mathbb{D}^n$ of even codimension at least 4, then F is the fixed set of an involution iff F is mod 2 homology acyclic. In this case, the smooth action is unique.*

Needless to say, much more can be said about these kinds of questions. The above is for illustration alone.

In addition to the extension of a group action across an h-cobordism, one can ask whether a group action can be propagated across a $\mathbb{Z}_{(n)}$-homology equivalence. In this case, the following theorems show that, in addition to the Wall obstruction, there is another obstruction relating to the normal invariant. The reader can consult Cappell-Weinberger [131] for the proof of the following.

Theorem 1.112. *(Cappell-Weinberger [131], Theorem 4.2) Let G be a finite group with order n. Let $(f, \partial f): (M^m, \partial M) \to (N^m, \partial N)$ be a $\mathbb{Z}_{(n)}$-homology equivalence of manifold pairs with $m \geq 6$. Let $\nu_{(n)}(f)$ be the normal invariant in $[N : F/Cat_{(n)}]$. Suppose that all spaces are simply connected and nonempty. In addition, suppose that G acts freely and $\mathbb{Z}[1/n]$-homotopically trivially on $(N, \partial N)$. Then there is a G-action on $(M, \partial M)$ with a homotopy between f and an equivariant map iff*

1. *$\sum_{i\geq 0}(-1)^i \mathrm{Sw}_G(|H_i(M_f; \mathbb{Z}|)$ and $\sum_{i\geq 0}(-1)^i \mathrm{Sw}_G(|H_i(M_{\partial f}; \mathbb{Z})|$ both vanish in $\widetilde{K}_0(\mathbb{Z}[G])$;*
2. *$\nu_{(n)}(f)$ belongs to the image of $[N/G : F/Cat_{(n)}] \to [N : F/Cat_{(n)}]$; i.e. the $\mathbb{Z}_{(n)}$-normal invariant is equivariant.*

Moreover, if (1) and (2) hold, then actions on $(N, \partial N)$ homotopy equivalent to the given one are in bijective correspondence with actions on $(M, \partial M)$ such that f can be taken to be equivariant.

Remark 1.113. *In Section 6.9 we will return to the problem of propagation for closed manifolds, after we have a deeper understanding of surgery for non-trivial L-groups. Extension across homology collars suggests an approach that uses $\mathbb{Z}_n$-homology h-cobordisms. The problem of classifying manifolds up to this particular equivalence relation is discussed in Section 8.2 on local surgery, and indeed has been applied to problems of propagation; however, we shall not pursue that theme in this book.*

1.5 WALL CHAPTER 9

Many mathematics problems can be successfully algebraicized, and in fact can be algebraicized in a seemingly trivial way: we can form a group generated by the problems, where two problems are equivalent if their difference can be solved. In this scenario, one always needs some non-trivial construction, such as the definition of the difference, or perhaps a non-trivial theorem, e.g. a cancellation result that says that a union of two problems, one of which is known to be solvable, is itself solvable iff both problems are solvable.

One example in topology in which this method is possible is the theory of Whitehead torsion, as developed in Cohen [168]. The Whitehead group of a CW complex X is constructed by finite complexes Y containing X as a deformation retract such that two such complexes Y_1 and Y_2 are equivalent if one can be obtained from the other using elementary expansions and collapses. This method develops a group $\mathrm{Wh}(X)$ for X, but it does not explain why $\mathrm{Wh}(X)$ depends only on the fundamental group $G = \pi_1(X)$ of X. It certainly does not explain why $\mathrm{Wh}(X)$ is isomorphic to the cokernel of the map

$$G \times \{\pm 1\} \to \mathrm{GL}_\infty(\mathbb{Z}[G]), \quad (g, \pm 1) \mapsto (\pm g)$$

which is our usual construction of the Whitehead group. The geometric version of $\mathrm{Wh}(X)$ is adequate, however, for proving sum and product formulas for torsion, but it has the notable advantage of easily generalizing to many other settings (e.g. proper, or equivariant, or topological).

As we mentioned in Section 1.2, Wall adopted a similar approach for the surgery obstruction groups $L_n(X)$ for $n \geq 5$. His reason was that, although he had given algebraic definitions for $L_n(\mathbb{Z}[\pi_1(X)])$, he could not find one for pairs of groups, i.e. for $L_n(\pi_1(X), \pi_1(Y))$, or worse, if Y were disconnected. However, by working geometrically he was able to construct a relative group for pairs (X, Y), where Y is a subspace of X. This group fits into a surgery exact sequence, and satisfies the obvious exact sequence of a pair:

$$\to L_n(\mathbb{Z}[\pi_1(X)]) \to L_n(\pi_1(X), \pi_1(Y)) \to L_{n-1}(\mathbb{Z}[\pi_1(Y)]) \to L_{n-1}(\mathbb{Z}[\pi_1(X)]) \to$$

The sequence would depend only on (1) the fundamental groups of the manifold and the components of the boundary (with inclusion homomorphisms), (2) the orientation character, and (3) the integer n mod 4 (on crossing with $\mathbb{CP}^2$ and using the five-lemma).

We will see in Chapter 8 that his definition for $L_*(\pi_1(X), \pi_1(Y))$ directly generalizes to a theory of L-groups for stratified spaces (at least for strongly stratified spaces). Additionally, the cobordism method developed by Wall directly foreshadows the simplicial methods of Casson and Quinn, which we will explain in Chapter 4. These simplicial methods create a topological theory for which fibrations responsible for the surgery exact sequence give an unexpected group structure on topological structure sets.

We describe Wall's definition in the absolute case and make a few comments. The verification of its key properties can be found in Wall [672].

Definition 1.114. *Let X be a space. An element ϑ of $L_n^h(X)$ consists of*

1. *a manifold pair $(P, \partial P)$ with a map $f : P \to X$,*
2. *a degree one normal map $\phi : (M, \partial M) \to (P, \partial P)$ for which the restriction map $\phi|_{\partial M} : \partial M \to \partial P$ is already a homotopy equivalence. (We are suppressing the normal data from our notation.)*

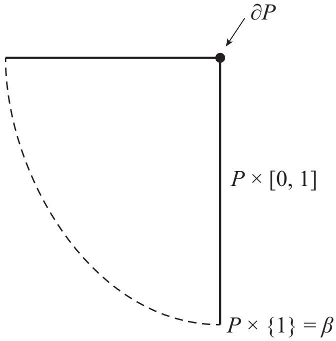


Figure 1.4: A schematic of a cobordism

At first glance, statement (2) seems to be defining elements in $L_n(P)$ and not in $L_n(X)$. However, statement (1) gives us a map from P to X, so that by functoriality we can push elements from $L_n(P)$ to $L_n(X)$.

One might ask why we need manifolds with boundary instead of just closed manifolds. There are a number of reasons. It turns out that closed manifolds do not provide enough elements in L-groups so that Wall realization is possible. This phenomenon is related to the oozing problem discussed in Section 6.7 and also to the fact that signature is not multiplicative in finite coverings of Poincaré complexes as it is for closed manifolds. We discuss the latter in Chapter 2. Morever, we would like the theory to accommodate problems of manifolds with boundary, working relative to the boundary.

Remark 1.115. *The theory described here makes sense in all manifold categories and gives the same L-groups. While clear from the proof of the surgery theorems, it is not a priori obvious. If we used smooth manifolds in the definition of the L-group, then it would not be obvious that topological surgery problems give rise to surgery obstructions! A posteriori, because of the proof of the surgery theorems on closed manifolds along the lines discussed in Section 1.2, this weird possibility does not occur. A recent paper that exploits the fact that topological surgery obstructions have smooth representatives is Weinberger-Xie-Yu [695].*[2]

It remains to construct an equivalence relation such that elements corresponding to solv-

[2] A very rare instance when Weinberger is alphabetically first in a triply authored paper.

able problems are trivial in the group. Since surgery obstructions measure the obstruction to finding a cobordism to homotopy equivalences, it makes sense that the required equivalence relation is based on bordism. In fact, we will consider cobordisms of pairs.

Definition 1.116. *A null-cobordism of an object, as above, consists of:*

1. *a Poincaré object Q with boundary ∂Q, so that $\partial Q = (P, \partial P) \cup (B, \partial P)$ is a decomposition of the boundary into a union of Poincaré pairs,*
2. *a degree one normal map $\Phi : (W, \partial W) \to (Q, \partial Q)$ from a manifold object such that $\partial W = (M, \partial M) \cup (N, \partial M)$ is a decomposition corresponding to that of ∂Q, and such that the restriction $\Phi|_N : N \to B$ is already a homotopy equivalence.*

It is an exercise to show that a difference $\vartheta - \vartheta'$ is null-cobordant iff there is a cobordism between ϑ and ϑ', and that a homotopy equivalence of pairs is trivial in this theory.

Remark 1.117. *Note that the presence of the boundary component $(B, \partial P)$ is the reason that the surgery problem $P \to P$ is still trivial even when P is not null-cobordant. In addition, Wall had to prove that every element of $L^h_n(X)$ can be represented by one for which $P \to X$ is a 1-equivalence.*

Once these pieces have been constructed, the π-π theorem then guarantees that, for any element ϑ of this form representing 0 in $L^h_n(X)$ considered as a surgery problem with range $(P, \partial P)$, there is a normal cobordism between ϑ and a homotopy equivalence of pairs. This result quickly implies the surgery exact sequence.

Remark 1.118. *The complexity of the objects needed for defining these groups for pairs in the same way will be twice as big as for the absolute L-groups. But the meaning should be clear; if L-groups are made of cycles, then the cycles for the pair are cycles on the boundary that are boundaries of chains in the bigger space, just as for the usual homology of a pair, and then considered up to bordism of such objects. Other than a theory for pairs, one can also define L^s-groups and other variants.*

1.6 ALGEBRAIC SURGERY THEORY

The algebraic theory of surgery is a variation of the idea of the previous section, namely that L-groups should be viewed as cobordism groups, but executed here in a completely algebraic setting. The theory is due to Ranicki [530], who vigorously developed it in many papers and books. It has a precursor in the work of Miščenko [466], which unfortunately did not deal with the prime 2 correctly. As our goal is merely to introduce the reader to these ideas, we will not discuss the complexities required at the prime 2. However, we hope that our explanation gives the reader some useful perspective.

Let Λ be a ring with anti-involution. An *n-dimensional algebraic Poincaré complex* C, or algebraic Poincaré pair (C, B), over Λ is a chain complex of finitely generated free (or projective or bound) Λ-modules $C_n \to \cdots \to C_0$ with a chain homotopy equivalence $\phi\colon C_* \to C^{n-*}$ to its dual $C^0 \to \cdots \to C^n$. The subtleties involved are in (1) the idea of symmetry and (2) quadratic refinement. Notice that the dual $D\phi$ of ϕ is another map $C_* \to C^{n-*}$. We want ϕ and $D\phi$ to be chain homotopic.

Let Φ be a chain homotopy between ϕ and $D\phi$. But then $D\Phi$ is another such chain homotopy. So again they should be chain homotopic, and the process can continue. The reader is right to be reminded of simplicial approximations to the diagonal. If 2 is inverted in Λ, we can replace ϕ by $\frac{1}{2}(\phi + D\phi)$ and obtain something self-dual, and there is no need to keep track of infinitely many higher homotopies. In addition, there is no concern about quadratic refinement, as we saw for the algebraic definition for L-groups.

Ranicki defined two different theories modulo decorations, the quadratic groups $L_n(\Lambda)$ and the symmetric groups $L^n(\Lambda)$, via the cobordism classes of such objects. Addition on these sets is given by direct sum. The former is relevant directly to surgery and has the quadratic refinement. The latter is an analogue of bordism of manifolds, rather than bordism of surgery problems, and only differs from the former in the issue of the quadratic refinement. They therefore agree when 2 is inverted in either Λ or in the L-group.

Ranicki proved was that, with the right definitions, one can perform surgery on quadratic chain complexes until they are concentrated in the middle dimension, but not necessarily on symmetric complexes. In other words, for $n = 2k$, there is just one group C_k with an isomorphism $\phi\colon C_k \to C^k$ with $\phi = D\phi$, i.e. exactly an element of $L_n(\Lambda)$, which can be obtained by taking the adjoint of the map $C_k \to C^k$ to arrive at a symmetric bilinear form $C_k \otimes C_k \to R$.

The equivalence relation here is given by cobordism. Here one can have just one nonzero group in the bounding chain complex $B = B_k$. Considering $L_k = \ker(C_k \to B_k)$ and the role of the duality, one sees that C_k is automatically the hyperbolic form on $L_k \oplus L^k$. Here L is called a Lagrangian for the quadratic form.

The odd L-groups are more difficult to define. There is a philosophy that, when one wishes to define a sequence of functors, the $(i + 1)$-st should consist of inequivalent ways of comparing trivializations of elements of the i-th functor. For example, the $(i + 1)$-st homotopy group $\pi_{i+1}(X)$ is the collection of maps of $\mathbb{S}^{i+1}$ into a space X, which is a pair of null-homotopies of a map of $\mathbb{S}^i$ into X. As another example, the K-group $K_1(R)$ is made of a pair of isomorphisms of a free module into some projective R-module.

From this point of view, one can construct $L_{2k+1}(\Lambda)$ from automorphisms of hyperbolic forms, which was Wall's point of view, or as a quadratic forms, necessarily hyperbolic, with a pair of Lagrangians. Alternatively, we can view them as a pair of null-cobordisms of an even-dimensional algebraic Poincaré complex. Ranicki uses the term *formation* to describe the latter.

Definition 1.119. *A nonsingular $(-1)^k$-quadratic formation (K, λ, μ, F, G) is a nonsingular $(-1)^k$-quadratic form (K, λ, μ) together with an ordered pair of Lagrangians*

F, G.

Definition 1.120. *The* $(2k+1)$*-dimensional quadratic L-group* $L_{2k+1}(\Lambda)$ *is a group of stable isomorphism classes of nonsingular* $(-1)^k$*-quadratic formations* (K, λ, μ, F, G) *over* Λ*, where stability is with respect to the formations such that either* F *and* G *are direct complements in* K *or share a common Lagrangian complement in* K*.*

The picture to have in mind is perhaps a Heegaard splitting in 3-manifold topology. The importance here is not in the splitting itself (the Heegaard surface), but in the pair of complexes of curves, i.e. the Lagrangians, even though in the Heegaard splitting one considers them "generated." For the trivial example of the sphere, the Lagrangians are orthogonal, like longitude and meridian in a torus.

This formulation of surgery theory is at times extremely useful. First of all, one can define surgery obstructions using chain complexes instead of concentrated ones without performing preliminary surgeries (although it was possible already in Wall's Chapter 9 approach). Second, the symmetric theory, which only differs slightly, i.e. at the prime 2, is an excellent place to hold invariants of manifolds. For example, Ranicki defines a pairing

$$L_n(\mathbb{Z}[\pi]) \otimes L^m(\mathbb{Z}[\Gamma]) \to L_{n+m}(\mathbb{Z}[\pi \times \Gamma])$$

that describes the effect of crossing a surgery problem with fundamental group π with a closed manifold with fundamental group Γ. One can also compute the symmetric groups $L^m(\mathbb{Z}[e])$:

$$L^m(\mathbb{Z}[e]) = \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 \mod 4, \\ \mathbb{Z}_2 & \text{if } m \equiv 1 \mod 4, \\ 0 & \text{if } m \equiv 2, 3 \mod 4. \end{cases}$$

Here the $L^0(\mathbb{Z}[e]) = \mathbb{Z}$ is the signature, and $L^2(\mathbb{Z}[e]) = \mathbb{Z}_2$ is the *de Rham invariant* of a $(4k+1)$-dimensional manifold, defined as the rank of 2-torsion in $H_{2k}(M^{4k+1})$ taken mod 2. Ordinarily L^*-groups are not 4-periodic, but they are 4-periodic in the simply connected case. Ranicki's pairing gives a simple approach to Morgan's product formula [472].

Remark 1.121. *Weiss [697] has defined a variation on symmetric L-groups denoted by* $VL^*(\pi)$*, where* V *stands for "visible," that is very useful and occasionally more computable. For example, whenever the Borel conjecture is true for a group* π*, one can compute* $VL^*(\pi)$ *homologically. We devote Section 5.1 to a discussion on the role of the Borel conjecture in surgery theory.*

An indirect application of the closeness of the connection between quadratic L_* and symmetric L^* is a general multiplicativity of signature-type invariants. For L_* it follows from the π-π theorem that the surgery obstruction of any pullback bundle which has no monodromy, i.e. is trivial on the 1-skeleton, is given by a product. For signatures, such products are much less clear, since their definition seems to use the global structure. However, on inverting 2, one can deduce it from the quadratic case. For higher signatures with $\pi_1 = \mathbb{Z}^n$ such multiplicativity was proved by Lusztig [414]. A general principle appears in Lück-Ranicki [410]. See also Weinberger [691].

Third, this geometric language for algebraic objects enables one to perform various processes like "glueing" together algebraic Poincaré complexes with boundary. In the next chapter, we will compute $L_1^h(\mathbb{R}[\mathbb{Z}_2])$ in this way. See Weinberger [693] for a manifold invariant of this sort. Of course, Wall's geometric definition of the relative L-groups can be repeated in this setting. Like Wall's Chapter 9 approach, these algebraic objects have the flexibility to be "spacified" so that algebraic surgery spectra can be built. We discuss spacifications in Chapter 4.

Chapter Two

Some calculations of L-groups

The previous chapter used surgery in the absence of any L-group computations and of the structure of the classifying spaces F/Cat. In order to make effective use of surgery, both issues have to be addressed. In Chapter 3, we will focus on the homotopy theory; this chapter concerns the L-groups themselves.

This chapter introduces the reader to some of the techniques that enter into L-theory. The nature of L-groups for groups π is very different if π is finite or if π is torsion-free. The latter case seems to be governed by rigidity phenomena and is conjecturally very closely related to group homology. Hints of these ideas will appear in this chapter (see in particular Section 2.7 for the discussion of splitting, for example). Finite groups, however, are studied algebraically, and most of the chapter will be dedicated to them.

The first case $L_*(\mathbb{Z}[e])$ was tacitly calculated by Kervaire-Milnor in their paper on exotic spheres [356]. They showed that surgery is always possible for odd-dimensional manifolds, but that there is a signature obstruction in dimensions $n \equiv 0 \bmod 4$, as well as a mod 2 invariant, called the *Arf invariant* in dimensions $n \equiv 2 \bmod 4$. These results are summarized in the table

$n \bmod 4$	0	1	2	3
$L_n(\mathbb{Z}[e])$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0

Since even-dimensional L-groups are related to quadratic forms with a certain kind of symmetry, we will review enough about quadratic forms over $\mathbb{Z}$ in the first section to explain the Kervaire-Milnor results. In the next section, we will describe some facts about symmetric bilinear forms over fields $\mathbb{F}$.

We will next exploit our knowledge of fields to obtain information about $L_*(\mathbb{Z}[\pi])$ when π is finite. The most obvious step is to consider the map $\mathbb{Z}[\pi] \to \mathbb{Q}[\pi]$ and use the Wedderburn decomposition of the latter to detect non-trivial elements. Indeed, one can go even farther to $\mathbb{R}[\pi]$, where the decomposition only involves $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. The theory of multisignatures and (essentially equivalently) the G-signature arising from these decompositions of $\mathbb{R}[\pi]$ is adequate for understanding $L_*(\mathbb{Z}[\pi])$ modulo torsion. We follow this discussion with a section that explains how to use other semisimple quotients of $\mathbb{Z}[\pi]$ to obtain useful invariants of L-groups. Here we were motivated by work of Pardon [497] and Davis [186].

The model of representation theory suggests a role for induction theory, which can leverage information about easy groups to more general ones. The induction theory for L-theory is Dress induction. It enables one, for example, to compute $L_*(\mathbb{Z}[\pi])$ for π of odd order in terms of its cyclic subgroups. It will be critical later for two important theorems: the solution to the spherical spaceform problem and the proof of the Borel conjecture for flat manifolds. In order to support the latter application, our approach is informed by the paper of Farrell-Hsiang [224], which is a variation on the original paper by Dress [207].

We then return for a few last remarks about finite groups. As is well known from number theory, quadratic forms should be studied adelically and use all completions; i.e. we need to combine all information arising from p-adic group rings. It would add another century of writing in order to include a complete discussion, so we will merely summarize some useful general features.

The classical Bass-Heller-Swan formula for $K_*(R[t, t^{-1}])$ in terms of $K_*(R)$ is sometimes called the *fundamental theorem of algebraic K-theory*. One can prove the analogous formula in L-theory algebraically, but we will adopt a geometrical approach (at least when R is an integral group ring), following the original approach of Shaneson [580], viewing $\mathbb{Z}[\pi][t, t^{-1}]$ as $\mathbb{Z}[\pi \times \mathbb{Z}]$. By iterating this perspective a number of times, we can compute $L_*(\mathbb{Z}[\mathbb{Z}])$ with a formula that very much resembles group homology or cohomology. The explanation for this resemblance lies in the Borel conjecture and the assembly map, which we will discuss in Section 4.4.

2.1 CALCULATING $L_*(\mathbb{Z}[e])$

To find smooth structures on the sphere, Kervaire and Milnor asked which framed cobordism classes of smooth n-manifolds, i.e. elements of π_n^S, contain homotopy spheres. To answer this question, they performed surgery, just as in the process described in Section 1.2, and found no obstructions except in dimensions 2 mod 4. The obstruction is the Kervaire-Arf invariant, which is related to the invariant that Kervaire used to construct the first-known PL manifold with no smooth structure.

When studying uniqueness, they had to perform surgery on a framed cobordism of a homotopy sphere to the sphere, and found another obstruction when the cobordism was of dimension 0 mod 4. A nonzero signature of the framed cobordism would prevent surgery. Indeed, this signature is the one used by Milnor in his original paper when he discovered the exotic 7-sphere.

For surgery theory, we need to understand $L_*(\Lambda)$ when $\Lambda = \mathbb{Z}[\pi]$ is the integer group ring for a group π. The involution on $\mathbb{Z}[\pi]$ is given by $\overline{\sum a_g g} = \sum a_g g^{-1}$ in the oriented case, and $\overline{\sum a_g g} = \sum a_g w(g) g^{-1}$ in the nonoriented case, where $w : \pi \to \mathbb{Z}_2$ defines the orientation. We immediately specialize to trivial π.

When π is trivial, then the involution is also trivial. To compute $L_0(\mathbb{Z}[e])$, we need to classify triples (P, λ, μ), where P is a free $\mathbb{Z}$-module and the form $\lambda: P \times P \to \mathbb{Z}$ is symmetric. Additionally, the quadratic form $\mu: P \to \mathbb{Z}$ satisfies

$$\lambda(x, y) = \mu(x + y) - \mu(x) - \mu(y),$$

and is determined by $\lambda: P \times P \to \mathbb{Z}$. By way of the formulas

$$\mu(x + x) - 2\mu(x) = \lambda(x, x)$$

and $\mu(2x) = 4\mu(x)$, we have $2\mu(x) = \lambda(x, x)$. Therefore the only symmetric bilinear forms under consideration are those for which $\lambda(x, x)$ is even for all $x \in P$. Conversely, given such a form, one obtains a well-defined element of the L-group defining $\mu(x) = \frac{1}{2}\lambda(x, x)$. These observations motivate the first half of the following definition.

Definition 2.1. *Let (P, λ, μ) be a quadratic form over $\mathbb{Z}$.*

1. *It is even or of Type II if λ takes on only even values; i.e. the diagonal terms of the representing matrix are all even integers. If it is not even, then it is called odd or of Type I.*
2. *It is definite if $\lambda(x, x)$ retains the same sign for all $x \in P$. Otherwise it is indefinite.*

Remark 2.2. *We will rely substantially on the treatment of quadratic forms in Serre [577]. Indeed the algebraic half of that book concerns quadratic forms over $\mathbb{Q}$, finite fields, the p-adic fields $\mathbb{Q}_p$, the Hasse principle, and their application to quadratic forms over $\mathbb{Z}$. This mathematics needs to be substantially generalized for the application to L-theory to go beyond a few rings (say $\mathbb{Z}[\mathbb{Z}_2^k]$), but Serre [577] and Borevich-Shaferevich [68] form the foundations for this subject.*

There are only finitely many definite quadratic forms over $\mathbb{Z}$ in any dimension, but their number grows very rapidly when the dimension increases. (See Milnor-Husemoller [459] for a discussion of the Siegel mass formula and its application to this problem.) However, for the purposes of L-theory, we can focus on indefinite forms, since the hyperbolic form $\mathcal{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is indefinite. Therefore by stabilizing, we can focus on the simpler indefinite case.

Indefinite forms of Type I are all equivalent to diagonal forms of the type $k(1) \oplus j(-1)$. While their forms are simple, they are irrelevant to us because of the μ-form. The calculation of $L_0(\mathbb{Z}[e])$ requires us to study Type II indefinite forms. They are not easy to find, but they are classical and arise in many places.

Example 2.3. *Let $n = 4k$ and let $V = \mathbb{Q}^n$ be endowed with the standard bilinear form $\sum_{i=1}^n x_i y_i$. Let $E_0 = \mathbb{Z}^n$ be the subgroup of V formed from the points with integer coordinates whose bilinear form is induced from that of V. Let E_1 be the submodule of E_0 formed from elements x such that $\lambda(x, x) = 0 \bmod 2$, i.e. $\sum_{i=1}^n x_i \equiv 0 \bmod 2$. Finally let E be the submodule of V generated by E_1 and $e = ({}^1\!/\!_2, \dots, {}^1\!/\!_2)$. An element $x \in V$*

belongs to E iff $2x_i \in \mathbb{Z}$ and $x_i - x_j \in \mathbb{Z}$ and $\sum_{i=1}^n x_i \in 2\mathbb{Z}$. Then $\lambda(x, e) = \frac{1}{2}$. Since $\lambda(e, e) = \frac{n}{2} = k$, it follows that the form $\lambda(x, y)$ takes on integer values on E. The quadratic form restricted to E is unimodular (see Serre [577]) and will be denoted by E_{4n}.

Example 2.4. *For the case of E_8, there are 240 elements $x \in E_8$ such that $\lambda(x, x) = 2$. If (e_i) denotes the canonical basis of $\mathbb{Q}^8$, these 240 elements are of the forms*

1. $\pm e_i + e_k$ *where* $i \neq k$,
2. $\frac{1}{2} \sum_{i=1}^8 \xi_i e_i$,

where $\xi_i = \pm 1$ and $\prod_{i=1}^8 \xi_i = 1$. One can take as a basis of E_8 the elements

1. $\frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + \cdots + e_7)$,
2. $e_1 + e_2$,
3. $e_i - e_{i-1}$ *for all* $i = 2, \ldots, 7$.

The corresponding matrix is

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Remark 2.5. *The space E_8 is remarkable and often used in combinatorics (see Viazovska [653]). It is positive-definite and 8-dimensional unimodular. In fact its signature is equal to 8.*

Theorem 2.6. *(Serre) Any indefinite unimodular quadratic form P of Type II over $\mathbb{Z}$ is isomorphic to a sum of copies of E_8 and $\mathcal{H}$. In other words, there are integers s and t such that $P \cong sE_8 \oplus t\mathcal{H}$.*

The $L_0(\mathbb{Z}[e])$ can be calculated by characterizing the isomorphism classes of Type II symmetric forms $\lambda\colon E \times E \to \mathbb{Z}$. Each form is isomorphic to some sum $sE_8 \oplus t\mathcal{H}$, where the hyperbolic forms $\mathcal{H}$ are set to 0 in the L-groups. Recall that the signature of $\mathcal{H}$ is 0.

Corollary 2.7. *There is an isomorphism $L_0(\mathbb{Z}[e]) \cong \mathbb{Z}$, where the isomorphism takes a quadratic form to its signature divided by 8.*

Remark 2.8. *The divisibility of the signature of a Type II unimodular quadratic form* *by 8 is called van der Blij's lemma, and there are of course proofs that do not depend on classification (e.g. in Serre [577] and Milnor-Husemoller [459]). We remark that the second Stiefel-Whitney class w_2 of a simply connected smooth 4-manifold vanishes iff the manifold is spin. In Chapter 3, we will discuss Rokhlin's theorem that such a manifold actually has signature divisible by 16, which implies that there are non-trivial PL structures of the torus in dimensions at least 5.*

We now turn to the computation of $L_2(\mathbb{Z}[e])$. Here we have a skew-symmetric bilinear form $\lambda\colon P \times P \to \mathbb{Z}$, so $\lambda(v, v) = 0$ for all v. The underlying module is even-dimensional,[1] and all information is encoded in the μ-form $\mu\colon \Lambda \to \mathbb{Z}_2$, which satisfies the equations $\mu(x + y) = \mu(x) + \mu(y) + \lambda(x, y)$ and $\mu(ax) = a\mu(x)$. As we show below, one can classify these forms precisely, and understand them by reducing mod 2, i.e. tensoring by $\mathbb{F}_2$ over $\mathbb{Z}_2$ to give an element of $L_2(\mathbb{F}_2)$.

Let $\mathcal{H}^{(0)}$ be the two-dimensional space with basis $\{x, y\}$ and form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with $\mu(x) = \mu(y) = 0$. Let $\mathcal{H}^{(1)}$ be the same as $\mathcal{H}^{(0)}$ except that $\mu(x) = \mu(y) = 1$. Note that $\mathcal{H}^{(0)}$ is precisely the form that is set to zero in the L-group $L_2(\mathbb{F}_2)$.

Lemma 2.9. *Let $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$ be defined as above.*

1. *Let P be a space with a quadratic form. Then $P \cong \mathcal{H}^{(0)} \oplus P'$ or $P \cong \mathcal{H}^{(1)} \oplus P'$ for some quadratic form P'.*
2. *We have an isomorphism $\mathcal{H}^{(0)} \times \mathcal{H}^{(0)} \cong \mathcal{H}^{(1)} \oplus \mathcal{H}^{(1)}$.*
3. *The direct sums $\bigoplus_{i=1}^{n} \mathcal{H}_i^{(0)}$ and $\mathcal{H}^{(1)} \oplus \bigoplus_{i=1}^{n-1} \mathcal{H}_i^{(0)}$ are not isomorphic.*

Proof.

1. Let $x \in P$ be nonzero and choose $y \in P$ with $\lambda(x, y) = 1$ by nonsingularity. If $\mu(x) = \mu(y)$, then we are done. Otherwise, we can assume that $\mu(x) = 0$ and $\mu(y) = 1$. Let $y' = x + y$ and so $\mu(x) = 0$ and $\mu(y') = 0$.
2. Let $\{x_1, y_1\}$ and $\{x_2, y_2\}$ be two bases for $\mathcal{H}^{(0)}$ and let $a_1 = x_1 + y_1 + x_2$, $b_1 = x_1 + y_1 + y_2$, $a_2 = x_1 + x_2 + y_2$, and $b_2 = y_1 + x_2 + y_2$. Then $\lambda(a_i, b_j) = 1$ if $i = j$ and $\mu(a_i) = \mu(b_i) = 1$ otherwise.
3. Let P be a space with a quadratic form and let $m(P)$ be the number of elements on P on which μ takes the value 0. Then by induction we have

$$m\left(\bigoplus_{i=1}^{n} \mathcal{H}_i^{(0)}\right) = 2^{2i-1} + 2^{i-1}$$

and

$$m\left(\mathcal{H}^{(1)} \oplus \bigoplus_{i=1}^{n-1} \mathcal{H}_i^{(0)}\right) = 2^{2i-1} - 2^{i-1}.$$

[1] Using Poincaré duality, this argument also shows that the Euler characteristic of a $(4k+2)$-dimensional orientable manifold is automatically even, a fact that we will use in Section 2.4.

Therefore the two are not isomorphic. □

Using the lemma, we can completely classify these forms since any space P is isomorphic to one of the form $\mathcal{H}^{(1)} \oplus \bigoplus_i \mathcal{H}_i^{(0)}$ or one of the form $\bigoplus_i \mathcal{H}_i^{(0)}$. Again, in the L-group $L_2(\mathbb{F}_2)$, the hyperbolic form $\mathcal{H}^{(0)}$ is set to 0.

Definition 2.10. *Let P be a finite-dimensional vector space over $\mathbb{Z}$ with a quadratic form. We define the Arf invariant of P as an element of $\mathbb{Z}_2$ as follows. If $P \cong \bigoplus_{i=1}^{n} \mathcal{H}_i^{(0)}$, we say that* $\mathrm{Arf}(P) = 0$*, and if $P \cong \mathcal{H}^{(1)} \oplus \bigoplus_{i=1}^{n-1} \mathcal{H}_i^{(0)}$, then* $\mathrm{Arf}(P) = 1$*. The assignment gives an isomorphism* $\mathrm{Arf} : L_2(\mathbb{F}_2) \to \mathbb{Z}_2$.

Another way to describe the Arf invariant is the following. Every nonsingular (-1)-symmetric form (P, λ) admits a symplectic basis $\{e_1, f_1, \dots, e_m, f_m\}$, i.e. a basis such that

1. $\lambda(e_i, f_i) = 1$ for all i;
2. $\lambda(e_i, f_j) = 0$ if $i \neq j$;
3. $\lambda(e_i, e_i)$ and $\lambda(f_i, f_i)$ are both zero for all i.

If (P, λ, μ) lies in $L_2(\mathbb{Z}[e])$, we then define the *Arf invariant* of (P, λ, μ) by

$$\mathrm{Arf}(P, \lambda, \mu) = \sum_{i=1}^{n} \mu(e_i)\mu(f_i)$$

as an element of $\mathbb{Z}_2$.

Theorem 2.11. *The Arf invariant defines an isomorphism* $\mathrm{Arf} : L_2(\mathbb{Z}[e]) \to \mathbb{Z}_2$ *given by* $[P, \lambda] \mapsto \mathrm{Arf}((P, \lambda) \otimes \mathbb{Z}_2)$.

Remark 2.12. *In fact, the same construction can equally be applied to a nonsingular form (Q, ϕ) over the field $\mathbb{F}_2$. Then we can define* $\mathrm{Arf}(Q, \phi) = \mathrm{Arf}((Q, \phi) \otimes \mathbb{Z}_2)$*. The identification of both $L_2(\mathbb{Z}[e])$ and $L_2(\mathbb{F}_2)$ with $\mathbb{Z}_2$ using these Arf maps gives a change of rings isomorphism $L_2(\mathbb{Z}[e]) \to L_2(\mathbb{F}_2)$ induced by the surjection $\mathbb{Z} \to \mathbb{F}_2$.*

2.2 ELEMENTARY WITT THEORY

When computing $L_0(\Lambda)$ with a trivial involution, we must understand symmetric bilinear forms $\lambda : P \times P \to \Lambda$ over Λ. In particular, when Λ is a field $\mathbb{F}$, the definition of $L_0(\mathbb{F})$ is precisely equal to that of the classical Witt group $\mathrm{Witt}(\mathbb{F})$. The L-theory for fields does not arise geometrically so often, but sometimes there is an indirect relation for fields with characteristic not 2. As usual, we can consider such λ very concretely as a symmetric matrix A with nonzero determinant. If we were working with $L_0^s(\mathbb{F})$ we would insist that the determinant of A be ± 1.

The Witt group for a field $\mathbb{F}$ has addition given by an orthogonal sum that coincides with the operation in $L_0(\mathbb{F})$. However, there is an additional tensor product that gives $\mathrm{Witt}(\mathbb{F})$ a ring structure. Excellent and varied references for Witt theory are Milnor-Husemoller [459], Pfister [508], and Lam [386]. In this setting, we will use the usual language of orthogonality that one learns in linear algebra.

Definition 2.13. *The rank function for a field $\mathbb{F}$ is the map $r : \mathrm{Witt}(\mathbb{F}) \to \mathbb{Z}_2$ assigning to a form $[P, \lambda]$ the dimension of P as a vector space over $\mathbb{F}$. Note that, for a Witt class, the dimension is only well defined modulo 2, since hyperbolic forms have even dimension. The map is a ring homomorphism. We define the fundamental ideal $I(\mathbb{F}) \subseteq \mathrm{Witt}(\mathbb{F})$ to be the kernel of the rank function.*

Proposition 2.14. *Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) \neq 2$, and assume that every element of $\mathbb{F}$ is a square. Then the rank homomorphism $f : \mathrm{Witt}(\mathbb{F}) \to \mathbb{Z}_2$ is an isomorphism.*

Proof. A form in $\mathrm{Witt}(\mathbb{F})$ has a basis for which the representing matrix A is diagonal, as one sees by a direct induction on dimension. Given the assumption on $\mathbb{F}$, we can normalize the basis so that the representing matrix is the identity. (Recall that the matrix is nonsingular, so there are no zeros on the diagonal.) The two-dimensional hyperbolic form is in particular equivalent to the 2×2 identity matrix, so the diagonal elements of our matrix can be removed in pairs, leaving only one of two possibilities. □

Corollary 2.15. *We have an isomorphism $\mathrm{Witt}(\mathbb{C}) \cong \mathbb{Z}_2$. If $\overline{\mathbb{F}_p}$ is the algebraic closure of a field $\mathbb{F}_p$ of odd characteristic, then $\mathrm{Witt}(\overline{\mathbb{F}_p}) \cong \mathbb{Z}_2$.*

Recall that, if A is a symmetric real matrix, then the *signature of A* is the number of positive eigenvalues minus the number of negative eigenvalues. Since the signature of the two-dimensional hyperbolic form is zero, it is well defined on $\mathrm{Witt}(\mathbb{F})$.

Proposition 2.16. *The signature map $\mathrm{sign} : \mathrm{Witt}(\mathbb{R}) \to \mathbb{Z}$ is an isomorphism.*

Proof. Over $\mathbb{R}$ we can diagonalize a symmetric matrix A so that all the diagonal elements are 1 or -1. We can remove them in pairs, since $(1) \oplus (-1)$ is hyperbolic, proving the result. □

Most fields $\mathbb{F}$ do not have the property that every element is a square; therefore the quotient $\mathbb{F}^*/(\mathbb{F}^*)^2$ is generally non-trivial. It obviously enters in the structure of $\mathrm{Witt}(\mathbb{F})$:

Definition 2.17. *Let $\mathbb{F}$ be a field. We define the discriminant function for $\mathbb{F}$ to be the map $\mathrm{disc} : I(\mathbb{F}) \to \mathbb{F}^*/(\mathbb{F}^*)^2$ given by*

$$[P, \lambda] \mapsto (-1)^{\mathrm{rank}(P)/2} \det(A),$$

where A is the matrix representing λ in some basis.

Remark 2.18. *We explain why the discriminant is well defined. A quadratic form* (P, λ) *in the fundamental ideal* $I(\mathbb{F})$ *has even rank so that* $\frac{1}{2}\mathrm{rank}(P)$ *is an integer. Since we are working modulo squares, the term* $\det(A)$ *is well defined on isomorphism classes of quadratic forms. Lastly, the term* $(-1)^{\mathrm{rank}(P)/2}$ *makes the entire quantity invariant under the addition of hyperbolic forms.*

The discriminant map is always surjective. Indeed, if $a \in \mathbb{F}^*$, then the discriminant of the two-dimensional form $(a) \oplus (-1)$ is a. The observation gives the following.

Proposition 2.19. *The Witt ring* $\mathrm{Witt}(\mathbb{Q})$ *of the rationals is infinitely generated.*

Witt discovered an important *cancellation* phenomenon in the Witt group of fields that does not hold even for $\mathbb{Z}$.

Theorem 2.20. *(Witt cancellation) Let* P_1, P_2, R *be quadratic spaces. If* $P_1 \oplus R \cong P_2 \oplus R$*, then* $P_1 \cong P_2$*.*

We say that λ is *isotropic* if there is a nonzero v with $\lambda(v, v) = 0$; otherwise it is *anisotropic*. One can use the greedy algorithm to simplify Witt classes, removing hyperbolic pairs one at a time, e.g. by finding vectors with $\lambda(v, v) = 0$, and then concentrate on the anisotropic inner product. We refer again to the standard references Milnor-Husemoller [459], Pfister [508], and Lam [386].

Proposition 2.21. *For the field* $\mathbb{F}_p$ *with odd p, the fundamental ideal is isomorphic to* $\mathbb{Z}_2$*.*

Proof. When char $\mathbb{F} \neq 2$, then we define a map $\rho\colon \mathbb{Z}_2 \cong \mathbb{F}^*/(\mathbb{F}^*)^2 \to I(\mathbb{F})$ by $a \mapsto (\mathbb{F}, a) \oplus (\mathbb{F}, -1)$. We use the fact that $(\mathbb{F}, ab) \oplus (\mathbb{F}, 1) = (\mathbb{F}, a) \oplus (\mathbb{F}, b)$ in $L_0(\mathbb{F})$ for all $a, b \in \mathbb{F}^*$. Note that, for all $a, b \in \mathbb{F}^*/(\mathbb{F}^*)^2$, we have

$$\begin{aligned} \rho(ab) &= (\mathbb{F}, ab) \oplus (\mathbb{F}, -1) \\ &= (\mathbb{F}, ab) \oplus (\mathbb{F}, 1) \oplus 2(\mathbb{F}, -1) \\ &= (\mathbb{F}, a) \oplus (\mathbb{F}, -1) \oplus (\mathbb{F}, b) \oplus (\mathbb{F}, -1) \\ &= \rho(a) \oplus \rho(b). \end{aligned}$$

Note that the fact above gives $\rho(-1) = (\mathbb{F}, -1) \oplus (\mathbb{F}, -1) = (\mathbb{F}, 1) \oplus (\mathbb{F}, 1)$ in $L_0(\mathbb{F})$. If $(\mathbb{F}, a_1) \oplus \cdots \oplus (\mathbb{F}, a_{2n}) \in I(\mathbb{F})$, then

$$\begin{aligned} \rho(a_1 \cdots a_{2n} - n \cdot 1) &= \rho(a_1 \cdots a_{2n}) \oplus n\rho(-1) \\ &= \rho(a_1 \cdots a_{2n}) \oplus 2n(\mathbb{F}, 1) \\ &= (\mathbb{F}, a_1 \cdots a_{2n}) \oplus (\mathbb{F}, -1) + 2n(\mathbb{F}, -1) \\ &= (\mathbb{F}, a_1 \cdots a_{2n}) \oplus (2n - 1)(\mathbb{F}, 1) \\ &= (\mathbb{F}, a_1) \oplus \cdots \oplus (\mathbb{F}, a_{2n}), \end{aligned}$$

so ρ is surjective. Lastly, the composition $d \circ \rho : \mathbb{F}^*/(\mathbb{F}^*)^2 \to I(\mathbb{F}) \to \mathbb{F}^*/(\mathbb{F}^*)^2$ given by $a \mapsto (\mathbb{F}, a) \oplus (\mathbb{F}, -1) \mapsto a$ is an isomorphism, so ρ is injective. □

We therefore have an exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Witt}(\mathbb{F}_p) \to \mathbb{Z}_2 \to 0.$$

The structure of $\mathrm{Witt}(\mathbb{F}_p)$, however, is sensitive to the nature of p. Fermat was familiar with the next lemma.

Lemma 2.22. *Let $\mathbb{F}$ be a finite field of odd characteristic with q elements. Then -1 is a square in $\mathbb{F}$ iff $q \equiv 1 \bmod 4$.*

Theorem 2.23. *Let $\mathbb{F}_q$ be the finite field of q elements with q odd. Then*

$$\mathrm{Witt}(\mathbb{F}_q) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q \equiv 1 \mod 4, \\ \mathbb{Z}_4 & q \equiv 3 \mod 4. \end{cases}$$

Proof. Suppose that $q \equiv 1 \bmod 4$. By the lemma, we know that -1 is a square in $\mathbb{F}_q$; i.e. there is $a \in \mathbb{F}_q$ such that $a^2 = -1$. Let $u \in \mathbb{F}_q^*$. Then $(\mathbb{F}, u) \oplus (\mathbb{F}, u) = (\mathbb{F}, u) \oplus (\mathbb{F}, a^2 u) = (\mathbb{F}, u) \oplus (\mathbb{F}, -u) = 0$ in $\mathrm{Witt}(\mathbb{F}_q)$. Hence every element has order at most 2; i.e. $\mathrm{Witt}(\mathbb{F}_q) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Suppose now that $q \equiv 3 \bmod 4$. Then -1 is not a square in $\mathbb{F}_q$. For the sake of contradiction, suppose that $(\mathbb{F}, 1) \oplus (\mathbb{F}, 1) = 0$ in $L_0(\mathbb{F}_q)$. Then $(\mathbb{F}, 1) \oplus (\mathbb{F}, 1) = (\mathbb{F}, 1) \oplus (\mathbb{F}, -1)$. By cancellation, we have $(\mathbb{F}, 1) = (\mathbb{F}, -1)$. Taking determinants, we find that $1 = -1$ in $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2$; i.e. -1 is a square in $\mathbb{F}_q$, a contradiction. Therefore $(\mathbb{F}, 1)$ has order 4 in $\mathrm{Witt}(\mathbb{F}_q)$ and $\mathrm{Witt}(\mathbb{F}_q) \cong \mathbb{Z}_4$. □

We refer to Milnor-Husemoller [459] for the next theorem and the definition of the map ∂_p (which is analogous to dévissage in K_0).

Theorem 2.24. *Let $\mathbb{P}$ be the set of positive primes in $\mathbb{Z}$. The sequence of abelian groups*

$$0 \to \mathrm{Witt}(\mathbb{Z}) \to \mathrm{Witt}(\mathbb{Q}) \xrightarrow{\oplus \partial_p} \bigoplus_{p \in \mathbb{P}} \mathrm{Witt}(\mathbb{F}_p) \to 0$$

is exact.

It is an exercise to show that this sequence splits, since $\mathrm{Witt}(\mathbb{Z}) \to \mathrm{Witt}(\mathbb{R})$ is an isomorphism. More generally, we have the following.

Proposition 2.25. *Suppose that D is a Dedekind domain with field of fractions $\mathbb{F}$. Let*

$\mathcal{M}$ be the collection of maximal ideals of D. Then the sequence of abelian groups

$$0 \to \mathrm{Witt}(D) \to \mathrm{Witt}(\mathbb{F}) \xrightarrow{\oplus \partial_{\wp}} \bigoplus_{\wp \in \mathcal{M}} \mathrm{Witt}(D/\wp)$$

is exact.

The last map $\oplus\partial_p$ is not necessarily surjective; there is a class group obstruction (see Milnor-Husemoller [459]).

One can take, for example, a number field for $\mathbb{F}$ and the ring of integers in $\mathbb{F}$ for D. In this case, the Witt terms $\mathrm{Witt}(D/\wp)$ are all of finite fields and $\mathbb{F} = D \otimes \mathbb{Q}$. Away from 2, the map on Witt theory is an isomorphism. This phenomenon is quite general, and much of Witt theory and L-theory is much simpler away from 2.

Example 2.26. *Consider an odd prime p and the field $\mathbb{F} = \mathbb{Q}(\cos(2\pi/p))$, which is a real subfield of $\mathbb{Q}(e^{2\pi i/p})$. This field arises in the study of manifolds with fundamental group $\mathbb{Z}_p$. For every k coprime to p, there is a Galois automorphism $\phi_k : \mathbb{F} \to \mathbb{F}$ that sends $\cos(2\pi/p)$ to $\cos(2\pi k/p)$. Note k and $-k$ give the same automorphism. Therefore, a quadratic form over $\mathbb{Q}(\cos(2\pi/p))$ has $\frac{p-1}{2}$ different signatures. In fact, there is an epimorphism $\mathbb{F}^* \to \mathbb{Z}_2^{(p-1)/2}$ that sends a nonzero element of $\mathbb{F}$ to its distribution of signs under different embeddings $\mathbb{F} \hookrightarrow \mathbb{R}$. This fact can be used to prove the following.*

Proposition 2.27. *If p is an odd prime, then there is a surjection*

$$\mathrm{Witt}(\mathbb{Q}(\cos(2\pi/p))) \to \mathbb{Z}^{(p-1)/2}.$$

In fact, it is an isomorphism modulo 2-torsion.

Remark 2.28. *Ultimately this map will be responsible for infinitely many homotopy lens spaces.*

Definition 2.29. *Let $\mathbb{F}$ be a field. An ordering on $\mathbb{F}$ is a partition of the nonzero elements of $\mathbb{F}$ into two subsets, the positive and negative elements, such that (1) the positive elements are closed under addition and multiplication, and (2) the negatives are exactly the negatives of the positives, i.e. v is positive iff $-v$ is negative.*

For every ordering of a field, one can define a signature of a given quadratic form. One can merely diagonalize, and count the number of positive elements on the diagonal minus the number of negative ones. The orderings of a number field $\mathbb{F}$ exactly correspond to the real embeddings $\mathbb{F} \hookrightarrow \mathbb{R}$. Therefore, all of the signature invariants of $\mathbb{F}$ correspond to ones induced from maps $\mathrm{Witt}(\mathbb{F}) \to \mathrm{Witt}(\mathbb{R})$ for different embeddings.

Remark 2.30. *As part of Artin's solution to Hilbert's 17th problem, he showed that a field has an ordering iff -1 is not a sum of squares in $\mathbb{F}$. Amazingly, Pfister showed that, if -1 is a sum of squares in a field, the minimum number of squares is a power of 2. Pfister also showed that the kernel of the total signature map, obtained by consider-*

ing all orderings, is always 2-torsion (for Witt($\mathbb{F}$)*). See Milnor-Husemoller [459] and Pfister [508].*

Remark 2.31. *For fields $\mathbb{F}$ that are not number rings, it is possible to achieve very high exponent 2-torsion. It is not particularly easy to find invariants that detect the elements; their nature is rather indirect, coming from decomposing forms into their anisotropic Witt equivalent representative, and counting the copies required for it to become isotropic. The exponent of the torsion of* Witt($\mathbb{R}(x_1, \dots, x_n)$) *grows with n, although, as of the writing of this book, one does not know how quickly.*

Although we discussed principally the symmetric case in this section, we end by mentioning skew-symmetric forms. Because every every skew-symmetric form over a field $\mathbb{F}$ is hyperbolic when $\mathrm{char}(\mathbb{F}) \neq 2$, we have the following.

Proposition 2.32. *Suppose that $\mathbb{F}$ is a field with* $\mathrm{char}(\mathbb{F}) \neq 2$*, and suppose also that the given involution is trivial. Then $L_2(\mathbb{F}) = 0$.*

2.3 L-THEORY OF FINITE GROUPS

In this section we will discuss what one can learn about L-theory of finite groups by comparing them to some simpler rings. As we have stated, we will continue to focus on even dimensions, and will start very crudely and describe the whole picture away from 2.

There is much known about the computations of $L_*(\mathbb{Z}[G])$ for finite G, starting with Wall's own sequence of papers [671, 674, 675, 677–679] and continuing with the work (in alphabetic order) of Bak, Carlsson, Connolly, Hambleton, Kolster, Milgram, Pardon, Taylor, Williams, and others. The computation techniques combine specific results in the classical theory of quadratic forms, algebraic number theory, algebraic groups, and representation theory, e.g. the induction of Dress [207], with general results in algebraic K- and L-theory. A very useful paper that explains these calculations in much more detail is Hambleton-Taylor [290].

We recall that, for the trivial involution on $\mathbb{R}$, we have $L_0(\mathbb{R}) \cong \mathbb{Z}$ and $L_2(\mathbb{R}) = 0$. However, the situation is very different when a field is endowed with a non-trivial involution. Non-trivial involutions are important to surgery theory because the integer group ring $\mathbb{Z}[G]$ has a natural non-trivial involution.

Proposition 2.33. *If $\mathbb{F}$ is a field with a non-trivial involution $-$, then $L_0(\mathbb{F}, -) \cong L_2(\mathbb{F}, -)$.*

Proof. Let v be an element of $\mathbb{F}$ that differs from $\overline{v}$. Then $w = \overline{v} - v$ satisfies the formula $\overline{w} = -w$. Therefore if λ is a (-1)-symmetric form, then $w\lambda$ is a (-1)-skew-symmetric form. The argument also works backwards. So there is a bijective correspondence between the two L-groups. □

Proposition 2.34. *Consider* $\mathbb{C}$ *with its natural conjugation involution. Then there is an isomorphism* $L_0(\mathbb{C}, -) \cong \mathbb{Z}$.

Proof. A (-1)-symmetric form is also known as a Hermitian form. We can diagonalize such a form and automatically the diagonal is real. As a result, the signature is defined as in the real symmetric case. Moreover, the map $L_0(\mathbb{R}) \to L_0(\mathbb{C}, -)$ is an isomorphism. □

The computation for $L_{2k}(\mathbb{Z}[G])$ requires some computations in representation theory. In particular, we need information about the Wedderburn decompositions of $\mathbb{F}[G]$. If $\mathbb{F}$ is a field of characteristic zero and G is a finite group, the group ring $\mathbb{F}[G]$ has a Wedderburn decomposition

$$\mathbb{F}[G] = R_1 \times \cdots \times R_k,$$

where each R_i is a simple matrix ring over some division ring. For the following, see Serre [578] or Curtis-Reiner [184].

Proposition 2.35. *The number* k *of simple factors in* $\mathbb{F}[G]$ *is equal to the number of isomorphism classes of irreducible* $\mathbb{F}$*-representations of* G.

For each simple finite-dimensional algebra D over $\mathbb{F}$, let $\alpha_D(\mathbb{F}, G)$ be the number of factors $S_j(\mathbb{F}, G)$ in $\mathbb{F}[G]$ with $D_j(\mathbb{F}, G) = D$. Therefore $\lambda(\mathbb{F}, G) = \sum_D \alpha_D(\mathbb{F}, G)$. We are mainly interested in the case with $\mathbb{F} = \mathbb{R}$.

Proposition 2.36. *Let* G *be a finite group, and decompose* $\mathbb{R}[G]$ *as above. The number* $\lambda(\mathbb{R}, G)$ *of simple factors* $S_j = S_j(\mathbb{R}, G) = M_{d_j}(D_j(\mathbb{R}, G))$ *is*

1. *the number of irreducible* $\mathbb{R}$*-representations of* G,
2. *the number of conjugacy classes of unordered pairs* $\{g, g^{-1}\}$ *of* G,
3. $\alpha_{\mathbb{R}}(\mathbb{R}, G) + \alpha_{\mathbb{C}}(\mathbb{R}, G) + \alpha_{\mathbb{H}}(\mathbb{R}, G)$, *where* $\alpha_D(\mathbb{R}, G)$ *is the number of* S_j *such that* $D_j(\mathbb{R}, G) = D$.

We first specialize to $G = \mathbb{Z}_m$ to study $L^h_{2k}(\mathbb{Z}[\mathbb{Z}_m])$, where $\mathbb{Z}[\mathbb{Z}_m]$ is given the usual involution by sending a generator g to g^{-1}. The even L-groups can be described in terms of representations: the ring $\mathbb{R}[\mathbb{Z}_m]$ decomposes into a product of fields, each of which is either $\mathbb{R}$ or $\mathbb{C}$. These pieces correspond to the irreducible representations of $\mathbb{Z}_m$. The reals $\mathbb{R}$ give a copy of $\mathbb{Z}$ in 0 mod 4, and the complexes $\mathbb{C}$ give a copy of $\mathbb{Z}$ in all even dimensions.

If m is odd, there is only one way to map $\mathbb{Z}[\mathbb{Z}_m]$ to $\mathbb{R}$, i.e. $g \mapsto 1$. If m is even, then there are two ways, i.e. $g \mapsto \pm 1$. In other words, the number of ways is

$$\alpha_{\mathbb{R}} = \begin{cases} 2 & \text{if} \quad m \equiv 0 \mod 2, \\ 1 & \text{if} \quad m \equiv 1 \mod 2. \end{cases}$$

Therefore, from the real representations, the L_0-group gains either $\mathbb{Z}$ or $\mathbb{Z}^2$. Since the L_2-group contains only skew-symmetric forms, it gains nothing from the real representations.

The irreducible $\mathbb{C}$-representations of $\mathbb{Z}_m$ are given by $\rho_j : \mathbb{Z}_m \to \mathbb{C}$ with $j \mapsto e^{2\pi i j/m}$. When one sends a generator g to an m-th root of unity, the geometric involution is sent to complex conjugation. By the previous proposition, each conjugacy class of complex embeddings gives a homomorphism of each $L_{2k}(\mathbb{R}[\mathbb{Z}_m])$ to $\mathbb{Z}$, and there are $[\frac{m-1}{2}]$ of them. We loosely use the term *signature map* to refer to each of these homomorphisms. The number of such representations is

$$\alpha_{\mathbb{C}} = \begin{cases} \frac{m-2}{2} & \text{if} \quad m \equiv 0 \mod 2, \\ \frac{m-1}{2} & \text{if} \quad m \equiv 1 \mod 2. \end{cases}$$

As a result, we have

$$\mathbb{R}[\mathbb{Z}_m] = \begin{cases} \mathbb{R}^2 \oplus \mathbb{C}^{(m-2)/2} & \text{if} \quad m \equiv 0 \mod 2, \\ \mathbb{R} \oplus \mathbb{C}^{(m-1)/2} & \text{if} \quad m \equiv 1 \mod 2. \end{cases}$$

The number $\lambda(\mathbb{R}, \mathbb{Z}_m)$ of simple factors of $\mathbb{R}[\mathbb{Z}_m]$ is $2 + \frac{m-2}{2} = \frac{m+2}{2}$ if m is even, and $1 + \frac{m-1}{2} = \frac{m+1}{2}$ if m is odd.

These calculations immediately give the following calculations:

$$L_0^h(\mathbb{R}[\mathbb{Z}_m]) = \begin{cases} \mathbb{Z}^{(m+2)/2} & \text{if} \quad m \equiv 0 \mod 2, \\ \mathbb{Z}^{(m+1)/2} & \text{if} \quad m \equiv 1 \mod 2, \end{cases}$$

$$L_2^h(\mathbb{R}[\mathbb{Z}_m]) = \begin{cases} \mathbb{Z}^{(m-2)/2} & \text{if} \quad m \equiv 0 \mod 2, \\ \mathbb{Z}^{(m-1)/2} & \text{if} \quad m \equiv 1 \mod 2. \end{cases}$$

Remark 2.37. *For a description of $L_{2k}^p(\mathbb{R}[\mathbb{Z}_m])$, we can combine the quadratic forms from each of the factors of the decomposition of $\mathbb{R}[\mathbb{Z}_m]$ and still have a projective module over $\mathbb{R}[\mathbb{Z}_m]$, as well as a quadratic form on the projective module. However, to construct a quadratic form in $L_{2k}^h(\mathbb{R}[\mathbb{Z}_m])$, we require that the dimensions of the component forms all be the same. Note that hyperbolic pairs have even dimension, so we only require forms from each factor to have the same dimension modulo 2.*

Once they have the same dimension, a Lagrangian for P will be automatically free. Therefore $L_{2k}^h(\mathbb{R}[\mathbb{Z}_m])$ injects into $L_{2k}^p(\mathbb{R}[\mathbb{Z}_m])$. As a result, the groups $L_{2k}^h(\mathbb{R}[\mathbb{Z}_m])$

and $L^p_{2k}(\mathbb{R}[\mathbb{Z}_m])$ are abstractly isomorphic free abelian of the same rank as calculated above, but the natural map $L^h_{2k}(\mathbb{R}[\mathbb{Z}_m]) \to L^p_{2k}(\mathbb{R}[\mathbb{Z}_m])$ is not an isomorphism. For example, when $n = 2$, the image of $L_0(\mathbb{R}[\mathbb{Z}_m])$ is of index 2, while the map on $L_2(\mathbb{R}[\mathbb{Z}_m])$ is an isomorphism. The odd L^p-groups for $\mathbb{R}[\mathbb{Z}_m]$ are trivial.

Remark 2.38. *As we mentioned above, the map $L^h_{2k}(\mathbb{R}[\mathbb{Z}_m]) \to L^h_{2k}(\mathbb{R}[\mathbb{Z}_m])$ has nontrivial cokernel. In fact, the comparison between $L^h_{2k}(\mathbb{R}[\mathbb{Z}_m])$ and $L^p_{2k}(\mathbb{R}[\mathbb{Z}_m])$ is governed here by the factors of $\mathbb{R}[\mathbb{Z}_m]$ and therefore encoded by $K_0(\mathbb{R}[\mathbb{Z}_m])$. In general there are exact sequences connecting the various types of L-groups, such as L^p, L^h, and L^s. They are called Ranicki-Rothenberg sequences, and will be discussed in the next section.*

Theorem 2.39. *If G is a finite group, then the L-groups of $\mathbb{Q}[G]$ and $\mathbb{R}[G]$ agree modulo 2-primary torsion. In fact, the common value for $L_n(\mathbb{Q}[G])[1/2] \cong L_n(\mathbb{R}[G])[1/2]$ is*

$$\sum_{\lambda(\mathbb{R},G)} \mathbb{Z}[1/2] \quad \text{if} \quad n \equiv 0 \mod 4,$$

$$\sum_{\alpha_{\mathbb{C}}(\mathbb{R},G)} \mathbb{Z}[1/2] \quad \text{if} \quad n \equiv 2 \mod 4.$$

If n is odd, then these L-groups are zero.

Proof. Indeed, the analysis given to $\mathbb{R}[\mathbb{Z}_m]$ is true for $L_{2k}(\mathbb{Q}[\mathbb{Z}_m])$. These signature maps are discrete-valued functions, defined on the quadratic forms taking values in $\mathbb{R}[\mathbb{Z}_m]$, and the nonsingularity is an open condition. Since $\mathbb{Q}[\mathbb{Z}_m]$ is dense in $\mathbb{R}[\mathbb{Z}_m]$, we can find a nearby $\mathbb{Q}[\mathbb{Z}_m]$-valued quadratic form that has the same multisignature as any given element of $L^h_{2k}(\mathbb{R}[\mathbb{Z}_m])$. Moreover, the obvious generalization of this works for all finite groups. □

For surgery we are interested in the L-theory of the integer group ring $\mathbb{Z}[G]$. Just as the localization theorem for Witt groups in the previous section shows that tensoring with $\mathbb{Q}$ has an effect only at the prime 2, the following theorem can be proved using the algebraic theory of surgery,

Theorem 2.40. *(Ranicki [535]) Let G be a group, infinite or finite. The localization map $L_*(\mathbb{Z}[G]) \to L_*(\mathbb{Q}[G])$ is an isomorphism modulo 2-primary torsion with any decoration; i.e. $L_*(\mathbb{Z}[G])[1/2] = L_*(\mathbb{Q}[G])[1/2]$.*

These signature maps completely describe the L-group $L_{2k}(\mathbb{Z}[\mathbb{Z}_m])$ as well modulo 2-torsion issues. In fact, except for the Kervaire invariant element in dimension 2, the even L-groups with decoration p and s do not have any 2-torsion. However, there is sometimes 2-torsion in L^h-groups, e.g. for $L^h_*(\mathbb{Z}[\mathbb{Z}_{29}])$. Additionally, the signature maps for these varying L-groups have different images; the images are the largest in the case of projective L-groups.

Ranicki's theorem can be used to prove the following.

Theorem 2.41. *For all finite π, the groups $L_i^h(\mathbb{Z}[\pi])$ are finitely generated abelian groups with no odd torsion. If i is odd, the groups are finite, and in even dimensions $2k$, aside from the 2-torsion, they are detected by a signature-type invariant lying in $R_{\mathbb{C}}(\pi)^{(-1)^k}$. Here $R_{\mathbb{C}}(\pi)$ is the ring of complex characters with an action by complex conjugation.*

Remark 2.42. *The signature-type invariant mentioned in the theorem is called the multisignature and will be discussed in a few pages.*

There are a few different ways of organizing the above information and each is useful in its own fashion.

(1) We can use a Wedderburn decomposition for $\mathbb{Q}[\pi]$ (or $\mathbb{R}[\pi]$) into (-1)-invariant pieces, and use the fact that L^p of a product is tautologously the product of the L-theory of the pieces. These pieces are matrix rings of various sorts.

(2) One observes that a matrix with matrix entries is just a larger matrix. In the case of $\mathbb{R}[\pi]$, the problem reduces to the study of $\mathbb{R}$, $\mathbb{C}$, and the quaternions $\mathbb{H}$. We will avoid discussion of the interaction of the involution with Wedderburn theory, as well as the quaternionic case, since the other approaches are somewhat more direct. Here is the table of the L-groups.

k	0	2
$L_k(\mathbb{R})$	$\mathbb{Z}$	0
$L_k(\mathbb{C})$	$\mathbb{Z}$	$\mathbb{Z}$
$L_k(\mathbb{H})$	$\mathbb{Z}$	$\mathbb{Z}_2$

Another way to organize this information is to take representations $\pi \to \mathrm{GL}_n(D)$ to obtain a map $L_k(\mathbb{Z}[\pi]) \to L_k(D)$, where D is some division ring. Then the available representations give a collection of invariants. We suspect that it was this point of view that led to the word *multisignature*, since there are many signatures.

2.3.1 The G-signature of manifolds

Finally, we mention another pair of perspectives which first arose in index theory, which can be used as an important tool in the calculations of examples. See Atiyah-Bott [24] and Atiyah-Singer [29]. We will review it first in this context.

We begin by stating the Hirzebruch signature formula, which expresses the signature of a closed oriented $4k$-dimensional smooth manifold in terms of characteristic classes.

Theorem 2.43. *(Hirzebruch) Let M^{4k} be a closed oriented Diff manifold. Then there is a homogeneous graded polynomial $L_k(p_1, \dots, p_k)$ in the Pontrjagin classes of M such that*

$$\mathrm{sig}(M) = \langle L_k(p_1, \dots, p_k), [M] \rangle.$$

Moreover, the total class $\mathcal{L} = 1 + L_1 + L_2 + \cdots$ is multiplicative for sums of bundles.

Remark 2.44. *We call L_k the k-th Pontrjagin L-class. We refer both to L_k and $\mathcal{L}$ as the Hirzebruch L-genus. See Appendix A.3.*

Our first goal is to extend Hirzebruch's formula to the Atiyah-Singer G-signature theorem, which makes computations in the presence of an action by a group G on M. Our discussion focuses on L-theory. For the application to L-theory, one can consider the surgery kernels with the action by the fundamental group of the manifold on them. In particular, we are interested in G-invariant symmetric or skew-symmetric forms over $\mathbb{R}$, i.e. quadratic forms with maps of G into their isometry group. Explicitly, given $\lambda : V \otimes V \to \mathbb{R}[G]$, we can express it as $\lambda(v, w) = \sum_{g \in G} a_g(v, w) g$ with appropriate coefficients $a_g(v, w)$. Define $\langle v, w \rangle = a_e(v, w)$. Note that

$$\begin{aligned} \langle gv, gv \rangle &= \lambda(gv, gw)_e \\ &= (g\lambda(v, w)g^{-1})_e \\ &= \lambda(v, w)_e. \end{aligned}$$

So G acts by isometries on the given inner product, and we obtain a G-invariant quadratic form in the G-equivariant Witt group. Conversely, given a G-invariant inner product $\langle \cdot, \cdot \rangle$, we can define $\lambda(v, w) = \sum_{g \in G} \langle gv, w \rangle$.

Remark 2.45. *Although we have seen that there are no interesting skew-symmetric forms for fields, i.e. they are unique up to isomorphism in every even dimension, there are indeed non-trivial examples in the equivariant case.*

Let G be a compact group, and consider a symmetric, nonsingular G-invariant inner product space (V, β) over $\mathbb{R}$. Define the operator $A : V \to V$ given by $\beta(x, y) = \langle x, Ay \rangle$ for all $x, y \in V$. This operator A commutes with the G-action on V, since

$$\langle x, Ay \rangle = \beta(x, y) = \beta(gx, gy) = \langle gx, Agy \rangle = \langle x, g^{-1}Agy \rangle$$

for all $x, y \in V$ and $g \in G$. All the eigenvalues of A are nonzero and real, so the positive and negative eigenspaces of A give a decomposition $V = V^+ \oplus V^-$. Both of these subspaces are G-invariant. Indeed, suppose that $v \in V$ is an eigenvector associated to an eigenvalue λ; therefore we have

$$\langle x, Agv \rangle = \beta(x, gv) = \beta(x, g\lambda A^{-1}v) = \beta(x, A^{-1}g\lambda v) = \langle x, \lambda gv \rangle,$$

demonstrating that gv is also an eigenvector of A with eigenvalue λ. We therefore have real representations $\sigma_{V^+} : G \to \mathrm{GL}(V^+)$ and $\sigma_{V^-} : G \to \mathrm{GL}(V^-)$. Note that

1. the characters of σ_{V^+} and σ_{V^-} are continuous functions of the inner product;
2. the space of all G-invariant inner products is connected;
3. the characters of the compact group G are discrete.

From these facts, we deduce that, up to isomorphism, the representations σ_{V^+} and σ_{V^-} are independent of the choice of inner product.

Definition 2.46. *If G is a finite group and $\mathbb{F}$ is a field, the representation ring $R_{\mathbb{F}}(G)$ consists of formal differences of isomorphism classes of finite-dimensional linear $\mathbb{F}$-representations of G. For the ring structure, addition is given by the direct sum of representations, and multiplication by their tensor product over $\mathbb{F}$. If $\mathbb{F} = \mathbb{R}$, we will write $RO(G)$. If $\mathbb{F} = \mathbb{C}$ we will suppress the field and simply write $R(G)$. The representation ring $R_{\mathbb{F}}(G)$ can also be described as the ring of all $\mathbb{Z}$-linear combinations of characters $\chi_\sigma : G \to \mathbb{F}$ of all the irreducible $\mathbb{F}$-representations σ of G.*

Definition 2.47. *Let $\beta : V \times V \to \mathbb{R}$ be a G-invariant symmetric bilinear form on a vector space V over $\mathbb{R}$. We define the G-signature of V to be difference*

$$\mathrm{sig}_G(V) = \sigma_{V^+} - \sigma_{V^-} \in RO(G).$$

Remark 2.48. *Now that we have a representation-valued invariant, it makes sense to consider the characters. If W is G-representation, then the character of $\chi_W : G \to \mathbb{F}$ is a class function, i.e. only depends on the conjugacy class of g. If χ_W is a real character, then $\chi_W(g) = \chi_W(g^{-1})$ for all $g \in G$. Much of representation theory is analyzed by way of the characters; for example, the number of real representations is exactly the number of conjugacy classes of unordered pairs of the form (g, g^{-1}).*

Let $g \in G$. By evaluating the virtual character of $\mathrm{sig}_G(V)$ at g, we obtain a function $\chi_V : G \to \mathbb{R}$ given by

$$\chi_V(g) = \mathrm{tr}(\sigma_{V^+}(g)) - \mathrm{tr}(\sigma_{V^-}(g)).$$

This quantity is often written $\mathrm{sig}_G(g, V)$ or $\mathrm{sig}(g, V)$ in the literature. We will often use this latter notation.

Remark 2.49. *Let G be the trivial group $\{e\}$. Then $RO(G) = \mathbb{Z}$ and* $\mathrm{sig}_G(V) = \dim_{\mathbb{R}}(V^+) - \dim_{\mathbb{R}}(V^-)$. *In this case we suppress the G in the notation and have* $\mathrm{sig}(V)$. *In general, if $g = e$ in G, then* $\mathrm{sig}(g, V) = \mathrm{sig}(V)$.

Remark 2.50. *If $G = \mathbb{Z}_2$, then there is also a second signature* $\mathrm{sig}(u, M)$ *where u is the non-trivial element of G. Here* $\mathrm{sign}(u, M) = 2\mathrm{sig}(M) - \mathrm{sig}(\widetilde{M})$.

Remark 2.51. *In the literature, both* $\mathrm{sig}_G(V) \in RO(G)$ *and* $\chi_V = \mathrm{sig}(\bullet, V) : G \to \mathbb{R}$ *are called the G-signature.*

For our purposes the most important example is the G-signature of a closed, connected, oriented Top $4k$-dimensional manifold M^{4k} endowed with an orientation-preserving action by a finite group G. We can endow the middle cohomology $H^{2k}(M)$ with any positive definite metric $\langle \cdot, \cdot \rangle$ on which G acts by isometries. Such a metric can always be found by an averaging process. With the usual cup product on $H^{2k}(M)$, we can define the *G-signature* $\mathrm{sig}_G(M) \in RO(G)$ *of the manifold* M^{4k} to be the G-signature of this cohomology ring.

For such Top manifolds of dimension $4k + 2$, the cup product is skew-symmetric, so its spectrum is purely imaginary. One can consider the difference between the positive imaginary part and the negative imaginary part to arrive at a definition for a G-signature.

Remark 2.52. *These vector spaces give rise to G-representations and hence can be subtracted in $R(G)$. Therefore the process defines a G-signature homomorphism given by*

$$\operatorname{sig}_G : L^s_{2k}(\mathbb{Z}[G]) \to RO^{(-1)^k}(G).$$

We will ease the notation and write $RO^{\pm}(G)$ with the understanding that $\pm = (-1)^k$.

Atiyah and Singer showed that, for smooth G-actions, the G-signature also has a characteristic class formula, but that it depends on M^g, i.e. the fixed set of g, as well as the manner in which g acts on the neighborhood of M^g. In the smooth case, this neighborhood can be considered a vector bundle, and the action of g decomposes it into eigenbundles. The action of the differential Dg on the normal bundle of the fixed set gives eigenvalues on the unit circle in $\mathbb{C}$. The eigenbundles corresponding to eigenvalues other than -1 have complex structures and therefore Chern classes; the (-1)-eigenspace has an Euler class. In particular, let $g \in G$ and let M^g be the subset of M which is fixed by g. The G-signature Theorem then states that $\operatorname{sig}(g, M)$ is the integral over M^g of a particular characteristic class of M^g.

Theorem 2.53. *(G-signature theorem [28]) Let G be a compact Lie group acting smoothly on a compact oriented smooth manifold M^{4k}, perhaps with boundary. Let $g \in G$, and let $L_k(g, TM|_{M^g}) \in \prod_i H^{4i}(M^g; \mathbb{R})$ be the characteristic class defined in Atiyah-Singer [28] that mixes those characteristic classes of the eigenbundles and the Hirzebruch L-genus associated to M^g. Then*[2]

$$\operatorname{sig}(g, M) = \langle L_k(g, TM|_{M^g}), [M^g] \rangle.$$

We will also express the right-hand expression as $L_k(g, M)$.

Corollary 2.54. *Let G and M be as above. If e is the identity element of G, then* $\operatorname{sig}(e, M) = \langle L_k(e, TM|_M), [M] \rangle = \operatorname{sig} M$. *If the action is free, i.e. $M^g = \emptyset$ for all non-trivial $g \in G$, then* $\operatorname{sig}(g, M) = 0$ *for all non-trivial $g \in G$.*

Corollary 2.55. *If $\sigma_{reg} : G \to \mathrm{GL}(V)$ is the regular representation on a vector space V, then* $\operatorname{tr}(\sigma_{reg}(g)) = 0$ *for all non-trivial $g \in G$. In particular* $\operatorname{sig}(\bullet, M) : G \to \mathbb{R}$ *is a multiple of* $\operatorname{tr} \circ \sigma_{reg} : G \to \mathbb{R}$*, both concentrated at just $g = e$.*

Remark 2.56. *In the special case where M^g is discrete, this formula requires contributions from each fixed point p, depending on the representation of g on the tangent space T_pM at p, and is given in Atiyah-Bott [24].*

[2] Here L_k is the notation for the L-genus, not for an L-group.

Remark 2.57. *This theorem was generalized by Wall to topological semifree actions on closed Top manifolds, which is the case required here. See §14B of Wall [682]. The assumption that M is closed is essential here, and motivates the definition of the ρ-invariant that we will see in Section 6.5.*

Remark 2.58. *Wall observed that this formula in the G-signature theorem remains true for semifree topological actions whose fixed sets happen to have a smooth normal bundle. In general, it is hard to make sense of such a formula for non-smooth actions. The fixed set M^g can be a quite pathological, and one cannot hope for an analysis using Atiyah-Bott-Singer as before. However, there are some more complicated G-signature theorems in non-smooth cases that can be found in Madsen-Rothenberg [421], T. Redman [Princeton PhD thesis], and Rosenberg-Weinberger [555].*

Remark 2.59. *One beautiful consequence of this formula is that, if M^g is empty, then* $\mathrm{sign}_g(M) = 0$. *In particular, for a free G-action, all the characters are 0 except for $g = e$. Therefore the G-signature is a multiple of the regular representation. In Section 5.3 we will give a simple topological proof of this fact using bordism theory.*

Example 2.60. *The Wall realization theorem implies that this vanishing does not occur for Poincaré complexes. Suppose that we start with a lens space L_p^{2k-1} and take a Wall realization of an element $\alpha \in L^h_{2k}(\mathbb{Z}[\mathbb{Z}_p])$ such that $\mathrm{sig}_G(\alpha)$ is representation with non-trivial character for some $g \neq e$. We can then glue the two ends together to produce a closed Poincaré complex whose universal cover has a $\mathbb{Z}_p$-signature that is not a multiple of the regular representation. It is a Poincaré complex, with a normal invariant by construction, but it is not homotopy equivalent to a manifold.*

Remark 2.61. *In the previous example, if we have $\alpha, \alpha' \in L^h_{2k}(\mathbb{Z}[\mathbb{Z}_p])$ such that $\mathrm{sig}_G(\alpha)$ and $\mathrm{sig}_G(\alpha')$ have different characters, then the homotopy lens spaces must be different. If they were isomorphic, one could glue these normal cobordisms together and obtain a closed manifold whose universal cover has a G-signature that is not a multiple of the regular representation.*

2.4 ODD L-GROUPS AND THE RANICKI-ROTHENBERG SEQUENCE

In the previous section, we saw that some calculations were easier for L^h and others for L^p, but that the two were related for the real group ring $\mathbb{R}[\pi]$. We now explicitly describe the relationship between L-groups with different decorations, and then use these ideas to give some information about odd L-groups.

Theorem 2.62. *Let G be a group. Then there are exact sequences*

$$\cdots \to \widehat{H}^{n+1}(\mathbb{Z}_2; \mathrm{Wh}(G)) \to L^s_n(\mathbb{Z}[G]) \to L^h_n(\mathbb{Z}[G]) \to \widehat{H}^n(\mathbb{Z}_2; \mathrm{Wh}(G)) \to \cdots$$

and

$$\cdots \to \widehat{H}^{n+1}(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[G])) \to L_n^h(\mathbb{Z}[G]) \to L_n^p(\mathbb{Z}[G]) \to \widehat{H}^n(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[G]) \to \cdots$$

that relate the L-groups $L_n^h(\mathbb{Z}[G])$ with simple and projective L-groups. Here the group $\widehat{H}^{n+1}(\mathbb{Z}_2; \Lambda)$ means the Tate cohomology of $\mathbb{Z}_2$ with coefficients in Λ.

The first sequence above is the *Rothenberg sequence*, which appears in Shaneson [580] with a more explicit description than Tate cohomology. In this section, we give a geometric explanation of this sequence. We will then describe the second sequence, called the *Ranicki-Rothenberg sequence*, and offer an application.

Let us start with the data needed to perform simple surgery. For L^h-surgery, one merely needs a Poincaré space X with the homotopy type of a finite complex. For simple surgery, we need additional structure on X, namely that the Poincaré duality map on the cellular chain complex of X has vanishing Whitehead torsion.

Definition 2.63. *Suppose that X is an n-dimensional Poincaré complex with fundamental group π, and let $\phi\colon C_*(X) \to C^{n-*}(X)$ be the chain equivalence on the based chain complexes given by Poincaré duality. If the Whitehead torsion $\tau(\phi)$ vanishes in* $\mathrm{Wh}(\pi)$*, we say that X is a simple Poincaré complex and ϕ is a simple isomorphism.*

Not every finite Poincaré complex X is homotopy equivalent to a simple one. The set of simple homotopy types is in bijective correspondence with $\mathrm{Wh}(\pi)$, which is often non-trivial. There is a condition on the torsion of homotopy equivalences between n-manifolds that one needs in order to explain the Ranicki-Rothenberg sequence.

Theorem 2.64. *If $f : X' \to X$ is a homotopy equivalence of n-dimensional Poincaré complexes with fundamental group π, let $\tau(X')$ and $\tau(X)$ be their self-duality torsions, respectively. Let $\tau = \tau(f)$ be the torsion of f. Then torsions $\tau(X')$ and $\tau(X)$ differ from each other by $\tau + (-1)^n\tau^*$ in* $\mathrm{Wh}(\pi)$*, where τ^* is the result of the involution on the Whitehead group induced by $g \mapsto w(g)g^{-1}$.*

Remark 2.65. *This term describes how f intertwines the self-duality map of X and X'. In particular, we have the formula $\tau = (-1)^{n+1}\tau^*$ for a homotopy equivalence between closed manifolds since their self-dualities have zero torsion.*

To determine whether X is homotopy equivalent to any simple Poincaré complex X with fundamental group π, one would start with a normal invariant and determine whether its surgery obstruction vanishes in $L_*^s(\mathbb{Z}[\pi])$.

Remark 2.66. *PL manifolds are simple Poincaré complexes; the proof follows from that of Poincaré duality using dual skeleta. In the Top category, the finiteness of the homotopy type of a compact manifold is already quite non-trivial. Indeed it has a canonical simple homotopy type (see Kirby-Siebenmann [361]). This finite structure satisfies simple duality in the above sense.*

Remark 2.67. *When the group π is cyclic, the group* $\text{Witt}(\pi)$ *is free abelian (see Bass [46]), and for the orientable case, the involution is trivial. Therefore, a homotopy equivalence between even-dimensional manifolds with such a fundamental group is automatically simple.*

We now explain the various maps in the Ranicki-Rothenberg sequence. Given a homotopy type X, we can choose any finite complex X' homotopy equivalent to X and determine $\tau(PD_{X'})$, where $PD_{X'}$ is the chain equivalence $\cap[X'] : C_*(X) \to C^{n-*}(X)$ guaranteed by Poincaré duality. Note that $PD_{X'}$ is chain homotopy equivalent to its dual, so $\tau(PD_{X'}) = (-1)^n \tau(PD_{X'})^*$. This self-duality changes as X is changed, so we obtain an element

$$\sigma(X) \in \{\tau | \tau = (-1)^n \tau^*\}/\{\tau + (-1)^n \tau^*\}.$$

(Note that anything of the form $\tau + (-1)^n \tau^*$ satisfies the $(-1)^n$-self-duality; it is called a *norm*.) By definition, this quotient is the Tate cohomology group $H^n(\mathbb{Z}_2; \text{Wh}(\pi))$. In other words, the Tate cohomology group is made up of symmetric elements modulo norms. Note that $\sigma(X)$ gives an obstruction to the existence of a homotopy equivalence from X to a closed manifold. The homomorphism

$$L_n^h(\mathbb{Z}[\pi]) \to H^n(\mathbb{Z}_2; \text{Wh}(\pi))$$

takes the surgery obstruction of any degree one normal invariant of X to $\sigma(X)$. It can be considered the torsion of the adjoint map in the condition (2) in the algebraic definition of the even L-group. In fact, the obstruction can be defined a priori without any normal invariant at all. The homomorphism is part of the Rothenberg sequence relating the L-groups L^s and L^h

$$\cdots \to H^{n+1}(\mathbb{Z}_2; \text{Wh}(\pi)) \to L_n^s(\mathbb{Z}[\pi]) \to L_n^h(\mathbb{Z}[\pi]) \to H^n(\mathbb{Z}_2; \text{Wh}(\pi)) \to \cdots$$

mentioned previously. See Shaneson [579] and Wall [672]. If the simplicity obstruction vanishes, then we can find a simple homotopy equivalent space X' for which the degree one normal map is suitable for simple surgery. It is not unique, since the possible choices for X are parametrized by $H^{n+1}(\mathbb{Z}_2; \text{Wh}(\pi))$.

The first map $H^{n+1}(\mathbb{Z}_2; \text{Wh}(\pi)) \to L_n^s(\mathbb{Z}[\pi])$ can be interpreted in the following way. Let W be an h-cobordism from X to another manifold X'. If the torsion of W is "correctly self-dual," then the map $X' \to X$ will be simple. Therefore, a map $F : W \to X \times [0, 1]$ gives a rel boundary surgery problem whose surgery obstruction can then be taken. Since the simple homotopy type can be modified by any torsion whose norm is trivial, the map to $L_n^s(\mathbb{Z}[\pi])$ can only be well-defined up to an element of $H^{n+1}(\mathbb{Z}_2; \text{Wh}(\pi))$.

Remark 2.68. *This duality also plays a role in the h-cobordism theorem. If (W, M, M') is an h-cobordism, then $\tau(W, M) = (-1)^{n+1} \tau(W, M')^*$. The torsion of the homotopy equivalence between M' and M is the norm of the torsion $\tau(W, M)$.*

The sequence that relates L^h-groups with L^p-groups is quite similar. Following the algebraic theory, we regard L-groups to be defined by cobordism classes of algebraic Poincaré complexes, where the chain complexes are based on finitely generated projective Λ-modules.

Definition 2.69. *A ring Λ is semisimple if every short exact sequence of Λ-modules splits, so every submodule of a free module is a summand. In other words, one can use the "truncation" of a $(2k+1)$-dimensional chain complex D given by*

$$0 \to \operatorname{im} \partial_{k+1} \to C_k \to C_{k-1} \to C_{k-2} \to \cdots \to C_0 \to 0$$

as a cobounding algebraic Poincaré complex.

We then have the theorem of Ranicki:

Theorem 2.70. *(Ranicki [534]) If Λ is semisimple, then the odd projective L-groups $L^p_{2k+1}(\Lambda)$ are trivial.*

Remark 2.71. *This null-cobordism is the formal analogue of the process used in the self-duality of the intersection homology of the cone on an odd-dimensional manifold (Goresky-MacPherson [266]).*

Therefore, the group $L^p_i(\mathbb{R}[\mathbb{Z}_2])$ vanishes unless $i \equiv 0 \bmod 4$. In this case, the isomorphism $L^p_i(\mathbb{R}[\mathbb{Z}_2]) \cong \mathbb{Z} \oplus \mathbb{Z}$ is given by the signatures over the real two factors of $\mathbb{R}[\mathbb{Z}_2]$. The K-group $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_2])$ is isomorphic to $\mathbb{Z}$, and the Tate cohomology is $\mathbb{Z}_2$ or 0, depending on the parity of the dimensions.

We now consider $L^h_i(\mathbb{R}[\mathbb{Z}_2])$. We have already discussed the case for even i in the previous section and saw that $L^h_2(\mathbb{R}[\mathbb{Z}_2])) = 0$ and $L^h_0(\mathbb{R}[\mathbb{Z}_2]))$ is the subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ where the two factors have the same parity. In odd dimensions $2k+1$, we have a pair of Poincaré complexes C and C' over $\mathbb{R}$ that bound some spaces D and D'. If $\chi(D) = \chi(D')$ we could combine them. We can also always change the coboundary by the Euler characteristic $\chi(X)$ of some closed R-Poincaré complex X of dimension $2k+2$. The Euler characteristic of an even-dimensional oriented manifold is even in dimensions 2 mod 4, but can take any value in dimension 0 mod 4 because of $\mathbb{CP}^2$. Therefore, we obtain

$$L^h_3(\mathbb{R}[\mathbb{Z}_2]) = 0 \text{ and } L^h_1(\mathbb{R}[\mathbb{Z}_2]) = \mathbb{Z}_2.$$

We have therefore calculated and verified the Ranicki sequence:

$$\cdots \to L^h_3(\mathbb{R}[\mathbb{Z}_2]) \to L^p_3(\mathbb{R}[\mathbb{Z}_2]) = 0 \to 0 \to L^h_2(\mathbb{R}[\mathbb{Z}_2]) \to$$

$$L^p_2(\mathbb{R}[\mathbb{Z}_2]) = 0 \to \mathbb{Z}_2 \to L^h_1(\mathbb{R}[\mathbb{Z}_2]) \to L^p_1(\mathbb{R}[\mathbb{Z}_2]) = 0 \to$$

$$0 \to L^h_0(\mathbb{R}[\mathbb{Z}_2]) \to L^p_0(\mathbb{R}[\mathbb{Z}_2]) = \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2 \to \cdots$$

where the Tate cohomologies are unlabeled but computed.

In general, we obtain the formula for the odd L-groups of a semisimple ring:

$$L^h_{2k+1}(\Lambda) \cong H^{2k}(\mathbb{Z}_2; \widetilde{K}_0(\Lambda))/\mathrm{im}\,(L^p_{2k+2}(\Lambda)).$$

In other words, the odd L^h groups are self-dual projective modules, modulo those that support symmetric or skew-symmetric forms. We note that the the K_0-group is also easy to compute, as all finitely generated modules are projective.

J. Davis wrote an elegant paper exploiting these ideas to prove Milnor's theorem that no dihedral group can act freely on a sphere, the significance of which is discussed in Section 6.6, as well as a number of other related results.

Definition 2.72. *(Davis [186]) Let X be a $(2k+1)$-dimensional Poincaré complex, and let A be a semisimple ring with involution $a \mapsto \overline{a}$. Then define the surgery semicharacteristic of (X, A) to be*

$$\chi_{1/2}(\widetilde{X}; A) = \sum_{i=0}^{k}(-1)^i[H_i(\widetilde{X}; A)] \in \widetilde{K}_0(A).$$

As $L^h_{2n+1}(A)$ is a quotient of $H^{2n+1}(\mathbb{Z}_2; \widetilde{K}_0(A))$, the semicharacteristic $\chi_{1/2}(\widetilde{X}; A)$ can be considered an element of $L^h_{2n+1}(A)$.

Theorem 2.73. *(Davis [186]) Let $f : M^{2k+1} \to X$ be a degree one normal map from a Top manifold M of dimension $2k+1$ to a Poincaré complex X. Let $\sigma : \mathcal{N}^{Top}(X) \to L_{2k+1}(\mathbb{Z}[\pi], w)$ be the usual surgery map, where $\pi = \pi_1(X)$. Given a homomorphism $(\mathbb{Z}[\pi], w) \to (A, -)$, we can regard A as a local coefficient system, i.e. a $\mathbb{Z}[\pi]$-module. If $j_* : L^h_{2k+1}(\mathbb{Z}[\pi], w) \to L^h_{2k+1}(A)$ is the induced map on L-theory, then the image $j_*(\sigma(M, f))$ of the surgery obstruction $\sigma(M, f)$ under j_* satisfies the equation*

$$j_*(\sigma(M, f)) = \chi_{1/2}(\widetilde{M}; A) - \chi_{1/2}(\widetilde{X}; A),$$

where $\widetilde{X}$ is the universal cover of X and $\widetilde{M}$ is the induced π-cover of M.

Denote by $\mathbb{F}_2[\pi]/\mathrm{rad}$ the largest semisimple quotient of the group ring $\mathbb{F}_2[\pi]$. Here rad is the Jacobson radical of the group ring $\mathbb{F}_2[\pi]$. By Wall [680], the surgery obstruction of a degree one normal map between two closed manifolds depends only on the 2-Sylow subgroup $\pi_{(2)}$ of $\pi = \pi_1(X)$. Therefore, since $\mathbb{F}_2[\pi_{(2)}]$ is a local ring, i.e. a ring with a unique left or right maximal ideal, we have $\widetilde{K}_0(\mathbb{F}_2[\pi_{(2)}]/\mathrm{rad}) = 0$. Therefore $L^h_{2k+1}(\mathbb{F}_2[\pi_{(2)}]/\mathrm{rad}) = 0$. These facts can be used to give a necessary condition for the existence of a degree one normal map between two closed manifolds. The reader can consult Sections 5.3 and 6.7 for related discussions.

This theory can be used to study the reduction of the surgery obstruction to $L_{2k+1}(\mathbb{Z}_{(\pi)}[\pi])$, and when the obstruction there vanishes, one obtains surgery problems whose kernels are finite groups prime to the order of the group. They lead to an L-theoretic variation on the idea of a Swan homomorphism, and we refer to the papers of Davis [186] for these ideas. A typical example of it preceded the theory, and is due to Pardon.

Theorem 2.74. *(Pardon [498]) Let* $k \geq 1$ *and* $f : M^{4k+3} \to X$ *be as above, and let* $\sigma : \mathcal{N}^{Cat}(X) \to L_{4k+3}(\mathbb{Z}[\pi])$ *be the surgery map. If* π *is cyclic of even order, then* $\sigma(f)$ *vanishes in* $L_{4k+3}(\mathbb{Z}[\pi])$ *iff the size* $|K_{2k+1}(M)|$ *of the kernel is congruent to 1 mod 8.*

The example of Cappell-Weinberger [132] of a Poincaré complex that is locally a manifold at every prime, but not itself a manifold, is verified using Pardon's method. The theory of Davis is very valuable for understanding homology propagation of group actions for closed simply connected manifolds.

2.5 TRANSFER AND DRESS INDUCTION

In the context of a covering space or a fiber bundle, we often have a notion of a *transfer* or *transfer map*. In this section we will introduce a few examples of transfer in the context of K-theory and the Whitehead group. Of course our most important examples are for the purposes of L-theory computations.

We will discuss various phenomena that arise when a functor is not only covariant, but also contravariant for subgroups of finite index. With this double functoriality, the functors become modules over an interesting ring of operations, giving them a structure which can yield interesting information. This structure will be important in the discussion of the spaceform problem in Section 6.6 and of almost flat manifolds in Chapter 7.

2.5.1 Transfer arguments

We first consider a general situation. Let $p : X \to Y$ be a degree n covering map. If $\sigma : \Delta^k \to Y$ is a singular k-simplex on Y, then the covering space theory provides n different lifts $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_n : \Delta^k \to X$ of σ. Define $\tau_k(\sigma)$ to be the singular k-chain $\widetilde{\sigma}_1 + \cdots + \widetilde{\sigma}_n$ on X. This map extends by linearity to a map $\tau : C_k(Y; R) \to C_k(X; R)$, where R is any commutative ring and $C_*(\bullet ; R)$ is the abelian group of singular simplices with coefficients in R. It is clear that the τ_k combine together to form a chain map $\tau : C_*(Y; R) \to C_*(X; R)$ that satisfies

$$p_*(\tau(x)) = n \cdot x,$$

where $p_* : C_*(X; R) \to C_*(Y; R)$ is the map on singular chains induced by p. The *umkehr* or *transfer map* $\tau : H_*(Y; R) \to H_*(X; R)$ is the map on homology induced by τ.

As a first example, let BG be the usual classifying space for principal G-bundles. We know that the Eilenberg-Maclane space $K(G,1)$ is an appropriate model for BG. For all commutative rings R, the (co)homology groups $H_k(BG;R) = H_k(K(G,1);R)$ are obviously functorial with respect to group homomorphisms $G \to H$. Associated to inclusions of subgroups $\Gamma \hookrightarrow G$ of finite index, the above construction yields a transfer map

$$i^* : H_k(K(G,1);R) \to H_k(K(\Gamma,1);R),$$

where $K(\Gamma,1)$ is considered to be a finite-sheeted cover of $K(G,1)$. Often this map is called the *classical transfer*.

Example 2.75. *Suppose G is a finite group acting simplicially on a space X, and suppose that K is a subgroup of G. Then there is the usual map $i_* : H_k(X/K) \to H_k(X/G)$ as well as the transfer $i^* : H_k(X/G) \to H_k(X/K)$.*

Proposition 2.76. *Suppose that K is trivial in the context above. Then the composition $i_* \circ i^* : H_k(X/G) \to H_k(X/G)$ is multiplication by $|G|$, i.e. $i_* \circ i^*(\sigma) = |G|\sigma$. The reverse composition $i^* \circ i_* : H_k(X/K) \to H_k(X/K)$ is given by $i^* \circ i_*(\sigma) = \sum g_*$ where the sum is of the induced map on homology.*

We immediately have the following corollaries.

Corollary 2.77. *If G is a finite group acting on a space X, and if $1/|G| \in R$, then $H_k(X/G;R) \cong H_k(X;R)^G$, where the superscript indicates G-invariance.*

Note that, if G is finite, the $(j+2)$-fold join $X = G * G * \cdots * G$ is a j-connected finite complex on which G acts freely. Therefore $H_k(BG,\mathbb{Z})$ is finitely generated for all $k \le j$. This finite generation is true for all k because j is arbitrary. As a consequence of the above argument, we obtain the following.

Corollary 2.78. *If G is finite, then the group $H_k(BG;\mathbb{Z})$ is finite for all $k \ge 0$. It can have p-torsion only if p divides $|G|$.*

Corollary 2.79. *If G_p is the p-Sylow subgroup of the finite G, the map $H_k(BG_p;\mathbb{Z}) \to H_k(BG;\mathbb{Z})$ is surjective on p-torsion, and the map $H_k(BG;\mathbb{Z}) \to H_k(BG_p;\mathbb{Z})$ is injective on p-torsion; i.e. it is injective when the map is restricted to the p-torsion component of the domain.*

Remark 2.80. *In Section 6.6 we will discuss the notion of cohomological periodicity. Another corollary of the above is the following. If $p \equiv 1 \bmod q$ and $G = \mathbb{Z}_p \rtimes \mathbb{Z}_q$ is a noncommutative metacyclic group, then the cohomology of G has period $2q$. An element $u \in \mathbb{Z}_p^*$ induces a map by the action of $\mathbb{Z}_q$ on $H^{2i}(B\mathbb{Z}_p) = \mathbb{Z}_p$ given by multiplication by u^i.*

Remark 2.81. *The algebraic topology story continues in several directions. One can extend from finite G to compact G. See Oliver [490], and Lewis-May-McClure [399],* *who explain how it follows naturally from the theory of $RO(G)$-graded functors. For non-free actions an extension is not possible for all generalized homology theories. However, for free actions of finite groups, the transfer makes sense for any cohomology theory (Kahn-Priddy [340] and Roush [565]). For free actions of compact Lie groups, the relevant definition and theory were developed by Becker-Gottlieb [51], who gave an interesting application. For this story, see Adams [6].*

Remark 2.82. *We note that, if Γ is an infinite discrete group, we can still develop all of these operations if we have a finite group G and a surjection $\Gamma \to G$. In Chapter 7, this idea is applied to the fundamental group of a flat manifold, where G is its holonomy. One uses induction to reduce problems involving complicated holonomy groups to ones involving smaller ones.*

Remark 2.83. *There are transfers associated to positive-dimensional fibers. They are important in work on the Farrell-Jones conjecture and stratified surgery. Unfortunately we will not discuss them in these notes.*

2.5.2 Dress induction

Induction theory found its origins in the work of Artin and Brauer on representation theory, and was placed into its most abstract setting by Dress. Since there is a close connection of L-theory to representation theory implemented by the multisignature, the idea of expanding its tools to surgery should seem promising. Dress induction involves the interplay of the various subgroups of finite index of a group G and the induction and restriction maps that link them. We will see its role in Section 6.6 when it is used to compute surgery obstructions in the topological spaceform problem. In this book, Dress induction will only be defined in the orientable case.

When H is a subgroup of the group G, then we can define a transfer on complex representation rings given by restriction $i^* : R(G) \to R(H)$, while induction is given by $i_* : R(H) \to R(G)$. Perhaps the most important theme in representation theory of finite groups is that any representation can be completely recovered by its character. In particular, representations of G are detected by the restrictions to cyclic subgroups; i.e. if $\mathfrak{C}$ denotes the class of cyclic subgroups of G, then the restriction map

$$R(G) \to \bigoplus_{C \in \mathfrak{C}} R(C) \tag{2.84}$$

is injective. It is however not surjective, since different conjugates of the same subgroup do not yield any additional information.

Dually there are also maps that combine to give an induction map

$$\bigoplus_{C \in \mathfrak{C}} R(C) \to R(G)$$

whose image, by the Artin induction theorem, is a subgroup of finite index. But it is not surjective; there we potentially do not understand all the representations. When we consider the cohomology of groups, we immediately see that cyclic subgroups are inadequate to achieve an isomorphism. For instance, the rank of the group $H^i(\mathbb{Z}_p \times \mathbb{Z}_p)$ grows linearly with degree, but there are only $p+1$ cyclic subgroups, and each only has bounded rank.

Brauer extended $\mathfrak{C}$ in the following way. If G is a finite group, let $\mathfrak{B}$ be the collection of subgroups of G of the form $H_1 \times H_2$ where H_1 is a p-group (for any p) and H_2 is a cyclic group.

Theorem 2.85. *(Brauer) If G is a finite group, then restriction map*

$$R(G) \to \bigoplus_{H \in \mathfrak{B}} R(H)$$

is injective and the induction map

$$\bigoplus_{H \in \mathfrak{B}} R(H) \to R(G)$$

is surjective. In neither case is an isomorphism guaranteed.

Remark 2.86. *If one tensors the representation rings with $\mathbb{Z}_{(p)}$ for a particular p, then one achieves the same injectivity and surjectivity results by replacing the generic p with the specific p in the subgroup H_1.*

Example 2.87. *For instance, from the fact that 2 times the trivial representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the sum of the induced representations of the trivial representations from each of the non-trivial cyclic subgroups, one can conclude that away from 2 any representation is a sum of representations induced from cyclic subgroups. Brauer induction can be viewed as a similar formula, but with integer coefficients, i.e. a formula for the trivial representation rather than for a multiple of it.*

Remark 2.88. *The proof of Brauer's theorem uses Frobenius reciprocity, which states that, if V is an H-representation and W is a G-representation, then $\mathrm{Hom}_H(V, i^*W) = \mathrm{Hom}_G(i_*V, W)$ or equivalently $\langle i_*V, W\rangle_G = \langle V, i^*W\rangle_H$ in terms of the character inner product. For example, see Serre [578] and Fulton-Harris [257].*

Remark 2.89. *If we want isomorphisms for the above maps, it is better to replace the above map by the inverse limit over the category of subgroups of G lying in $\mathfrak{B}$ with morphisms given by conjugation and inclusions.*

Remark 2.90. *This problem is much worse when we apply these ideas in K-theory because the relevant groups contain torsion, and we lose information if only cyclic subgroups are considered. See Swan [633] for the relevance of Brauer induction to this problem.*

According to Dress [207], the same reasoning applies in L-theory, although one needs a somewhat larger class of subgroups. In the place of products, one needs semidirect products. The following is a variant of Dress's original theorem as generalized by Farrell-Hsiang.

Theorem 2.91. *Let $\Gamma \to G$ be a surjection, and denote by Γ_K the inverse image of K. Let $\mathfrak{J}$ be the collection of subgroups of G that are semidirect products $K = K_1 \rtimes K_2$, where K_1 is a p-group and K_2 is a cyclic group. For any ring R, the map*

$$L_k(R\Gamma) \to \bigoplus_{K \in \mathfrak{J}} L_k(R\Gamma_K)$$

is injective for each k. If one tensors with $\mathbb{Z}_{(p)}$ then we only need the subgroups corresponding to that p; therefore, for rational calculations, one can restrict attention to cyclic subgroups. Here $R\Gamma$ is the group ring associated to R and Γ, not any kind of representation ring.

We can express induction more formally in the following way. Let *Groups* denote the category of finite groups whose morphisms are the monomorphisms, and let F be an arbitrary contravariant functor from *Groups* into the category *Abelian* of abelian groups such that, for any inner conjugation $i_g : G \to G$ by an element $g \in G$, the map induced by the functor F is the identity $i_L = F(i_g) : F(G) \to F(G)$.

Let $\mathfrak{J}$ be an arbitrary class of finite groups closed with respect to subgroups and isomorphic images. If G is a finite group, let $\mathfrak{J}(G)$ be the collection of all subgroups of G that belong to $\mathfrak{J}$. Consider the subgroup

$$\varprojlim_{K \in \mathfrak{J}(G)} F(K) \subseteq \prod_{H \in \mathfrak{J}(G)} F(K).$$

The product of all the restriction maps $\phi : F(G) \to \prod_{H \in \mathfrak{J}(G)} F(K)$ maps $F(G)$ into $\varprojlim_{H \in \mathfrak{J}(G)} F(K)$. We say that L is $\mathfrak{J}$ *computable* if the map ϕ induces an isomorphism

$$\phi' : F(G) \to \varprojlim_{H \in \mathfrak{J}(G)} F(K).$$

Let $\mathfrak{C}$ be the class of cyclic groups. For each prime p, let $\mathfrak{P}_p$ be the class of p-elementary groups, i.e. groups of the form $K_1 \times H_2$, where K_1 is cyclic and K_2 is a p-group. Let $\mathfrak{H}_p$ be the class of p-hyperelementary groups, i.e. groups that contain a normal cyclic subgroup with p-power index. For any abelian group A, let $A[1/2] \equiv A \otimes \mathbb{Z}[1/2]$ and $A_{(p)} \equiv A \otimes \mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q} : (n, p) = 1\}$. Tensoring with $\mathbb{Z}_{(p)}$ eliminates everything with p-torsion.

Theorem 2.92. *(Dress [207]) For the functor $F = L(R\Gamma)$ when $\Gamma \to G$ is a surjection, as in the previous theorem, we have the following:*

1. *$F \otimes \mathbb{Q}$ is $\mathfrak{C}$ computable;*

2. *$F[{}^1\!/\!{}_2] \equiv F \otimes \mathbb{Z}[{}^1\!/\!{}_2]$ is $\bigcup_{p\neq 2} \mathfrak{P}_p$ computable;*
3. *$F_{(2)} \equiv F \otimes \mathbb{Z}_{(2)}$ is $\mathfrak{H}_2$ computable;*
4. *F is $\mathfrak{H}_2 \cup \bigcup_{p\neq 2} \mathfrak{P}_p$ computable.*

Remark 2.93. *If G is a group, denote by $\mathfrak{H}_2(G)$ the collection of subgroups of G that are in the class $\mathfrak{H}_2$, and denote by $\mathfrak{C}(G)$ the collection of subgroups of G that are in the class $\mathfrak{C}$. Note that, if G is a group of odd order, then $K \in \mathfrak{H}_2(G)$ iff K has a normal cyclic subgroup with 2-power index iff K is cyclic iff $H \in \mathfrak{C}(G)$. So for odd-order groups one only requires the computations for cyclic odd-order groups.*

We note the following result, which (1) explains why L-theory could fit into this framework, but (2) indicates a difficulty in applying it too directly to the nonoriented setting.

Theorem 2.94. *(Taylor) Let G be a group and let $g \in G$. Denote by $c_g : G \to G$ the map that conjugates by g, and let w be a homomorphism $w : G \to \mathbb{Z}_2$. Then the induced map $c_{g*} : L_*(\mathbb{Z}[G]) \to L_*(\mathbb{Z}[G])$ is the map that multiplies by $w(g)$.*

Proof. We give a geometric argument in the case $L_0(\mathbb{Z}[G], w)$ when the L-theory element is represented by a surgery problem on a $4k$-dimensional manifold. Given a surgery problem, one obtains a quadratic form after surgery that determines the surgery obstruction. However, a choice is made by a path from the basepoint to each immersed sphere. If we exchange all of these paths by precomposing by the loop g, the intersection number is multiplied by $w(g)$. The loop at the basepoint is also conjugated by g. Therefore the two surgery problems are the same. □

Theorem 2.95. *Let G be a finite group. Then we have an equality*

$$L^h_*(\mathbb{Z}[G]) = \varprojlim L^h_*(\mathbb{Z}[K]),$$

where K runs over the conjugacy classes in G of 2-hyperelementary subgroups, and the maps are the restrictions. In particular, the surgery obstruction for a particular normal invariant is zero in $L^h_*(\mathbb{Z}[G])$ iff its image under the map $L^h_*(\mathbb{Z}[G]) \to L^h_*(\mathbb{Z}[K])$ is zero for each 2-hyperelementary subgroup K of G.

Therefore the study of $L^h_*(\mathbb{Z}[G])$ for G finite is reduced to the problem of computing the L-groups for 2-hyperelementary groups.

Remark 2.96. *Dress's theorem is a general theorem about modules over the equivariant Witt ring GW, and therefore includes the theorems above. In Chapter 4 we will put all of surgery theory in an L-theoretic frame, so topological structure sets will be a type of relative L-group and therefore also satisfy Dress induction (see Nicas [482]). The same also holds for G-manifolds and their isovariant structure sets $S^{Top}_G(M \text{ rel } M_{sing})$ relative to their singular sets (see Section 8.7).*

2.6 SQUARES

Dress induction is a powerful tool for converting calculations from harder groups to easier ones. So far the only integral group ring that we have examined is $\mathbb{Z}[e]$. We took an approach that is based on extremely precise information about the quadratic form theory. In general, it is rare for this kind of information to be available.

To compute the L-theory of other group rings, we can compare them to rings that are easier to manage. In algebraic K-theory and L-theory there are various Mayer-Vietoris sequences associated to squares with appropriate properties that will enable us to perform calculations. See the beginning of Milnor [457] and Wall [671].

This section focuses on the computation of $L_*(\mathbb{Z}[\mathbb{Z}_2^\pm])$, and then discusses the arithmetic square that is used for general calculations. The starting point is the pullback square

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}_2^+] & \xrightarrow{g\mapsto 1} & \mathbb{Z} \\ {\scriptstyle g\mapsto -1}\downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{F}_2 \end{array}$$

Remark 2.97. *Note that the maps for $\mathbb{Z}_2^-$ do not preserve the involution, so we cannot use the square to compute its L-theory.*

We recall that $L_0(\mathbb{F}_2) = L_2(\mathbb{F}_2) \cong \mathbb{Z}_2$ and $L_1(\mathbb{F}_2) = L_3(\mathbb{F}_2) = 0$. The quadrad gives rise to a twelve-term exact sequence

$$\begin{array}{ccccccccccc} L_0(\mathbb{Z}[\mathbb{Z}_2^+]) & \longrightarrow & L_0(\mathbb{Z}[e])^2 & \xrightarrow{0} & L_0(\mathbb{F}_2) & \longrightarrow & L_3(\mathbb{Z}[\mathbb{Z}_2^+]) & \longrightarrow & \cancel{L_3(\mathbb{Z}[e])^2} & \longrightarrow & \cancel{L_3(\mathbb{F}_2)} \\ \uparrow & & & & & & & & & & \downarrow \\ \cancel{L_1(\mathbb{F}_2)} & \longleftarrow & \cancel{L_1(\mathbb{Z}[e])^2} & \longleftarrow & L_1(\mathbb{Z}[\mathbb{Z}_2^+]) & \longleftarrow & L_2(\mathbb{F}_2) & \twoheadleftarrow & L_2(\mathbb{Z}[e])^2 & \longleftarrow & L_2(\mathbb{Z}[\mathbb{Z}_2^+]) \end{array}$$

where the slash mark indicates that the given groups are trivially zero.

We need to compute some of the arrows. First we explain why $L_0(\mathbb{Z}[e]) \to L_0(\mathbb{F}_2)$ induced by the surjection $\mathbb{Z} \to \mathbb{Z}_2$ is the zero map. The generator of $L_0(\mathbb{Z}[e])$ is the Type II E_8-form $\lambda : \mathbb{Z}^8 \times \mathbb{Z}^8 \to \mathbb{Z}$. Taking the mod 2 representation of E_8, which is also Type II, we can endow it with the form with $\mu \equiv 1$ (note that condition 3b in Definition 1.30 is satisfied). If $\{\alpha_1, \ldots, \alpha_4, \beta_1, \ldots, \beta_4\}$ is a symplectic basis, then the Arf invariant $\mathrm{Arf} : L_2(\mathbb{Z}[e]) \to \mathbb{Z}_2$ of this form is $\sum_{i=1}^4 \mu(\alpha_i)\mu(\beta_i)$ which is 0.

Therefore $L_0(\mathbb{Z}[\mathbb{Z}_2^+]) \cong L_0(\mathbb{Z}[e])^2 = \mathbb{Z}^2$ and $L_3(\mathbb{Z}[\mathbb{Z}_2^+]) \cong L_0(\mathbb{F}_2) = \mathbb{Z}_2$. In addition, the map $L_2(\mathbb{Z}[e])^2 \to L_2(\mathbb{F}_2)$ is surjective because $L_2(\mathbb{Z}[e])$ contains the same Arf invariant as $L_2(\mathbb{F}_2)$ by Remark 2.12. Therefore $L_2(\mathbb{Z}[\mathbb{Z}_2^+]) \cong \mathbb{Z}_2$ and $L_1(\mathbb{Z}[\mathbb{Z}_2^+]) = 0$.

The L-groups for $\mathbb{Z}_2^-$ are 2-periodic, with a periodicity induced by multiplying by the bilinear form on $\mathbb{RP}^2$. The L-groups are $\mathbb{Z}_2$ in even dimensions, detected by the Arf invariant, and vanish in odd dimensions. The proofs use either the methods of arithmetic squares in Wall that we mentioned in Remark 2.100, or the Browder-Livesay approach that we will discuss in Section 6.2.

In summary we have the following:

	$\mathbb{Z}_2^+$	$\mathbb{Z}_2^-$
$L_0(\mathbb{Z}[\pi])$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_2$
$L_1(\mathbb{Z}[\pi])$	0	0
$L_2(\mathbb{Z}[\pi])$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$L_3(\mathbb{Z}[\pi])$	$\mathbb{Z}_2$	0

Note the interesting element in $L_3(\mathbb{Z}[\mathbb{Z}_2^+]) \cong \mathbb{Z}_2$ that arises from an Arf invariant one dimension lower. If one took the Kervaire surgery problem $M^{4k+2} \to \mathbb{S}^{4k+2}$ with nonzero simply connected surgery obstruction and crossed with a circle, one would obtain an element in $L_3(\mathbb{Z}[\mathbb{Z}])$ which pushes forward to this non-trivial element of $L_3(\mathbb{Z}[\mathbb{Z}_2])$. The isomorphism $L_0(\mathbb{Z}[\mathbb{Z}_2^+]) \cong \mathbb{Z} \times \mathbb{Z}$ is given by the simply connected obstruction of the problem and its two-fold cover. In any case, we know that the two copies of $\mathbb{Z}$ come from the multisignature.

For an odd prime p, let $\xi_p = e^{2\pi i/p}$. A similar square can be used to reduce the L-group computations for $\mathbb{Z}[\mathbb{Z}_p]$ to the quadratic theory of the number ring $\mathbb{Z}[\xi_p]$:

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}_p] & \longrightarrow & \mathbb{Z}[\xi_p] \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{F}_p \end{array}$$

This square is often called a *Rim square*.

However, for composite numbers n, the situation is more complicated. One needs to use a number of different number rings associated to the roots of unity dividing n. The "finite glue" of the diagram, i.e. the instructions described by the right vertical and bottom right arrows, is called the *conductor* in Bass [46] and has a more elaborate structure.

In Section 2.3 we stated some results about projective L-groups. Here we give some additional calculations without proof.

Theorem 2.98. *The projective L-groups $L^p(\mathbb{Z}[\mathbb{Z}_n^+])$ are given in the following table.*

	n odd	n even
$L_0(\mathbb{Z}[\pi])$	$\mathbb{Z}^{(n+1)/2}$	$\mathbb{Z}^{(n+2)/2}$
$L_1(\mathbb{Z}[\pi])$	0	0
$L_2(\mathbb{Z}[\pi])$	$\mathbb{Z}^{(n-1)/2}$	$\mathbb{Z}_2 \times \mathbb{Z}^{(n-2)/2}$
$L_3(\mathbb{Z}[\pi])$	0	$\mathbb{Z}_2$

The odd-term L^h-groups and L^s-groups are isomorphic. Because of the Rothenberg sequence, the even L^h-groups may have torsion depending on $\widetilde{K}_0(\mathbb{Z}[\pi])$. It is a nice fact, essentially Herbrand's lemma in class field theory, that L^s-groups are easier to understand. They are isomorphic to the L^p-groups, although not necessarily by the natural isomorphism.

Dress induction, which states that the L-group of the integer group ring $\mathbb{Z}[\pi]$ can be computed from that of its 2-hyperelementary subgroups, gives the following striking result. It is very useful for many purposes, in particular if one studies actions on manifolds by odd-order groups.

Theorem 2.99. *If π is an odd-order group, then the odd L-groups $L_{2k+1}(\mathbb{Z}[\pi])$ are all trivial. The even L-groups $L_n(\mathbb{Z}[\pi])$ are detected by the multisignature and the simply connected surgery obstruction.*

Remark 2.100. *The main tool for actual detailed calculations is the arithmetic square, a descendent of the Hasse principle for quadratic forms over number rings. Hasse principles do not always hold for the applications to L-theory, and Galois cohomology also enters. In the diagram*

$$\begin{array}{ccc} \mathbb{Z}[G] & \longrightarrow & \mathbb{Q}[G] \\ \downarrow & & \downarrow \\ \mathbb{Z}[\widehat{G}] & \longrightarrow & \mathbb{Q}[\widehat{G}] \end{array}$$

the bottom line is the completion of $\mathbb{Z}$ and $\mathbb{Q}$ with respect to the primes in $\mathbb{Z}$. We refer to Hambleton-Taylor [290] for a fairly recent review of this subject.

2.7 USING SPLITTING THEOREMS

In Section 1.3 we discussed the Browder-Wall codimension one splitting problem as an application of the π-π theorem. Our goal here is to explain inductive L-theory calculations for groups of the form $G \times \mathbb{Z}$, using the Farrell fibering theorem [219], which gives necessary and sufficient conditions under which a manifold fibers over a circle, or equivalently the splitting theorems of Farrell-Hsiang [222]. Let M be a manifold of dimension ≥ 5 with a codimension one submanifold P. Suppose that $\pi_1(M) \cong G \times \mathbb{Z}$ with $\pi_1(P) = G$, and that $\mathrm{Wh}(G \times \mathbb{Z}) = 0$. We will show how to construct a map $\alpha(P)\colon L_{n+1}(\mathbb{Z}[G \times \mathbb{Z}]) \to L_n(\mathbb{Z}[G])$.

Before discussing L-theory let us discuss K-theory. Let G be a finitely presented group and $\mathbb{Z}[G]$ its integral group ring. Let $C(\mathbb{Z}[G])$ be the Grothendieck group of pairs $[P, \nu]$, where P is a projective $\mathbb{Z}[G]$-module and ν is a nilpotent endomorphism of P. The forgetful functor ignoring the nilpotent endomorphism gives a map $C(\mathbb{Z}[G]) \to \widetilde{K}_0(\mathbb{Z}[G])$ with kernel $\widetilde{C}_0(\mathbb{Z}[G])$. There is an isomorphism

$$\mathrm{Wh}(G \times \mathbb{Z}) \cong \mathrm{Wh}(G) \times \underbrace{\widetilde{K}_0(\mathbb{Z}[G]) \times \widetilde{C}_0(\mathbb{Z}[G])}_{C(\mathbb{Z}[G])} \times \widetilde{C}_0(\mathbb{Z}[G])$$

given by Bass-Heller-Swan [47]. Projecting onto the first copy of $\widetilde{C}_0(\mathbb{Z}[G])$ and using the decomposition $C(\mathbb{Z}[G]) = \widetilde{C}_0(\mathbb{Z}[G]) \times \widetilde{K}_0(\mathbb{Z}[G])$, we have a map $p\colon \mathrm{Wh}(G \times \mathbb{Z}) \to C(\mathbb{Z}[G])$.

We shall now discuss the analogue of this formula in L-theory, due to Shaneson. It will not have an analogue of the C_0 term.

Theorem 2.101. *(Shaneson [580]) There is a map*

$$\alpha\colon L_n^s(\mathbb{Z}[G \times \mathbb{Z}]) \to L_{n-1}^h(\mathbb{Z}[G])$$

along with a short exact sequence

$$0 \to L_n^s(\mathbb{Z}[G]) \to L_n^s(\mathbb{Z}[G \times \mathbb{Z}]) \xrightarrow{\alpha} L_{n-1}^h(\mathbb{Z}[G]) \to 0$$

that is exact and splits (unnaturally). The splitting is given by taking the product of a surgery problem with a circle.

We will write the proof using the language of splitting, because of intended later generalizations.

The Farrell-Hsiang theorem allows twisted products, and later we will describe how the results change with a twist.

1. Suppose that Q^n is a smooth or PL manifold of dimension $n \geq 6$ with a codimension one submanifold P with $\partial P = \partial Q \cap P$ meeting ∂Q transversely.
2. Suppose that $\pi_1(P) = G$ and $\pi_1(Q) = G \times \mathbb{Z}$.
3. Suppose that we have a homotopy equivalence $\phi\colon (M, \partial M) \to (Q, \partial Q)$ transverse to $(P, \partial P)$ restricting to a homotopy equivalence

$$(\partial M, \phi^{-1}(\partial P)) \to (\partial Q, \partial P).$$

In this case we say that $\phi|_{\partial M}$ is *split along* ∂P. Note of course that $\pi_1(M) =$

$G \times \mathbb{Z}$ also.

4. Suppose that there also exists a homotopy equivalence $\psi : (M, \psi^{-1}(P)) \to (Q, P)$ transverse to P that is homotopic to the above $\phi : (M, \partial M) \to (Q, \partial Q)$ relative to ∂M. Then we say that ϕ is *splittable along* P.

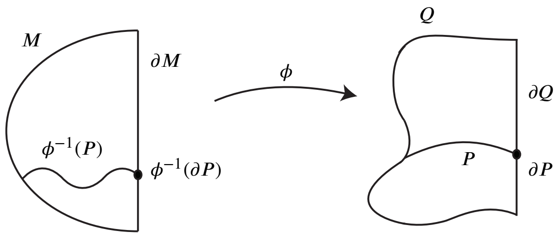


Figure 2.1: A splittable map

Theorem 2.102. *(Farrell-Hsiang [222]) Let $\phi : (M, \partial M) \to (Q, \partial Q)$ be the above homotopy equivalence, and let $p : \mathrm{Wh}(G \times \mathbb{Z}) \to C(\mathbb{Z}[G])$ be the given projection map. The map ϕ is splittable along P iff $p(\tau(\phi)) = 0$ in $C(\mathbb{Z}[G])$, where $\tau(\phi) \in \mathrm{Wh}(G \times \mathbb{Z})$ is the Whitehead torsion of ϕ.*

Theorem 2.103. *Let $\tau(\phi)$ be the Whitehead torsion given in the previous theorem. The map $\phi : M \to Q$ is splittable along P iff $\tau(\phi)$ lies in the image of $i_* : \mathrm{Wh}(G) \to \mathrm{Wh}(G \times \mathbb{Z})$. In particular, if $\tau(\phi) = 0$, then ϕ is splittable along P. In fact, the map ϕ is splittable iff $\tau(\phi)$ lies in the image of the map $i_* : \mathrm{Wh}(G) \to \mathrm{Wh}(G \times \mathbb{Z})$ induced by the inclusion $i : G \to G \times \mathbb{Z}$.*

We now limit our discussion to product manifolds. Consider a product manifold $X = P \times \mathbb{S}^1$ with $\pi_1(P) = G$ and $n \geq 5$ (or $n \geq 6$ if there is boundary). Let (M, ϕ, F) be a class in $\mathcal{N}^{Cat}(X)_{\mathrm{rel}}$. Since we assume that $\phi|_{\partial M}$ is a simple homotopy equivalence, its torsion vanishes. So by Theorem 2.103, we may assume that $\phi|_{\partial M}$ is splittable and that ϕ is transverse to P. Note here that P is a codimension one submanifold of X. The following is a form of the Wall realization theorem.

Theorem 2.104. *Suppose that $n \geq 7$ and K^{n-2} is a closed PL manifold. Let $P^{n-1} = K \times I$. We consider the manifold $P^{n-1} \times \mathbb{S}^1$. Let $\beta \in L_n^s(\mathbb{Z}[G \times \mathbb{Z}])$. Then $\beta = \sigma(M, \phi, F)$ for some $(M, \phi, F) \in \mathcal{N}^{PL}(P \times \mathbb{S}^1)$, where $\phi|_{\partial\_M} : \partial\_M \to K \times \{0\} \times \mathbb{S}^1$ is a PL homeomorphism.*

In the following, we present a way to relate an L-group of degree n and one of degree $n-1$. In the event that the groups in question have vanishing Whitehead torsion, the decorations on the L-groups can be removed and decomposition results can be deduced.

Definition 2.105. *Consider the situation in the previous theorem. Let $\beta \in L_n^s(\mathbb{Z}[G \times \mathbb{Z}])$ and let (M, ϕ, F) be the given element in the preimage of the surgery map σ. Then*

$(\phi^{-1}(P), \phi|_{\phi^{-1}(P)}, F|_{\phi^{-1}(P)})$ belongs to $\mathcal{N}^{PL}(P)$. Define

$$\alpha_P(M,\phi,F) = \sigma(\phi^{-1}(P), \phi|_{\phi^{-1}(P)}, F|_{\phi^{-1}(P)})$$

in $L^h_{n-1}(\mathbb{Z}[G])$. We will then define $\alpha(K): L^s_n(\mathbb{Z}[G\times\mathbb{Z}]) \to L^h_{n-1}(\mathbb{Z}[G])$ by $\alpha(K)(\beta) = \alpha_{K\times I}(M,\phi,F)$.

In other words, having split along the codimension one manifold, one can also split the degree one normal map, and obtain a new rel boundary surgery problem. One interpretation of this map is that, if we cannot solve the codimension one surgery problem, then we cannot solve the original surgery problem.

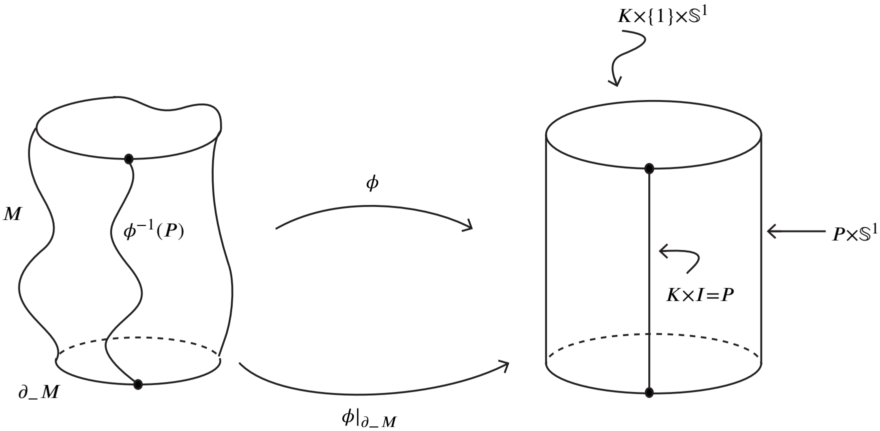


Figure 2.2: Splitting with a cobordism

The splitting addresses this exact situation. Once we have split, then we can open up along a splitting, as in the proof of the Browder-Wall theorem. We obtain then an element of L^s. This map splits, and the Shaneson theorem is proved.

Remark 2.106. *We can generalize this calculation to $G \rtimes \mathbb{Z}$. We use α to denote the automorphism of G. The generalization is an exact sequence*[3]

$$\cdots \to L^A_n(G) \to L^s_n(G) \to L_n(G \rtimes \mathbb{Z}) \to L^A_{n-1}(\mathbb{Z}[G]) \to L^s_{n-1}(\mathbb{Z}[G]) \to \cdots$$

where A is the subgroup of the Whitehead group of elements where $\tau = \alpha(\tau)$ and the maps $L^A_n(\mathbb{Z}[G]) \to L^s_n(\mathbb{Z}[G])$ are induced by $1-\alpha$. In other words, one performs surgery to obtain a homotopy equivalence with torsion in this subgroup; there is a suitable Rothenberg sequence comparing it to L^s and L^h. It might remind the reader of the Wang sequence for the homology of a bundle over a circle. When we study the assembly map, this analogy will be seen to be more than merely formal.

[3]Of course the middle term is $L_h(\mathbb{Z}[G \rtimes \mathbb{Z}])$ but we omit the $\mathbb{Z}$ to reduce cumbersome notation.

Now we can compute $L_*(\mathbb{Z}[\mathbb{Z}^k])$ inductively. Note that, since $\mathrm{Wh}(\mathbb{Z}^k) = 0$, there is no difference between L^h and L^s.

Corollary 2.107. *We have*

$$L_n(\mathbb{Z}[\mathbb{Z}^k]) \cong \sum_{0\le \ell \le k} \binom{k}{\ell} L_{n-\ell}(\mathbb{Z}[e]),$$

where the L-groups are 4-periodic.

Proof. Since the Whitehead group of $\mathbb{Z}^r$ is zero for all r, it follows that there is no difference between L^s and L^h. By the splitting of the sequence above, we have an isomorphism $L_n(\mathbb{Z}[\mathbb{Z}^k]) \cong L_{n-1}(\mathbb{Z}[\mathbb{Z}^{k-1}]) \oplus L_n(\mathbb{Z}[\mathbb{Z}^{k-1}])$. Using induction on n, we have

$$L_n(\mathbb{Z}[\mathbb{Z}^k]) \cong \sum_{0\le \ell \le k-1} \binom{k-1}{\ell} L_{n-1-\ell}(\mathbb{Z}[e]) \oplus \sum_{0\le \ell \le k-1} \binom{k-1}{\ell} L_{n-\ell}(\mathbb{Z}[e]),$$

which can be simplified to the desired sum. □

Remark 2.108. *In general, we can use the same method to compute $L_*(\mathbb{Z}[G\times\mathbb{Z}^-])$ by comparing the groups with $L_*(\mathbb{Z}[G])$.*

As pointed out in Wall [672], we have the following calculations:

G	$L_0(\mathbb{Z}[G])$	$L_1(\mathbb{Z}[G])$	$L_2(\mathbb{Z}[G])$	$L_3(\mathbb{Z}[G])$
$\mathbb{Z}^+\times\mathbb{Z}_2^+$	$\mathbb{Z}^2\oplus\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\mathbb{Z}^+\times\mathbb{Z}_2^-$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}^+\times\mathbb{Z}^-$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$

Remark 2.109. *In the case of nonorientable manifolds, the non-trivial orientation map $w\colon \pi_1(M)\to\mathbb{Z}_2$ is not necessarily unique. Different orientation maps could yield different L-groups. Consider the groups $G=\mathbb{Z}_2^- * \mathbb{Z}_2^-$ and $G'=\mathbb{Z}_2^+ * \mathbb{Z}_2^-$. Disregarding 2-torsion, i.e. tensoring with $\mathbb{Z}[1/2]$, we know that $L_n(\mathbb{Z}[\mathbb{Z}_2^+])$ is $\mathbb{Z}^2$ for $n\equiv 0 \bmod 4$ and is trivial otherwise. Similarly we know that $L_n(\mathbb{Z}[\mathbb{Z}_2^-])$ is 0 for all n. Then we have the sequences, still tensoring by $\mathbb{Z}[1/2]$,*

$$0\to L_1(\mathbb{Z}[G])\to L_0(\mathbb{Z}[e])\to L_0(\mathbb{Z}[\mathbb{Z}_2^-])\times L_0(\mathbb{Z}[\mathbb{Z}_2^-]),$$
$$0\to L_1(\mathbb{Z}[G'])\to L_0(\mathbb{Z}[e])\to L_0(\mathbb{Z}[\mathbb{Z}_2^+])\times L_0(\mathbb{Z}[\mathbb{Z}_2^-]).$$

A simple check will show that $L_1(\mathbb{Z}[G])\cong\mathbb{Z}$ and $L_1(\mathbb{Z}[G'])\cong 0$.

In the case of $G=\mathbb{Z}$ with the non-trivial involution, we resort to the same exact sequences, this time with $w(g)=-1$ and trivial H. Here α_* is still the identity, so that

$1 - w(g)\alpha_*$ is multiplication by 2.

Therefore we have the following summary:

$$\begin{array}{rcl} L_0(\mathbb{Z}[\mathbb{Z}^-]) & \cong & \mathbb{Z}_2 \\ L_1(\mathbb{Z}[\mathbb{Z}^-]) & \cong & 0 \\ L_2(\mathbb{Z}[\mathbb{Z}^-]) & \cong & \mathbb{Z}_2 \\ L_3(\mathbb{Z}[\mathbb{Z}^-]) & \cong & \mathbb{Z}_2 \end{array}$$

The invariant given in $L_0(\mathbb{Z}[\mathbb{Z}^-])$ is given by $\frac{1}{8}$ of the signature. The Arf invariant occurs again in $L_2(\mathbb{Z}[\mathbb{Z}^-])$ and the codimension one Arf invariant appears in $L_3(\mathbb{Z}[\mathbb{Z}^-])$.

Remark 2.110. *If the $\mathbb{Z}$ is orientation-reversing and all relevant Whitehead groups vanish, then the sequence becomes*

$$\to L_n(\mathbb{Z}[G]) \to L_n(\mathbb{Z}[G]) \to L_n(\mathbb{Z}[G \times \mathbb{Z}^-]) \to L_{n-1}(\mathbb{Z}[G]) \to L_{n-1}(\mathbb{Z}[G]) \to .$$

When we split the manifold, the two boundary components are oriented the same way. Therefore, the maps $L_n(\mathbb{Z}[G]) \to L_n(\mathbb{Z}[G])$ are multiplication by 2, and twice the obstruction in $L_{n-1}(\mathbb{Z}[G \times \mathbb{Z}^-])$ automatically vanishes.

Remark 2.111. *The more general codimension one splitting theorem of Cappell [111] gives calculations for much more general classes of groups, such as surface groups or solvable Baum-Solitar groups*

$$\langle a, b | aba^{-1} = b^n \rangle$$

for which $L_1(\mathbb{Z}[G]) \cong \mathbb{Z}_{n-1}$ when $n \neq 1$, giving examples of odd torsion in L-theory.

Chapter Three

Classical surgery theory

This chapter will complete a first view of surgery theory: we analyze the structure sets for PL and Top surgery using the surgery exact sequence and L-group computations. To understand the Top category, we critically need an analysis of the PL structure set of the torus, which can be achieved as an application of the structure of F/PL and the calculation of $L_*(\mathbb{Z}[\mathbb{Z}^n])$ from the last chapter. Together with the local contractibility of homeomorphism spaces by Černavskiĭ and Edwards-Kirby, we obtain the homotopy structure of F/Top and also the Kirby-Seibenmann obstruction to triangulation.

The calculation also shows that high-dimensional homotopy tori are themselves tori, an important special case of the Borel conjecture. The final section of the chapter shows that the proper analogue of the Borel conjecture is false, an illustration of the strength just of the machinery developed to this point without any additional calculational input.

In Section 3.5 we study a classical question from the beginnings of surgery theory: whether all finite H-spaces are homotopy equivalent to closed manifolds. We sketch some of the results proved on this question because they do not depend on difficult surgery calculations, but instead rely largely on homotopy-theoretic considerations that have already appeared elsewhere.

3.1 LOW DIMENSIONS AND SMOOTHING THEORY

Very low-dimensional manifolds are smoothable: there is no difference between smooth, PL, or topological manifolds in dimension ≤ 3. The three-dimensional case is due to Bing and Moïse. In dimension 4, the great discovery of the early 1980s due to the contrasting work of Donaldson and Freedman shows that Top and the other categories are completely different from each other. However, PL and Diff continue to be equivalent through dimension 7.

In this section we will discuss PL versus Diff.

The major theoretical result in this area is the Hirsch-Mazur smoothing theory [316]. It is an h-principle for the classification of smoothings. (See the appendices for the relevant definitions.)

Theorem 3.1. *A PL manifold M has a smoothing iff its stable normal PL block bundle has a lift to BO:*

$$\begin{array}{ccc} & & BO \\ & & \downarrow \\ M & \longrightarrow & BPL \end{array}$$

Moreover concordance classes of smoothings of M, when they exist, correspond to the different lifts; i.e. they are in a bijective correspondence with $[M : PL/O]$.

Corollary 3.2. *The obstructions to existence of a smooth structure on a PL manifold lie in $H^i(M; \pi_{i-1}(PL/O))$ and to uniqueness in $H^i(M; \pi_i(PL/O))$.*

This statement can be proved by induction over the regular neighborhood handlebody structure associated to a triangulation of M.

Remark 3.3. *The failure of this result in Top in dimension 4 is very striking. Although contractible, the Euclidean space $\mathbb{R}^4$ has uncountably many differential structures (Taubes [634]). There is also a contractible 4-manifold whose boundary is the Poincaré homology sphere (Freedman [252]); it cannot be smoothed by Rokhlin's theorem, which we will see in the next section.*

The proof of the smoothing theorem has two aspects: one is tangential and will yield a theorem involving $BO_n \to BPL_n$, and the second is the effect of stabilization detected by an Euler class in both categories. We will not prove the theorem here; our discussion of the Kirby-Siebenmann obstruction later in the chapter provides an argument that holds in dimensions at least 5.

Remark 3.4. *If one studies locally linear group actions, there is a smoothing theory in the tangential sense, but not in the normal sense. See Lashof-Rothenberg [389].*

Theorem 3.5. *(Cerf [146] and Kervaire-Milnor [356]) The space PL/O is 6-connected.*

In other words, PL manifolds of dimension at most 6 are triangulable with unique differential structures. PL manifolds in dimension 7 are smoothable, but not necessarily uniquely. Recall that $\mathbb{S}^7$ has 28 differential structures, according to Kervaire-Milnor [356].

The hard case is $\pi_4(PL/O)$, which uniquely smooths PL 4-manifolds. The calculation ultimately relies on the statement that every diffeomorphism of the 3-sphere extends over the 4-ball. A much deeper fact due to Hatcher is that $\mathrm{Diff}(\mathbb{S}^3) \simeq O_4$ [292]. One can then rely on surgery theory to give the remaining homotopy groups, using calculations discussed later in the chapter. For example, one needs to know that a homotopy 4-sphere bounds a smooth contractible 5-manifold (a surgery-theoretic fact). In any case, we are only interested in the above result through dimension 4, because the analysis of F/PL with surgery gives information in dimensions at least 5.

Corollary 3.6. *The map $F/O \to F/PL$ is an equivalence on homotopy groups through dimension 5.*

In other words, the low-dimensional homotopy of F/PL can be studied using the seemingly easier situation of F/O. The space F/O is easier because it relies on the study of the homotopy fiber of the map $BO \to BF$, the classical J-homomorphism comparing the known homotopy groups of O (known by Bott periodicity) to the stable stems π_n^S. The analysis of the J-homomorphism in all dimensions was a great achievement of K-theory, but low dimensions can be analyzed using the Serre spectral sequence. We record the result in the table below. Indeed, the calculation by Serre of π_3^S was needed in the original proof of Rokhlin's theorem. In the next section, we will see the importance of Rokhlin's theorem in the structure of F/PL.

n	$\pi_n(BO)$	$\pi_n(BF)$	$J : \pi_n(BO) \to \pi_n(BF)$	$\pi_n(F/O) \cong \pi_n(F/PL)$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \xrightarrow{\sim} \mathbb{Z}_2$	0
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \xrightarrow{\sim} \mathbb{Z}_2$	$\mathbb{Z}_2$
3	0	$\mathbb{Z}_2$	$0 \to \mathbb{Z}_2$	0
4	$\mathbb{Z}$	$\mathbb{Z}_{24}$	$\mathbb{Z} \twoheadrightarrow \mathbb{Z}_{24}$	$\mathbb{Z}$
5	0	0	$0 \to 0$	0
6	0	0	$0 \to 0$	$\mathbb{Z}_2$

Remark 3.7. *We give a caveat. Despite the equivalence of Diff and PL through dimension 6, there are differences between these categories when one asks geometric questions. For example, there is a unique PL embedding of $\mathbb{S}^3$ in $\mathbb{S}^6$ by the Zeeman unknotting theorem, but Haefliger knots give infinitely many smooth embeddings of $\mathbb{S}^3$ in $\mathbb{S}^6$ [279].*

3.2 THE HOMOTOPY GROUPS OF F/PL

Following Sullivan, the homotopy type of F/PL is analyzed using the surgery exact sequence together with the PL Poincaré conjecture. The homotopy groups of PL/O are reflected in the existence of exotic differential structures on the sphere; i.e. the Poincaré conjecture in the smooth category should be understood as being false, i.e. with respect to diffeomorphism instead of homeomorphism.

Using the PL Poincaré conjecture, Sullivan shows that F/PL has a relatively transparent structure. The one interesting complication, a single k-invariant linking the homotopy groups $\pi_2(F/PL)$ and $\pi_4(F/PL)$, is a consequence of Rokhlin's theorem. We say that a smooth manifold is *spin* if its first and second Stiefel-Whitney classes both vanish.

Theorem 3.8. *(Rokhlin [550]) The signature of a closed spin 4-manifold is a multiple of 16.*

Rokhlin's Theorem will be used to complete the analysis of F/PL, while F/Top has a different structure because the analogue of Rokhlin's Theorem in Top fails.

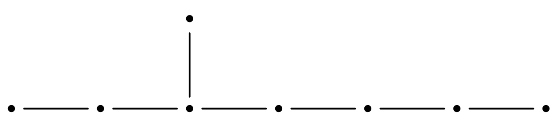
3.2.1 Milnor and Kervaire manifolds

Given two rank n vector bundles E_1 and E_2 over smooth n-manifolds M_1 and M_2, we can glue together their disk bundles $D(E_1)$ and $D(E_2)$ in a process called *plumbing*. We plumb them together by glueing coordinate charts $\mathbb{D}^n \times \mathbb{D}^n$ for M_1 and $\mathbb{D}^n \times \mathbb{D}^n$ for M_2 to each other by interchanging the fiber and base directions. See the figure below. Bundles can be plumbed together at multiple points, if necessary. Plumbing is the main tool for proving the Wall realization theorem 1.41.

One can plumb any finite number of times to achieve a manifold $(M^{2n}, \partial M^{2n})$ with boundary. If one plumbs $E(\xi_1 \downarrow \mathbb{S}^{2n})$ and $E(\xi_2 \downarrow \mathbb{S}^{2n})$ together, the self-intersections are determined by the Euler classes of ξ_1 and ξ_2, and the intersections are determined by the number of points at which the bundles are plumbed. They are encoded in an intersection matrix A. The simplest situation is the use of a tree to plumb spheres, only plumbing different spheres bundles to each other at most once. Then the manifold obtained is a $(2n-1)$-connected $4n$-manifold; in fact, it is homotopically a wedge of spheres $\mathbb{S}^{2n}$. Poincaré duality can be used to prove that the matrix A is nonsingular, i.e. $\det(A) = \pm 1$, iff the boundary of the manifold has the $\mathbb{Z}$-homology of the sphere. The statement holds because of the relationship of Poincaré duality with intersection, as well as the fact that a homology spherical boundary, indicates that $M \cup \text{cone}(\partial M)$ has Poincaré duality. Of course, if $n \geq 3$, then the boundary is simply connected by van Kampen's theorem, and the Hurewicz isomorphism theorem would then show that it is a homotopy sphere.

The key plumbing construction used for this purpose appears in Kervaire-Milnor [356]. Together with some small modifications, this method can be used to construct two very important families of PL manifolds, called the *Milnor and Kervaire manifolds*, which we now describe.

Construction 3.9. *The Milnor E_8-manifold is constructed as follows. Let $m \geq 2$ and plumb together eight copies of the disk bundle $\tau(\mathbb{S}^{2m})|_{\mathbb{D}^{2m}}$ according to the diagram E_8 with Euler class 2 given by*



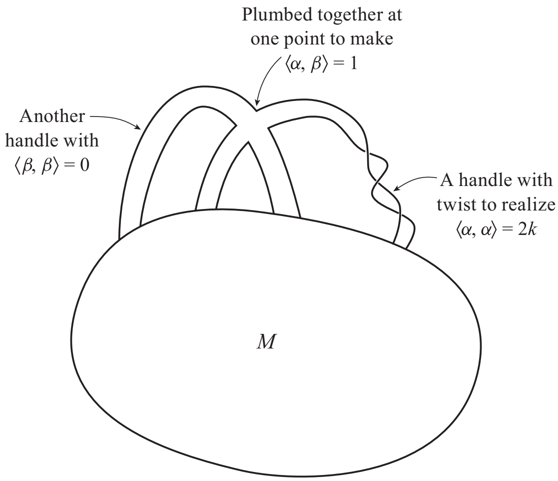


Figure 3.1: The plumbing construction

The diagram produces a determinant 1 intersection matrix (cf. Example 2.3)

$$E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Let $(M^{4m}, \partial M^{4m})$ be the $4m$-dimensional manifold with boundary obtained from E_8. Since $\pi_1(\partial M)$ is trivial for $m \geq 2$, the Poincaré conjecture implies that the boundary is PL homeomorphic to a sphere $\mathbb{S}^{4m-1}$. Let M_E^{4m} be the closed $4m$-dimensional PL manifold formed by attaching a cone to M along ∂M, called the Milnor E_8-manifold. It is never smoothable, nor is it PL cobordant to a differentiable manifold by Brumfiel-Madsen-Milgram [102].

Remark 3.10. *Because of the Hopf invariant one theorem, manifolds that are $(n-1)$-connected and $2n$-dimensional always have quadratic forms of Type II, unless $n = 1, 2, 4$. In these dimensions, one can use the bundles over $\mathbb{S}^n$ with odd Hopf invariant to construct Milnor manifolds with any nonsingular quadratic form of any type. One needs to perform some additional surgeries to make it simply connected if the diagram of the plumbing is not a tree. Of course, if the form were Type I, then the degree one map $M^{4m} \to \mathbb{S}^{4m}$ would not be covered by bundle data. For Type II forms, there is bundle data as these manifolds are almost parallelizable, i.e. their tangent bundles are* *trivial on every proper subset.*

Remark 3.11. *When $m = 1$, the boundary is not simply connected and M_E^4 is not a manifold.*

Remark 3.12. *For suitable r, the r-fold connected sum $M_E^{4m} \# \cdots \# M_E^{4m}$ is smoothable (see Kervaire-Milnor [355]). This fact is connected to the finiteness of the number of differentiable structures on the sphere. In fact, one can use finiteness to show that, for some integer r, the r-fold connected sum is smoothable. Often we will have in mind a connected sum of some number of the Milnor manifold associated to E_8 if we do not specify.*

Construction 3.13. *The Kervaire manifolds are constructed by a version of plumbing using immersed handles that allows for a non-trivial μ-form in the sense of $L_2(\mathbb{Z}[e])$, using the unique skew-symmetric form on a two-dimensional $\mathbb{Z}$-module. The construction gives a smooth manifold M of dimension $4m + 2$ with boundary. The Kervaire manifold M_A^{4m+2} is obtained from M by attaching a cone to its boundary ∂M. It is typically PL but not smoothable.*

Remark 3.14. *The 6-dimensional Kervaire manifold is smoothable, and therefore the argument given below for the 5-dimensional PL Poincaré conjecture works smoothly as well, and explains why the 5-sphere has a unique smooth structure.*

Remark 3.15. *Kervaire showed that the manifold K_A^{10} is a PL manifold without any differentiable structure [351]. Subsequent study of the Arf invariant for K_A^{4m+2} has connected it to important stable homotopy theory. Browder [83] showed that, if K_A^{4m+2} is smoothable, then the dimension must be of the form $2^k - 2$. The Kervaire manifolds are not PL cobordant to a differentiable manifold according to Brumfield-Madsen [102]. Recent results by Hill-Hopkins-Ravenel [307] prove that the same non-smoothability phenomenon holds even when the dimension is $2^k - 2$ for $k \geq 8$. It is known that the manifolds are smoothable in dimensions 2, 6, 14, 30, and 62 (cf. Barratt-Mahowald [38]). Therefore the only unresolved case at the writing of this book is in dimension 126.*

Theorem 3.16. *Let M_E^{4m} and K_A^{4m+2} be the Milnor and Kervaire manifolds as above. Let $\sigma : \mathcal{N}^{PL}(\mathbb{S}^{4m}) \to L_{4m}(\mathbb{Z}[e])$ be the surgery map for $\mathbb{S}^{4m}$. There is a degree one normal map $f_{4m} : M_E^{4m} \to \mathbb{S}^{4m}$ such that $\sigma(M_E, f_{4m}) = 1$. Similarly, there is a degree one normal map $g_{4m+2} : K_A^{4m+2} \to \mathbb{S}^{4m+2}$ with image 1 in $L_2(\mathbb{Z}[e])$ under the surgery map $\mathcal{N}^{PL}(\mathbb{S}^{4m+2}) \to L_2(\mathbb{Z}[e])$.*

Proof. There is a map $f_{4m} : M_E^{4m} \to \mathbb{S}^{4m}$ whose induced map on homology sends the fundamental class $[M_E^{4m}]$ to the fundamental class $[\mathbb{S}^{4m}]$, so has degree one. The normal data come from the construction of M to be parallelizable outside cone point, i.e. M_E^{4m} is almost parallelizable. The punctured Milnor manifold M^{4n} constructed by plumbing tangent bundles of the sphere immerses into Euclidean $4n$-space, and is therefore parallelizable; indeed the statement is true whenever the quadratic form is of Type II. The signature of the map is the signature of the associated intersection matrix, which can be computed to be 8. Therefore the surgery map gives $\sigma(M_E, f_{4m}) = \frac{1}{8}\operatorname{sig}(B) = 1$. In the other case, a computation of the Arf invariant leads to a similar result. □

Remark 3.17. *The previous theorem has significant ramifications in the study of simply connected manifolds. Suppose that X is a simply connected Poincaré complex of dimension $4k$ with a PL manifold structure $g : N^{4k} \to X^{4k}$. Suppose that its surgery obstruction $\sigma(g)$ is nonzero in $L_{4k}(\mathbb{Z}[e]) \cong \mathbb{Z}$. As above, let $f_{4k} : M_E^{4m} \to \mathbb{S}^{4m}$ be the degree one normal invariant from the Milnor manifold with surgery obstruction 1. Therefore, the connected sum $g\#f_{4k} : N^{4k}\#K_A^{4m} \to X\#\mathbb{S}^{4m} = X$ is now a manifold structure of X with surgery obstruction $\sigma(g\#f_{4k}) = \sigma(g) \pm 1$. By adding sufficiently many Milnor manifolds by connected sum to the domain, one can obtain a manifold structure $N' \to X$ with zero surgery obstruction.*

3.2.2 Higher homotopy groups

We now begin to pave the way to a calculation of the homotopy groups $\pi_n(F/PL)$. We start with the Alexander trick, a basic result in geometry, in order to execute the easiest step in the proof of the generalized Poincaré conjecture.

Lemma 3.18. *(Alexander trick) If $\mathbb{D}$ and $\mathbb{D}'$ are n-disks and $f : \partial\mathbb{D} \to \partial\mathbb{D}'$ is a PL homeomorphism, then f extends to a PL homeomorphism from $\mathbb{D}$ to $\mathbb{D}'$.*

Remark 3.19. *Even though every step of the homotopy can be smoothed, there is an overall singularity in the isotopy at the origin, which shows that the Alexander trick is strictly a PL and Top result.*

We now state the very important Generalized Poincaré conjecture. The theorem states that the standard n-sphere $\mathbb{S}^n$ is *PL rigid* for $n \geq 5$. In other words, the PL structure set only contains one element, namely the standard sphere itself.

Theorem 3.20. *(Generalized PL Poincaré conjecture, Smale [596] and Stallings) Let $n \geq 5$. If M is a PL manifold homotopy equivalent to the standard $\mathbb{S}^n$, then M is PL homeomorphic to $\mathbb{S}^n$; i.e. the structure set $S^{PL}(\mathbb{S}^n)$ is trivial.*

In the case $n \geq 6$ and Cat = PL or Diff, the Cat h-cobordism theorem implies that, if M is a Cat manifold that is homotopically equivalent to the n-sphere $\mathbb{S}^n$, then M can be expressed as a union $\mathbb{D}_1^n \cup_g \mathbb{D}_2^n$ of two Cat disks identified along their boundaries by a Cat isomorphism g. At this point we can prove the Poincaré conjecture by extending g to a homeomorphism between the two disks using the Alexander trick.

Remark 3.21. *In dimension 5, one uses surgery, together with the fact that $\pi_5(F/PL) = 0$, to show that any homotopy sphere is normally cobordant to the sphere. If the normal cobordism has non-trivial surgery obstruction in $L_6(\mathbb{Z}[e]) = \mathbb{Z}_2$, then by the discussion* *in Remark 3.17, we can connect sum with a smoothable Kervaire 6-manifold to kill this obstruction; i.e. the map* $\mathcal{N}^{PL}(\mathbb{S}^5 \times [0,1]) \to L_6(\mathbb{Z}[e])$ *is surjective.*

At this point, the computation of $\pi_i(F/PL)$ is apparent from the surgery exact sequence. We begin with the following:

$$\cdots \to L_{i+1}(\mathbb{Z}[e]) \to \mathcal{S}^{PL}(\mathbb{S}^{i-1} \times [0,1] \text{ rel } \partial) \to \pi_i(F/PL) \to L_i(\mathbb{Z}[e]) \to \cdots$$

We can identify $\mathcal{S}^{PL}(\mathbb{S}^{i-1} \times [0,1] \text{ rel } \partial)$ with $\mathcal{S}^{PL}(\mathbb{S}^i)$ using the Alexander trick and arrive at

$$\cdots \to L_{i+1}(\mathbb{Z}[e]) \to \mathcal{S}^{PL}(\mathbb{S}^i) \to \pi_i(F/PL) \to L_i(\mathbb{Z}[e]) \to \mathcal{S}^{PL}(\mathbb{S}^{i-1}) \to \cdots$$

The PL Poincaré conjecture tells us that $\mathcal{S}^{PL}(\mathbb{S}^i) = 0$ for $i \geq 5$, so we conclude that $\pi_i(F/PL) \to L_i(\mathbb{Z}[e])$ is an isomorphism for $i \geq 6$. Note that the rightmost part of the sequence is

$$\cdots \to L_6(\mathbb{Z}[e]) \to \mathcal{S}^{PL}(\mathbb{S}^5) \to \pi_5(F/PL) \to L_5(\mathbb{Z}[e]).$$

Since $L_5(\mathbb{Z}[e]) = 0$, it follows that $\pi_5(F/PL) = 0$. In other words, for $n \geq 5$, we have

$$\pi_n(F/PL) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \mod 4, \\ \mathbb{Z} & \text{if } n \equiv 2 \mod 4, \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

3.2.3 Homotopy of F/PL in low dimensions

To compute the homotopy groups of F/PL in low dimensions, we rely on an understanding of the spaces PL/O and F/O, first recalling that there is an exact sequence

$$\pi_n(PL/O) \to \pi_n(F/O) \to \pi_n(F/PL) \to \pi_{n-1}(PL/O)$$

of homotopy groups of classifying spaces. We know that PL/O is 6-connected by Theorem 3.5. It therefore follows that $\pi_n(F/PL) \cong \pi_n(F/O)$ for all $n \leq 5$, so in these dimensions it suffices to compute $\pi_n(F/O)$. To this end, we use the long exact sequence

$$\cdots \to \pi_{n+1}(BO) \to \pi_{n+1}(BF) \to \pi_n(F/O) \to \pi_n(BO) \to \pi_n(BF) \to \cdots$$

induced by the fibration $F/O \to BO \to BF$.

A quick calculation using the exact sequences and the data presented in the table in Section 3.1 shows that the odd homotopy groups $\pi_1(F/PL)$ and $\pi_3(F/PL)$ are trivial, and that $\pi_2(F/PL) \cong \mathbb{Z}_2$ and $\pi_4(F/PL) \cong \mathbb{Z}$. Although the result is more like a corollary, we will call it a theorem to emphasize its importance.

Theorem 3.22. *The homotopy groups $\pi_n(F/PL)$ and L-groups $L_n(\mathbb{Z}[e])$ are isomorphic in every dimension.*

Note that we have proved that the groups $\pi_n(F/PL)$ and $L_n(\mathbb{Z}[e])$ are *abstractly* isomorphic in dimensions ≤ 5. We should examine the surgery maps $\pi_n(F/PL) \to L_n(\mathbb{Z}[e])$ in greater detail. Since the groups are zero in odd dimensions, we only need to examine the cases $n = 2$ and $n = 4$. The groups are $\mathbb{Z}_2$ in the first case and $\mathbb{Z}$ in the second, so the maps are either zero or injective. We end the section by resolving this question.

For the $n = 2$ case, we require the following result to show that the Arf invariant has a non-trivial image.

Proposition 3.23. *There is a degree one normal map $(\mathbb{T}^2, \varepsilon^2) \to (\mathbb{S}^2, \varepsilon^2)$ with Arf invariant 1.*

Proof. Give the torus the Lie group invariant framing. With this framing it represents, using the Pontrjagin construction, the non-trivial element of $\pi_2^S \cong \mathbb{Z}_2$. □

The domain of a degree one normal map to $\mathbb{S}^4$ is a smooth spin 4-manifold, which by Rokhlin's Theorem has signature divisible by 16. It suffices to identify such a manifold which has signature exactly 16. Having done so, we then see that the map $\pi_4(F/PL) \to L_4(\mathbb{Z}[e]) \cong \mathbb{Z}$ is not an isomorphism, but rather multiplication by 2.

Consider the quotient $\mathbb{T}^4/\sim$ where the equivalence identifies each point of the torus with its conjugate. It is an orbifold with Euler characteristic 8 and signature zero. The singular points have neighborhoods of the cone $c(\mathbb{RP}^3)$ on $\mathbb{RP}^3$. Taking $\mathbb{RP}^3$ as the unit tangent bundle to $\mathbb{S}^2$, we can replace $c(\mathbb{RP}^3)$ by the tangent disk bundle to $\mathbb{S}^2$. The combinatorial resolution of the 16 singularities yields a closed, spin 4-manifold $K3$ admitting a unique smooth or PL structure. By a van Kampen argument, we can show that it is simply connected. The Euler characteristic of $K3$ is 24 and the signature is 16. See Huybrechts [333]. The middle homology $H_2(K3; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{22}$. This manifold has the structure of an algebraic variety called the *Kummer surface*. Its intersection form is represented in an appropriate basis by the unimodular matrix

$$E_8 \oplus E_8 \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where E_8 is the 8×8 matrix given in Construction 3.9.

These analyses give us the following.

Corollary 3.24. *The map $\pi_4(F/PL) \to L_4(\mathbb{Z}[e])$ is injective, in fact multiplication by 2, and the map $\pi_2(F/PL) \to L_2(\mathbb{Z}[e])$ is an isomorphism. Therefore the equations at the very end of Subsection 3.2.2 apply for all $n \geq 1$.*

3.3 SOME PL AND DIFF EXAMPLES

In the next section we will give the precise structure of F/PL. In this section, we will see a few examples that do not require precise information beyond the homotopy groups. We will also perform one smooth calculation, namely the calculation of the the group of homotopy spheres (Kervaire-Milnor [355]).

3.3.1 PL examples

If the PL surgery exact sequence were true in dimension 3 (it is not), then we could have applied it to $\mathbb{S}^3$ to arrive at the segment of the surgery exact sequence given by

$$\pi_4(F/PL) \to L_4(\mathbb{Z}[e]) \to \mathcal{S}^{PL}(\mathbb{S}^3) \to \pi_3(F/PL).$$

Since the leftmost map is multiplication by 2, it would follow that $\mathcal{S}^{PL}(\mathbb{S}^3) \cong \mathbb{Z}_2$. Of course we know by the Poincaré conjecture that this result is false. However, Shaneson [580] shows in his thesis that this phantom $\mathbb{Z}_2$ arises in the computation of $\mathcal{S}^{PL}(\mathbb{T}^2 \times \mathbb{S}^3)$, which we now calculate; i.e. there is a homotopy $\mathbb{T}^2 \times \mathbb{S}^3$ that “thinks" that it is of the form $\mathbb{T}^2 \times \Sigma$, where Σ is a suitable counterexample to the Poincaré conjecture (which of course does not exist).

Proposition 3.25. *The PL structure set of the product $\mathbb{T}^2 \times \mathbb{S}^3$ is given by $\mathcal{S}^{PL}(\mathbb{T}^2 \times \mathbb{S}^3) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

Proof. We will first compute the four outer groups in the surgery exact sequence for $\mathbb{T}^2 \times \mathbb{S}^3$ given by

$$[\Sigma(\mathbb{T}^2 \times \mathbb{S}^3) : F/PL] \to L_6(\mathbb{Z}[\mathbb{Z}^2]) \to \mathcal{S}^{PL}(\mathbb{T}^2 \times \mathbb{S}^3) \to [\mathbb{T}^2 \times \mathbb{S}^3 : F/PL] \to L_5(\mathbb{Z}[\mathbb{Z}^2]).$$

Recall that $\pi_k(F/PL) = \mathbb{Z}_2$ when $k \equiv 2 \bmod 4$ and $\pi_4(F/PL) = \mathbb{Z}$ when $k \equiv 0 \bmod 4$. First of all, we know from Theorem 2.7 that $L_6(\mathbb{Z}[\mathbb{Z}^2]) = L_6(\mathbb{Z}[e]) \oplus 2L_5(\mathbb{Z}[e]) \oplus L_4(\mathbb{Z}[e]) = \mathbb{Z}_2 \oplus \mathbb{Z}$ and $L_5(\mathbb{Z}[\mathbb{Z}^2]) = L_5(\mathbb{Z}[e]) \oplus 2L_4(\mathbb{Z}[e]) \oplus L_3(\mathbb{Z}[e]) = \mathbb{Z} \oplus \mathbb{Z}$.

Since F/PL is an infinite loop space, there is Y for which $F/PL = \Omega Y$. Also we know that $\Sigma\mathbb{T}^2 \simeq \mathbb{S}^3 \vee \mathbb{S}^2 \vee \mathbb{S}^2$. Then

$$\begin{aligned}
[\mathbb{T}^2 : F/PL] &= [\mathbb{T}^2 : \Omega Y] \\
&= [\Sigma\mathbb{T}^2 : Y] \\
&= [\mathbb{S}^3 \vee \mathbb{S}^2 \vee \mathbb{S}^2 : Y] \\
&= \pi_3(Y) \oplus \pi_2(Y) \oplus \pi_2(Y) \\
&= \pi_2(F/PL) \oplus \pi_1(F/PL) \oplus \pi_1(F/PL) \\
&= \mathbb{Z}_2.
\end{aligned}$$

Using the identity $\Sigma(X \times Y) = \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$, we also have

$$\begin{aligned}
[\mathbb{T}^2 \times \mathbb{S}^3 : F/PL] &= [\Sigma(\mathbb{T}^2 \times \mathbb{S}^3) : \Omega Y] \\
&= [\Sigma\mathbb{T}^2 \vee \Sigma\mathbb{S}^3 \vee \Sigma(\mathbb{T}^2 \wedge \mathbb{S}^3), Y] \\
&= [\Sigma^2\mathbb{T} \vee \mathbb{S}^4 \vee \mathbb{S}^5 \vee \mathbb{S}^5, Y] \\
&= \mathbb{Z}_2 \oplus \pi_4(Y) \oplus \pi_5(Y) \oplus \pi_5(Y) \\
&= \mathbb{Z}_2 \oplus \pi_3(F/PL) \oplus \pi_4(F/PL) \oplus \pi_4(F/PL) \\
&= \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}.
\end{aligned}$$

Lastly we have

$$\begin{aligned}
[\Sigma(\mathbb{T}^2 \times \mathbb{S}^3) : F/PL] &= [\Sigma\mathbb{T}^2 : F/PL] \oplus \pi_4(F/PL) \oplus \pi_5(F/PL) \oplus \pi_5(F/PL) \\
&= [\mathbb{S}^3 \vee \mathbb{S}^2 \vee \mathbb{S}^2 : F/PL] \oplus \mathbb{Z} \\
&= \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}.
\end{aligned}$$

The above sequence becomes

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \to \mathbb{Z}_2 \oplus \mathbb{Z} \to \mathcal{S}^{PL}(\mathbb{T}^2 \times \mathbb{S}^3) \to \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

where the map $\pi_4(F/PL) \to L_4(\mathbb{Z}[e])$ discussed above appears in the first map. It is easily seen that $\mathcal{S}^{PL}(\mathbb{T}^2 \times \mathbb{S}^3) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where one copy of $\mathbb{Z}_2$ appears as the cokernel of the first map and the other appears as the kernel of the last. □

We offer now two straightforward simply connected examples.

Theorem 3.26. *If $n+m \geq 5$ and either m and n is even, then $\mathcal{S}^{PL}(\mathbb{S}^m \times \mathbb{S}^n)$ is non-trivial. Otherwise it is trivial.*

Proof. Note first that there is a map $\mathbb{S}^{m+1} \vee \mathbb{S}^{n+1} \vee \mathbb{S}^{m+n+1} \to \Sigma(\mathbb{S}^m \times \mathbb{S}^n)$, where map $\mathbb{S}^{m+n+1} \to \Sigma(\mathbb{S}^m \times \mathbb{S}^n)$ is obtained by a Thom space map of the inclusion $\mathbb{S}^m \times \mathbb{S}^n \hookrightarrow \mathbb{S}^{m+n+1}$. One can easily check that this map is a homology equivalence and therefore a homotopy equivalence. Since F/PL is an infinite loop space, we let Z be a space with $F/PL = \Omega Z$. Note that $[\cdot : F/PL]$ is a generalized cohomology theory, so by the Künneth theorem we have $[\mathbb{S}^m \times \mathbb{S}^n : F/PL] = [\mathbb{S}^m \times \mathbb{S}^n : \Omega Z] = [\Sigma(\mathbb{S}^m \times \mathbb{S}^n) : Z] = [\mathbb{S}^{m+1} \vee \mathbb{S}^{n+1} \vee \mathbb{S}^{m+n+1} : Z] = [\Sigma(\mathbb{S}^m \vee \mathbb{S}^n \vee \mathbb{S}^{m+n}) : Z] = [\mathbb{S}^m \vee \mathbb{S}^n \vee \mathbb{S}^{m+n} : \Omega Z] = [\mathbb{S}^m \vee \mathbb{S}^n \vee \mathbb{S}^{m+n} : F/PL] = \pi_{m+n}(F/PL) \oplus \pi_m(F/PL) \oplus \pi_n(F/PL)$. The surgery obstruction for $\mathbb{S}^m \times \mathbb{S}^n$ projects onto the first summand, since $\pi_{m+n}(F/PL) = L_{m+n}(\mathbb{Z}[e])$. Therefore the surgery exact sequence is given by

$$\mathcal{S}^{PL}(\mathbb{S}^m \times \mathbb{S}^n) \to \pi_{m+n}(F/PL) \oplus \pi_m(F/PL) \oplus \pi_n(F/PL) \to L_{m+n}(\mathbb{Z}[e]).$$

The kernel of the surgery map is therefore $\pi_m(F/PL) \oplus \pi_n(F/PL) \cong L_m(\mathbb{Z}[e]) \oplus L_n(\mathbb{Z}[e])$, which is non-trivial if either m or n is even. So the structure set $\mathcal{S}^{PL}(\mathbb{S}^m \times \mathbb{S}^n)$ is non-trivial in this case. In fact, if m or $n \equiv 0 \bmod 4$, then the structure set is infinite. This map $\mathcal{S}^{PL}(\mathbb{S}^m \times \mathbb{S}^n) \to L_m(\mathbb{Z}[e]) \oplus L_n(\mathbb{Z}[e])$ can be shown to be an isomorphism. □

Theorem 3.27. *Let M be a simply connected 5-manifold or even a Poincaré complex. Then $\mathcal{S}^{PL}(M)$ is finite.*

Proof. By Poincaré duality, we have $H^4(M;\mathbb{Z}) \cong H_1(M;\mathbb{Z}) = 0$. In the next section, we will see that there is a fibration $K(\mathbb{Z},4) \to F/PL^{[5]} \to K(\mathbb{Z}_2,2)$ that gives rise to an exact sequence

$$H^4(M;\mathbb{Z}) \to [M^5 : F/PL^{[5]}] \to H^2(M;\mathbb{Z}_2)$$

and $[M^5 : F/PL^{[5]}] = [M^5 : F/PL]$ since M^5 is 5-dimensional. Therefore we have a sequence $H^4(M;\mathbb{Z}) \to [M : F/PL] \to H^2(M;\mathbb{Z}_2)$ which is exact in the middle term. Since $H^4(M;\mathbb{Z}) = 0$, it follows that the map $[M : F/PL] \to H^2(M;\mathbb{Z}_2)$ is injective.

The long exact sequence associated to the fibration $K(\mathbb{Z},4) \to F/PL^{[5]} \to K(\mathbb{Z}_2,2)$ extends to the right, giving

$$0 = H^4(M;\mathbb{Z}) \to [M^5 : F/PL^{[5]}] \to H^2(M;\mathbb{Z}_2) \to H^5(M;\mathbb{Z}).$$

The last map factors through $H^4(M;\mathbb{Z}_2) = 0$ via the Bockstein map $\beta : H^i(M;\mathbb{Z}_2) \to H^{i+1}(M;\mathbb{Z})$ associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. Therefore $[M : F/PL] \cong H^2(M;\mathbb{Z}_2)$. The surgery exact sequence gives $[\Sigma M : F/PL] \to L_6(\mathbb{Z}[e]) \to \mathcal{S}^{PL}(M) \to [M : F/PL] = H^2(M;\mathbb{Z}_2)$. The first map is surjective, so there is an injection $\mathcal{S}^{PL}(M) \to H^2(M;\mathbb{Z}_2)$. Since this latter group is finite, it follows that $\mathcal{S}^{PL}(M)$ is finite. □

Remark 3.28. *See Barden [36] and Smale [597] for the classification of simply connected 5-manifolds.*

Remark 3.29. *It is an exercise, using the Puppe sequence of the cofibration $\mathbb{CP}^{n-1} \to \mathbb{CP}^n \to \mathbb{S}^{2n}$ and the surgery exact sequence, to show inductively that for $n \geq 3$ we have*

$$\mathcal{S}^{PL}(\mathbb{CP}^n) \cong \mathcal{S}^{PL}(\mathbb{CP}^{n-1}) \oplus L_{2n-2}(\mathbb{Z}[e]),$$

and also that $\mathcal{S}^{PL}(\mathbb{CP}^3)$ is infinite, and all possible values of p_1 occur as the Pontrjagin class of some element in the structure set. We will do this calculation in a less ad hoc way later.

3.3.2 Additional Diff examples

Exotic spheres are frequently difficult to understand, but the story is a beautiful one. We first have the classic result of Kervaire and Milnor [355].

Theorem 3.30. *(Kervaire and Milnor [356]) If $n \geq 5$, then $\mathcal{S}^{Diff}(\mathbb{S}^n)$ is finite.*

Proof. Let F_k be the set of homotopy equivalences $\mathbb{S}^{k-1} \to \mathbb{S}^{k-1}$ and let F be the direct limit. The group F/O fits into an exact sequence

$$\pi_n(O) \to \pi_n(G) \to \pi_n(F/O) \to \pi_{n-1}(O).$$

By taking the long exact sequence associated to the fibration

$$\Omega^{k-1}\mathbb{S}^{k-1} \to F_k \to \mathbb{S}^{k-1}$$

and taking limits, we can conclude that $\pi_n(F) = \pi_n^S$, which is shown by Serre to be finite, so F has finite homotopy as well. By Bott we know that $\pi_n(O)$ is finite unless $n \equiv 3 \bmod 4$, in which case $\pi_n(O) \cong \mathbb{Z}$. and therefore we know that $\pi_n(F/O)$ is finite unless $n \equiv 0 \bmod 4$, in which case $\pi_n(F/O)$ is the direct sum of $\mathbb{Z}$ with some finite group H. We now analyze the Diff surgery exact sequence for spheres:

$$L_{n+1}(\mathbb{Z}[e]) \to \mathcal{S}^{Diff}(\mathbb{S}^n) \to \pi_n(F/O) \to L_n(\mathbb{Z}[e]).$$

There are then four cases to consider.

Case 1: If $n \equiv 1 \bmod 4$, then we have $\mathbb{Z}_2 \to \mathcal{S}^{Diff}(\mathbb{S}^n) \to \pi_n(F/O) \to 0$, so $\mathcal{S}^{Diff}(\mathbb{S}^n)$ is finite.

Case 2: If $n \equiv 2 \bmod 4$, then we have $0 \to \mathcal{S}^{Diff}(\mathbb{S}^n) \to \pi_n(F/O) \to \mathbb{Z}_2$, so again $\mathcal{S}^{Diff}(\mathbb{S}^n)$ is finite.

Case 3: If $n \equiv 0 \bmod 4$, then we have $0 \to \mathcal{S}^{Diff}(\mathbb{S}^n) \to \pi_n(F/O) \to \mathbb{Z}$, where $\pi_n(F/O)$ is the direct sum of $\mathbb{Z}$ and a finite group H. Because the coefficient of p_k in the Hirzebruch polynomial L_k is nonzero, the rationalization $\pi_n(F/O) \otimes \mathbb{Q} \to \mathbb{Q}$ of the last map is an isomorphism $\mathbb{Q} \to \mathbb{Q}$ (see Milnor-Stasheff [461]). By exactness we know that $\mathcal{S}^{Diff}(\mathbb{S}^n) \otimes \mathbb{Q}$ is trivial; i.e. $\mathcal{S}^{Diff}(\mathbb{S}^n)$ is finite.

Case 4: If $n \equiv 3 \bmod 4$, then the sequence, extended to the left by one additional term, is

$$\pi_{n+1}(F/O) \to \mathbb{Z} \to \mathcal{S}^{Diff}(\mathbb{S}^n) \to \pi_n(F/O) \to 0.$$

The rationalization of the sequence gives

$$\mathbb{Q} \to \mathbb{Q} \to \mathcal{S}^{Diff}(\mathbb{S}^n) \otimes \mathbb{Q} \to 0.$$

The first map is an isomorphism by the previous case. Therefore $\mathcal{S}^{Diff}(\mathbb{S}^n) \otimes \mathbb{Q}$ is again trivial, so $\mathcal{S}^{Diff}(\mathbb{S}^n)$ is finite. Indeed, Kervaire and Milnor describe the size of this finite group in terms of homotopy groups of spheres (and the J-homomorphism, as we will soon explain) and two other pieces. They analyze the map $\pi_{4k}(F/O) \to \mathbb{Z}$ in terms of Bernoulli numbers, and are left with an ambiguity about the map $\pi_{4k+2}(F/O) \to \mathbb{Z}_2$ which can influence the size of $\mathcal{S}^{Diff}(\mathbb{S}^n)$ by a factor of 2 when $n \equiv 1$ or $2 \bmod 4$. Here we have the famous Kervaire invariant problem. These questions determine the highly connected Kervaire and Milnor PL $2n$-manifolds that are smoothable. See Levine [397] and the discussion in Construction 3.13 for references to the history of this story. □

The remaining part of understanding exotic spheres is studied in the unpublished work of Kervaire-Milnor (see Levine [397]) and is related to L-theory; this paper studies the "signature piece."

Remark 3.31. *The structure of F/O is a complicated story, depending on work of Adams (on the J-homomorphism) and Quillen and Sullivan, both of whom gave a proof of the Adams conjecture. We will describe this work in Section 4.6.*

At this point, the reader might be led to the following speculation (that the second author first heard explicitly articulated by Quinn). For PL and Top, the classifying space F/Cat is relatively easy to understand, and therefore we are quite good at classifications within a homotopy type. For Diff, the classifying space BSO is well understood, but F/O is not. Perhaps one should aim for classification without first fixing a homotopy type. There are indeed successes of this type. We mention two.

Example 3.32. *The first is due to Wall, who used quadratic forms to classify $(n-1)$-connected $2n$-manifolds up to connected sum with an exotic sphere. Note that, except in dimensions 4, 8, and 16, all quadratic forms must be of Type II because of the Hopf invariant one theorem. Indeed, if X is a generator of $H^n(M;\mathbb{Z})$, we can attach cells to kill all but the Poincaré dual of X. This process will produce a complex $X \simeq \mathbb{S}^n \cup e^{2n}$ whose inner product structure would contradict the Hopf invariant one theorem. Wall's classification can be achieved because one can explicitly manipulate the handlebody structures of these manifolds, as Smale did in his proof of the generalized Poincaré conjecture. The PL and Top classifications of these manifolds are harder because of $\pi_{2n-1}(\mathbb{S}^n)$.*

Example 3.33. *A second approach, due to Kreck [366], still requires some homotopy theory, but less. His idea is still to perform surgery on a map $M \to X$, but to require homotopy theory only through the middle dimension (so in the Milnor example, one would have just a wedge of sphere with specified rank) and then use Poincaré duality to provide the missing information. In some circumstances this lack of a need for homotopy theory is highly beneficial. The reader might seek conditions for which homogeneous spaces G/H are diffeomorphic or homotopy equivalent. The latter seems no simpler than the former. Kreck and Stolz gave some interesting examples, for instance, of homeomorphic homogeneous manifolds that are not diffeomorphic [371, 372]. A brief discussion of Kreck surgery can be found in Section 8.1.*

3.4 THE HOMOTOPY TYPE OF F/PL

Understanding the classifying space F/PL involves more than merely knowing its homotopy groups. Sullivan determined the homotopy type, showing that it has a transparent structure that allows for very clean computations. The result was never published by Sullivan himself, although his seminar notes were widely distributed. Madsen-Milgram present the proof in their book [420]. The calculations of $F/PL_{(2)}$ and $F/PL[1/2]$ both require some facts about the Thom spectrum MSO and the latter involves a connection between bordism and K-theory. Basic notions of spectra are reviewed in Section 4.4.

3.4.1 Localizing F/PL at 2

The classifying space F/PL is an H-space and has homotopy groups $\pi_n(F/PL) \cong L_n(\mathbb{Z}[e])$. It can be localized, and Sullivan proves that $F/PL_{(2)}$ has an infinite product structure almost entirely into Eilenberg-Maclane spaces. This section will be devoted to outlining the proof for this decomposition.

We recall oriented bordism $\Omega_j^{SO}(X)$, which is a generalized homology theory of X. Elements are compact smooth j-manifolds M^j equipped with continuous maps $f : M \to X$ (see Conner-Floyd [172] and Appendix A). Two elements (M, f) and (N, g) in $\Omega_j^{SO}(X)$ are *equivalent* if there is a cobordism (W^{j+1}, M, N) between M and N equipped with a map $F : W \to X$ that restricts to f and g on the boundaries.

This Ω_j^{SO}, which is also called MSO, is obviously a homotopy functor and satisfies all of the Eilenberg-Steenrod conditions aside from the dimension axiom. As such, it is a generalized homology theory. See the appendices for basic facts about MSO. We shall use Thom's theorem that $MSO_{(2)}$ is Eilenberg-Maclane, or more precisely that $MSO_n(X) \otimes \mathbb{Z}_{(2)} \to H_n(X; \mathbb{Z}_{(2)})$ is surjective.

In this section, we will determine the homotopy type of $F/PL_{(2)}$ by studying the group $MSO_{(2)}(F/PL_{(2)})$. See Rourke-Sullivan [564], Morgan-Sullivan [475], and Madsen-Milgram [420]. Let $h_{2n} : \pi_{2n}(F/PL_{(2)}) \to H_{2n}(F/PL_{(2)}; R)$ be the Hurewicz map, where $R = \mathbb{Z}_{(2)}$ or $R = \mathbb{Z}_2$ depending on the context. Also let γ_{2n} be a generator of $\pi_{2n}(F/PL_{(2)})$.

Lemma 3.34. *There are elements*

$$\rho'_{4n} \in \mathrm{Hom}(H_{4n}(F/PL_{(2)}; \mathbb{Z}_{(2)}), \mathbb{Z}_{(2)}) \quad \text{and} \quad \tau'_{4n-2} \in \mathrm{Hom}(H_{4n-2}(F/PL_{(2)}; \mathbb{Z}_2), \mathbb{Z}_2)$$

such that $\rho'_{4n}(h_{4n}(\gamma_{4n})) = 1$ *and* $\tau'_{4n-2}(h_{4n-2}(\gamma_{4n-2})) = 1$.

Proof. Let

$$h_{MSO_{(2)}} : \pi_{4n}(F/PL_{(2)}) \to (MSO_{(2)})_{4n}(F/PL_{(2)})$$

and

$$h_{MO} : \pi_{4n-2}(F/PL_{(2)}) \to MO_{4n-2}(F/PL_{(2)})$$

be the Hurewicz-Thom maps that take $f : \mathbb{S}^n \to F/PL_{(2)}$ to the pair $\{\mathbb{S}^n, f\}$ in the appropriate bordism groups. The surgery obstruction $\sigma_n : [\mathbb{S}^n : F/PL_{(2)}] \to L_n(\mathbb{Z}[e])$

is a bordism invariant, so the maps

$$\sigma_{4n} \colon \pi_{4n}(F/PL_{(2)}) \to \mathbb{Z}_{(2)}$$

and

$$\sigma_{4n-2} \colon \pi_{4n-2}(F/PL_{(2)}) \to \mathbb{Z}_2$$

can be factored into compositions

$$\pi_{4n}(F/PL_{(2)}) \xrightarrow{h_{MSO_{(2)}}} (MSO_{(2)})_{4n}(F/PL_{(2)}) \xrightarrow{\sigma'_{4n}} \mathbb{Z}_{(2)}$$

and

$$\pi_{4n-2}(F/PL_{(2)}) \xrightarrow{h_{MO}} MO_{4n-2}(F/PL_{(2)}) \xrightarrow{\sigma'_{4n-2}} \mathbb{Z}_2.$$

These Hurewicz-Thom maps factor through homology in the following ways:

$$\begin{array}{ccc} \pi_{4n}(F/PL_{(2)}) & \xrightarrow{\quad h_{MSO_{(2)}} \quad} & (MSO_{(2)})_{4n}(F/PL_{(2)}) \\ & {\scriptstyle h_{4n}} \searrow \qquad \swarrow {\scriptstyle \alpha_{MSO_{(2)}}} & \\ & H_{4n}(F/PL_{(2)}; \mathbb{Z}_{(2)}) & \end{array}$$

and

$$\begin{array}{ccc} \pi_{4n-2}(F/PL_{(2)}) & \xrightarrow{\quad h_{MO} \quad} & MO_{4n-2}(F/PL_{(2)}) \\ & {\scriptstyle h_{4n-2}} \searrow \qquad \swarrow {\scriptstyle \alpha_{MO}} & \\ & H_{4n-2}(F/PL_{(2)}; \mathbb{Z}_2) & \end{array}$$

Let β_{MO} and $\beta_{MSO_{(2)}}$ be sections of α_{MO} and $\alpha_{MSO_{(2)}}$. Altogether we have the conglomerate diagrams given by

$$\begin{array}{ccccc} & & \pi_{4n}(F/PL_{(2)}) & \xrightarrow{\qquad \sigma_{4n} \qquad} & \mathbb{Z}_{(2)} \\ & {\scriptstyle h_{4n}} \swarrow & & {\scriptstyle h_{MSO_{(2)}}} \searrow \qquad \nearrow {\scriptstyle \sigma'_{4n}} & \\ H_{4n}(F/PL_{(2)}; \mathbb{Z}_{(2)}) & & \xrightarrow{\qquad \beta_{MSO_{(2)}} \qquad} & (MSO_{(2)})_{4n}(F/PL_{(2)}) & \end{array}$$

and

$$\begin{array}{ccccc} & & \pi_{4n-2}(F/PL_{(2)}) & \xrightarrow{\sigma_{4n-2}} & \mathbb{Z}_2 \\ & \swarrow^{h_{4n-2}} & & \searrow^{h_{MO}} & \nearrow_{\sigma'_{4n-2}} \\ H_{4n-2}(F/PL_{(2)};\mathbb{Z}_2) & & \xrightarrow[\beta_{MO}]{} & & MO_{4n-2}(F/PL_{(2)}) \end{array}$$

The surgery maps σ_{4n} and σ_{4n-2} take the generators γ_k of $\pi_k F/PL_{(2)}$ to 1 in their respective ranges. Let $\rho'_{4n} = \sigma'_{4n} \circ \beta_{MSO_{(2)}} : H_{4n}(F/PL_{(2)};\mathbb{Z}_{(2)}) \to \mathbb{Z}_{(2)}$ and $\tau'_{4n-2} = \sigma'_{4n-2} \circ \beta_{MO} : H_{4n-2}(F/PL_{(2)};\mathbb{Z}_2) \to \mathbb{Z}_2$. Then we finally have $\rho'_{4n}(h_{4n}(\gamma_{4n})) = 1$ and $\tau'_{4n-2}(h_{4n-2}(\gamma_{4n-2})) = 1$, as required. □

Proposition 3.35. *For each $n \geq 2$ there are maps $\rho_{4n} : F/PL_{(2)} \to K(\mathbb{Z}_{(2)}, 4n)$ and $\tau_{4n-2} : F/PL_{(2)} \to K(\mathbb{Z}_2, 4n-2)$ that induce isomorphisms on π_{4n} and π_{4n-2} respectively.*

Proof. Note that the universal coefficient theorem will give the sequences

$$\begin{aligned} \operatorname{Ext}(H_{4n-1}(F/PL_{(2)};\mathbb{Z}_{(2)}),\mathbb{Z}_{(2)}) &\to H^{4n}(F/PL_{(2)};\mathbb{Z}_{(2)}) \\ &\to \operatorname{Hom}(H_{4n}(F/PL_{(2)};\mathbb{Z}_{(2)}),\mathbb{Z}_{(2)}) \to 0 \end{aligned}$$

and

$$\begin{aligned} \operatorname{Ext}(H_{4n-3}(F/PL_{(2)};\mathbb{Z}_2),\mathbb{Z}_2) &\to H^{4n-2}(F/PL_{(2)};\mathbb{Z}_2) \to \\ &\operatorname{Hom}(H_{4n-2}(F/PL_{(2)};\mathbb{Z}_2),\mathbb{Z}_2) \to 0. \end{aligned}$$

By Lemma 3.34, the elements ρ'_{4n} and τ'_{4n-2} lie in these two Hom groups, which by surjectivity can be pulled back to elements ρ_{4n} and τ_{4n-2} in $H^{4n}(F/PL_{(2)};\mathbb{Z}_{(2)})$ and $H^{4n-2}(F/PL_{(2)},\mathbb{Z}_2)$ satisfying the required properties. □

Theorem 3.36. *(Sullivan [622]) The 2-local homotopy type of F/PL satisfies*

$$F/PL_{(2)} \simeq E^4 \times \prod_{n\geq 2} K(\mathbb{Z}_{(2)}, 4n) \times K(\mathbb{Z}_2, 4n-2)$$

where E^4 is a complex for which $\pi_k(E^4) = 0$ for all $k \geq 5$ and $\pi_j(E^4) \cong \pi_j(F/PL_{(2)})$ for all $j \leq 4$.

Proof. Attach cells of dimension 6 and higher to $F/PL_{(2)}$ to produce a complex E^4 with $\pi_k(E^4) = 0$ for all $k \geq 5$. The inclusion $i : F/PL_{(2)} \to E^4$ induces isomorphisms on π_j for all $j \leq 4$. Let

$$P = \prod_{n\geq 2} K(\mathbb{Z}_2, 4n-2) \times K(\mathbb{Z}_{(2)}, 4n).$$

The maps in Proposition 3.35 give a map $\Phi : F/PL_{(2)} \to P$ inducing isomorphisms on π_k for all $k \geq 5$, and $\pi_j(P) = 0$ for all $j \leq 4$. Consider $\phi : i \times \Phi : F/PL_{(2)} \to E^4 \times P$. This map induces an isomorphism on all homotopy groups, so it is a homotopy equivalence by the Whitehead theorem. □

We now discuss the structure of E^4. In general, let $0 \to A \to B \to C \to 0$ be an exact sequence of abelian groups and let M be a space. There is a long exact sequence in cohomology given by

$$\cdots \to H^{i-1}(M;C) \to H^i(M;A) \to H^i(M;B) \to H^i(M;C) \xrightarrow{\beta} H^{i+1}(M;A) \to \cdots$$

where β is the boundary map. The *Bockstein map* is the boundary map $\beta : H^i(M;\mathbb{Z}_2) \to H^{i+1}(M;\mathbb{Z})$ for the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. We then have a composition $H^2(M;\mathbb{Z}_2) \xrightarrow{Sq^2} H^4(M;\mathbb{Z}_2) \xrightarrow{\beta} H^5(M;\mathbb{Z})$.

If E^4 is the space above, denote by $E^{[r]}$ the r-th Postnikov approximation to E^4. For all n there is a fibration $K(\pi_n(E), n) \to E^{[n]} \to E^{[n-1]}$. The obstructions to a cross section lie in the groups $H^{i+1}(E^{[n-1]}; \pi_i(K(\pi_n(E), n)))$ for all i. The only possibly nonvanishing group occurs when $i = n$. A primary obstruction $k^{n+1} \in H^{n+1}(E^{[n-1]}; \pi_n(E))$, called the *k-invariant*, can be interpreted as a function $k^{n+1} : E^{[n-1]} \to K(\pi_n(E), n+1)$.

Theorem 3.37. *(Sullivan) The space E^4 is the 2-stage Postnikov system with non-trivial homotopy groups $\pi_2(E^4) = \mathbb{Z}_2$ and $\pi_4(E^4) = \mathbb{Z}_{(2)}$. In addition its k-invariant is $\beta\, Sq^2 \in H^5(K(\mathbb{Z}_2;2), \mathbb{Z}_{(2)})$, where β is the Bockstein operator.*

Below is the Postnikov system for E^4. Recall that $\pi_k(E^4) = 0$ for all $k \geq 5$.

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
K(\pi_4(E),4) & \longrightarrow & E^{[4]} & \xrightarrow{k_6} & K(\pi_5(E),6) \\
 & & \downarrow & & \\
K(\pi_3(E),3) & \longrightarrow & E^{[3]} & \xrightarrow{k_5} & K(\pi_4(E),5) \\
 & & \downarrow & & \\
K(\pi_2(E),2) & \longrightarrow & E^{[2]} & \xrightarrow{k_4} & K(\pi_3(E),4) \\
 & & \downarrow & & \\
K(\pi_1(E),1) & \longrightarrow & E^{[1]} & \xrightarrow{k_3} & K(\pi_2(E),3)
\end{array}$$

Most of these spaces are trivial for E^4, and the Postnikov system reduces to

$$
\begin{array}{ccccc}
K(\mathbb{Z},4) & \longrightarrow & E^{[4]} & & \\
 & & \big\downarrow & & \\
 & & E^{[3]} & \xrightarrow{k_5} & K(\mathbb{Z},5) \\
 & & \big\| & & \\
K(\mathbb{Z}_2,2) & \xrightarrow{\cong} & E^{[2]} & & \\
 & & \big\downarrow & & \\
 & & E^{[1]} & \xrightarrow{k_3} & K(\mathbb{Z}_2,3)
\end{array}
$$

There is a fibration $K(\mathbb{Z},4) \to E^{[4]} \to K(\mathbb{Z}_2,2)$, assembled by the k-invariant $k_5 : E^{[3]} \to K(\mathbb{Z},5)$. We can then write $E^{[4]} \simeq K(\mathbb{Z}_2,2) \times_{k_5} K(\mathbb{Z},4)$. Note that the fibration $K(\mathbb{Z},4) \to E^{[4]} \to K(\mathbb{Z}_2,2)$ can be extended to give

$$K(\mathbb{Z},4) \to E^{[4]} \to K(\mathbb{Z}_2,2) \to K(\mathbb{Z},5).$$

Theorem 3.38. *(Sullivan) The induced map $H^2(E^{[4]};\mathbb{Z}_2) \to H^5(E^{[4]};\mathbb{Z})$ is given by the composition of the Bockstein map β with Sq^2.*

Proof. To accommodate Rokhlin's theorem about the signature in dimension 4, the space F/PL should have a non-trivial k-invariant, and βSq^2 is the only possibility (see Madsen-Milgram [420]), since it is a loop map. □

3.4.2 Localizing F/PL away from 2

Recall that, for a compact space X, the KO-group $K(X)$ is the quotient $F(X)/E(X)$, where $F(X)$ is the free abelian group generated by isomorphism classes of real vector bundles over X and $E(X)$ is the subgroup of $F(X)$ generated by elements of the form $[V]+[W]-([V]\oplus[W])$. This $KO(X)$ has a ring structure given by an interior tensor product. The *reduced KO-ring* $\widetilde{KO}(X)$ is given by the kernel of the natural projection $KO(X) \to KO(pt) \cong \mathbb{Z}$.

For an extensive discussion of KO, see Lawson-Michelsohn [390]. To begin, we give some results involving the localization of KO away from 2.

Theorem 3.39. *(see Conner-Floyd [172]) If X is a CW complex, then there is an isomorphism $MSO_*(X)\otimes_{MSO_*(pt)}\mathbb{Z}[1/2] \cong KO_*(X)\otimes\mathbb{Z}[1/2]$, where we use the signature to make $\mathbb{Z}[1/2]$ into an MSO-module.*

Theorem 3.40. *(Yosimura [714]) There is a "universal coefficient" surjective map $KO[1/2]^n(X) \to \mathrm{Hom}(KO[1/2]_n(X),\mathbb{Z}[1/2])$ with kernel given by*

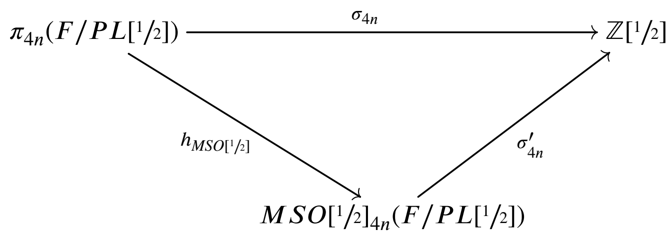
$$\mathrm{Ext}(KO[1/2]_{n-1}(X),\mathbb{Z}[1/2]).$$

Remark 3.41. *The map is given by associating to M the symbol of the signature operator of M. See Atiyah [22] or Higson-Roe [306].*

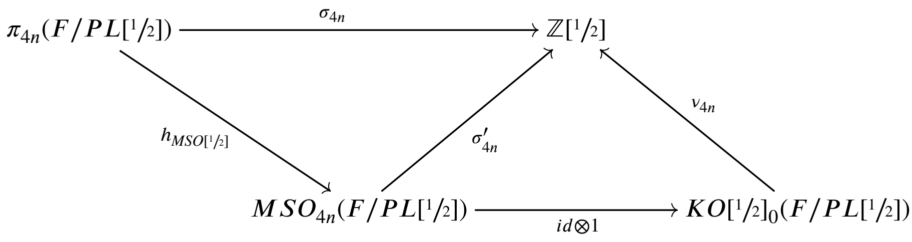
Theorem 3.42. *The odd homotopy type of F/PL is given by $F/PL[1/2] \simeq BO[1/2]$.*

Proof. Since localization commutes with homotopy, it follows that $\pi_n(F/PL[1/2]) = \pi_n(F/PL)[1/2]$, which equals $\mathbb{Z}[1/2]$ if $n \equiv 0 \bmod 4$ and is zero otherwise. The homotopy groups of $BO[1/2]$ are the same. We therefore wish to find a map $F/PL[1/2] \to BO[1/2]$ that realizes an isomorphism on $4k$-dimensional homotopy groups for all k. We break up the argument into a number of steps.

Step 1: Now MSO is the spectrum that classifies oriented bordism, and the surgery obstruction $\sigma_{4n} : \pi_{4n}(F/PL[1/2]) = [\,\mathbb{S}^{4n} : F/PL[1/2]] \to \mathbb{Z}[1/2]$ is an oriented bordism invariant. Therefore it factors through the localization of MSO as follows, just as before:



The map σ'_{4n} factors through $KO[1/2]_0(F/PL[1/2])$:



since $\mathrm{sig}(M \times N) = \mathrm{sig}(M)\mathrm{sig}(N)$. Note that the map on the bottom factors through

$$\Omega^{SO}_{4*}(F/PL[1/2]) \to \Omega^{SO}_{4*}(F/PL[1/2]) \otimes_{\Omega_*(pt)} \mathbb{Z}[1/2].$$

Step 2: Since $\nu_{4n} \in \mathrm{Hom}(KO[1/2]_0(F/PL[1/2]), \mathbb{Z}[1/2])$, it lifts by Theorem 3.39 to an element $\widetilde{\nu}$ in $KO[1/2]^0(F/PL[1/2])$. The element $\widetilde{\nu}$ can be seen as a map $F/PL[1/2] \to BO[1/2] \times \mathbb{Z}[1/2]$. Composed with projection, this map gives a map $\nu' : F/PL[1/2] \to BO[1/2]$, seen as an element of $\widetilde{KO}[1/2]^0(F/PL[1/2])$. This ν' induces isomorphisms on all homotopy groups because we know how it behaves with respect to the MSO-

Hurewicz homomorphism, and therefore demonstrates that $F/PL[{}^1\!/\!_2]$ and $BO[{}^1\!/\!_2]$ are indeed homotopy equivalent. $\square$

3.4.3 Combining information

Note that, given the homotopy type of $F/PL_{(2)}$, we obtain

$$F/PL_{\mathbb{Q}} = \Pi_{j\geq 1} K(\mathbb{Q}, 4j)$$

after inverting all primes. It also follows from the homotopy groups and Hopf's theorem (à la Serre). From the usual localization diagram, we have the following:

Theorem 3.43. *The integral homotopy type of F/PL fits into the diagram*

$$\begin{array}{ccc} F/PL & \longrightarrow & E \times K(\mathbb{Z}_2, 6) \times K(\mathbb{Z}_{(2)}, 8) \times \cdots \\ \downarrow & & \downarrow \\ BO[{}^1\!/\!_2] & \longrightarrow & \prod_{j\geq 1} K(\mathbb{Q}, 4j) \end{array}$$

Remark 3.44. *The bottom line is given by the Chern character of the complexification. The proof is straightforward from the construction of the maps from F/PL to each of these spaces.*

Remark 3.45. *The space F/PL is an H-space. As a result $[X : F/PL]$ is a finitely generated abelian group when X is a finite CW complex. The free part is abstractly isomorphic to the direct sum of the free parts of $H^{4*}(X;\mathbb{Z})$.*

For example, if X is rationally trivial and 2-connected, we abstractly have an isomorphism of abelian groups

$$\mathcal{N}^{PL}(X) \cong \bigoplus_{i\geq 1} H^{4i}(X;\mathbb{Z}_{(2)}) \oplus \bigoplus_{i\geq 1} H^{4i-2}(X;\mathbb{Z}_2) \oplus (KO(X) \otimes \mathbb{Z}[{}^1\!/\!_2]).$$

We finish the section with some computations on complex projective spaces $\mathbb{CP}^n$. Recall that the punctured complex projective space $\mathbb{CP}^n_0$ retracts to $\mathbb{CP}^{n-1}$, i.e. as $\mathbb{CP}^n$ is the orbit space of the equivariant $\mathbb{S}^1$-manifold $\mathbb{S}^1 * \cdots * \mathbb{S}^1$.

Lemma 3.46. *Here are some basic facts about complex projective space.*

1. *We have an isomorphism $[\mathbb{CP}^2 : F/PL] \cong \mathbb{Z}$.*
2. *If $n \geq 3$, then $KO(\mathbb{CP}^n) \otimes \mathbb{Z}[{}^1\!/\!_2] \cong \mathbb{Z}[{}^1\!/\!_2]^{[n/2]}$.*

Proof. We give a proof of (1). Let F be the fiber of $\beta : K(\mathbb{Z}_2, 4) \to K(\mathbb{Z}, 5)$. In fact $F = K(\mathbb{Z}, 4)$. The fiber sequence $K(\mathbb{Z}, 4) \to F \to K(\mathbb{Z}_2, 4)$ gives rise to another short exact sequence when applying $[\mathbb{CP}^2, -]$ to it, namely $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. Now Sq^2 induces a map of short exact sequences which is the identity on the $\mathbb{Z}$-terms on the left and also the identity on the $\mathbb{Z}_2$ terms on the right, since $Sq^2 : H^2(\mathbb{CP}^2; \mathbb{Z}_2) \to H^4(\mathbb{CP}^2; \mathbb{Z}_2)$ is an isomorphism. It follows that the group $[\mathbb{CP}^2 : F/PL]$ is isomorphic to $\mathbb{Z}$. □

Proposition 3.47. *The following identities hold for complex projective spaces $\mathbb{CP}^n$. Let Y^4 be the two-stage Postnikov tower appearing in the homotopy type of F/PL.*

1. *If $n \geq 3$, then $\mathcal{S}^{PL}(\mathbb{CP}^n) = \mathbb{Z}^k \oplus \mathbb{Z}_2^m$, where $k = \left[\frac{n-1}{2}\right]$ and $m = \left[\frac{n-2}{2}\right]$.*
2. *If $n \geq 2$, then $[\mathbb{CP}^n : Y^4] = \mathbb{Z}$.*
3. *If $n \geq 2$, then $[\mathbb{CP}^n : F/PL] = \mathbb{Z}^q \oplus \mathbb{Z}_2^r$, where $q = \left[\frac{n}{2}\right]$ and $r = \left[\frac{n-1}{2}\right]$.*

This result immediately follows from the structure of F/PL. We can use it to compute $\mathcal{S}^{PL}(\mathbb{CP}^n)$, since we have $\mathcal{S}^{PL}(\mathbb{CP}^n) \cong \mathcal{S}^{PL}(\mathring{\mathbb{CP}}^n) \cong [\mathbb{CP}^{n-1} : F/PL]$ by the π-π theorem.

The copies of $\mathbb{Z}_2$ and $\mathbb{Z}$ are called *splitting invariants*. They determine whether a homotopy equivalence $f : M \to \mathbb{CP}^n$ is homotopic to a map g that is transverse to $\mathbb{CP}^k$ such that $g|_{g^{-1}(\mathbb{CP}^k)}$ is a homotopy equivalence.

3.5 FINITE H-SPACES

Named after Hopf, an *H-space* is a topological space X, which we will generally assume to be connected in this section, together with a continuous map $\mu : X \times X \to X$ with a basepoint e so that $\mu(e, x) = \mu(x, e) = x$ for all x in X. The maps $\mu(e, x)$ and $\mu(x, e)$ are sometimes only required to be homotopic to the identity through basepoint-preserving maps, but it does not make a difference because of the homotopy extension property. An H-space is not necessarily a group even up to homotopy because associativity can fail, for example. Needless to say, every topological group is an H-space.

The fundamental group of an H-space is abelian by the following argument. Let X be an H-space with identity e and let α and β be loops at e. Define a map $F : [0, 1] \times [0, 1] \to X$ by $F(s, t) = \alpha(s)\beta(t)$. Then $F(s, 0) = F(s, 1) = \alpha(s)e$ is homotopic to α, and $F(0, t) = F(1, t) = e\beta(t)$ is homotopic to β. It is clear how to define a homotopy from $\alpha\beta$ to $\beta\alpha$.

The multiplicative structure of an H-space has some implications on its homology and cohomology groups. For example, if X is a path-connected H-space with finitely generated cohomology groups, then the cohomology ring must be a Hopf algebra. (Indeed, Hopf invented Hopf algebras to study the cohomology of Lie groups.) Dually there is a Pontrjagin product on the homology groups of an H-space. Background information about H-spaces can be found in Stasheff [614] and Lin [401].

There are many examples of H-spaces: Lie groups, finite topological groups, finite loop spaces, Eilenberg-Maclane spaces, the localization of an odd-dimensional sphere localized at an odd prime, and many other families of spaces. In this section we will study sufficient conditions for a finite H-space to be homotopy equivalent to a Cat manifold. It is an old conjecture that they all are.

This section will be largely expository and is here mainly to showcase one application of the π-π theorem in a theorem by Bauer-Kitchloo-Notbohm-Pedersen [49] toward the end. It also gives us a chance to exploit Zabrodsky mixing, which we used for propagation.

3.5.1 Properties of H-spaces

At one time, it was believed that all simply connected finite H-spaces were products of Lie groups and the 7-sphere. Zabrodsky shattered this belief by the method of *mixing*.

In this section, we sketch a few points in the theory of H-spaces, omitting the proofs of theorems when they are basically algebraic topological. To start, we briefly discuss the Hopf invariant. Let $n \geq 2$ and $\phi\colon \mathbb{S}^{2n-1} \to \mathbb{S}^n$ be continuous. We then form the complex $C_\phi = \mathbb{S}^n \cup_\phi \mathbb{D}^{2n}$ whose cohomology $H^i(C_\phi;\mathbb{Z})$ is $\mathbb{Z}$ for $i = 0, n, 2n$ and 0 otherwise. If α is a generator for $H^n(C_\phi;\mathbb{Z})$ and β is a generator for $H^{2n}(C_\phi;\mathbb{Z})$, then there is an integer h_ϕ for which $\alpha \cup \alpha = h_\phi \beta$. This h_ϕ is the *Hopf invariant* of ϕ. The nonexistence part of the following theorem of Adams is established by proving the equivalent statement that there is no element of Hopf invariant 1 in $\pi_{2n-1}(\mathbb{S}^n)$ when $n \neq 2, 4, 8$.

Theorem 3.48. *(Adams [1]) The only spheres that are H-spaces are $\mathbb{S}^0$, $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$.*

Each of these spaces forms an H-space by our viewing it as the subset of norm-one elements of the reals, complexes, quaternions, and octonions, respectively, and using the multiplication operations from these algebras. In fact, it is clear that $\mathbb{S}^0$, $\mathbb{S}^1$, and $\mathbb{S}^3$ are (Lie) groups with these multiplications. But $\mathbb{S}^7$ is not a group in this way because octonion multiplication is not associative, nor can it be given any other continuous multiplication for which it is a group.

On the other hand, for all even spheres $\mathbb{S}^n$, there is a map of Hopf invariant 2, and it is not hard to see that every $\mathbb{S}^{2k+1}[1/2]$ is an H-space as we now show.

Proposition 3.49. *Let X be a sphere of dimension n and let $X[1/2]$ be its localization away from 2. Then $X[1/2]$ is an H-space iff n is odd.*

Proof. The necessity is a trivial cohomology algebra calculation. For sufficiency, we want to extend a map $\mathbb{S}^n \vee \mathbb{S}^n \to \mathbb{S}^n$ to $\mathbb{S}^n \times \mathbb{S}^n \to \mathbb{S}^n$. The obstruction to such an extension is given by the Whitehead product $[i, i]$ of the identity map on $\mathbb{S}^{2k-1}$ in

$\pi_{2n-1}(\mathbb{S}^n)$. The Whitehead product satisfies $[i, i] = -[i, i]$ and so has order 2. When we invert 2, there are no obstructions. □

Remark 3.50. *We say a few words about the Whitehead product here. If A and B are spaces, there is a cofiber sequence $A * B \xrightarrow{w} \Sigma A \vee \Sigma B \to \Sigma A \times \Sigma B$. Here w is a binary operation $[\Sigma A : X] \times [\Sigma B : X] \to [A * B : X]$. If $\alpha : \Sigma A \to X$ and $\beta : \Sigma B \to X$, then there is a diagram*

$$\begin{array}{ccc} A * B & \xrightarrow{[\alpha,\beta]} & X \\ & {\scriptstyle w}\searrow \quad \nearrow{\scriptstyle (\alpha,\beta)} & \\ & \Sigma A \vee \Sigma B & \end{array}$$

Then $[\alpha, \beta] = 0$ iff $(\alpha, \beta) : \Sigma A \vee \Sigma B \to X$ extends up to homotopy to a map $[\Sigma A \times \Sigma B : X]$. We recall that the Whitehead product is endowed with a graded commutativity with $[p, q] = (-1)^{|p||q|}[q, p]$.

The necessity above was considerably extended by Hopf. It is the famous result for which he introduced the Hopf algebra.

Theorem 3.51. *(Hopf [320]) The only possible finite-dimensional rational cohomology algebra compatible with an H-space structure is that of a product of odd-dimensional spheres, i.e. an exterior algebra on finitely many odd-dimensional classes. Equivalently, an H-space X with finite-dimensional rational cohomology has the rational homotopy type of a product of odd-dimensional spheres.*

Remark 3.52. *If one allows infinite cohomological dimension, then Hopf proves also that the rational cohomology algebra is a product of a free polynomial algebra with a free exterior algebra; i.e. the H-space is rationally a product of Eilenberg-MacLane spaces.*

Theorem 3.53. *(Zabrodsky [718]) There is a finite H-space that is not a product of Lie groups or spheres.*

Proof. The construction of such an H-space is simple. Let $P_2 = \{2, 3\}$ and P_1 be the set of the remaining primes. Let $S = \mathbb{S}^3 \times \mathbb{S}^5 \times \mathbb{S}^7 \times \mathbb{S}^{11}$ and $T = \mathrm{SU}(6)$. These spaces rationally agree. One can mix them. Since $S[1/2]$ is an H-space, one can check (see Zabrodsky [718]) that the pullback

$$\begin{array}{ccc} & & S[1/6] \\ & & \downarrow \\ T_{(6)} & \longrightarrow & \prod_1^5 K(\mathbb{Q}, 2m+1) \end{array}$$

is an H-space that is neither of these Lie groups (because of the structure of these spaces), and, in fact, is not Lie at all, by inspecting the list of compact Lie groups. □

The integral structure of H-spaces is much more subtle and has been extensively studied. After the examples of Hilton-Roitberg [309] and the ones of Zabrodsky [718] of H-spaces not homotopy equivalent to Lie groups, Browder asked whether finite or finitely dominated H-spaces are necessarily homotopy equivalent to closed (smooth) parallelizable manifolds. This question was suggested by the following.

Theorem 3.54. *(Browder [79]) If X is a finite H-space, then X satisfies Poincaré duality; i.e. X is a Poincaré complex.*

Remark 3.55. *This theorem does not include finiteness, i.e. a finitely dominated H-space could potentially have a non-trivial Wall finiteness obstruction.*

Recall that a Diff manifold M is *parallelizable* if its tangent bundle TM is trivial, and it is *stably parallelizable* if $TM \oplus \mathbb{R}^n$ is trivial for some $n \geq 1$.

Proposition 3.56. *(Browder-Spanier [95] and Spanier [603]) Every finite H-space X admits a degree one Diff normal map $f: M \to X$ where M is a stably parallelizable Diff manifold.*

This proposition follows from the following.

Theorem 3.57. *(Browder-Spanier [95]) If X is a finite H-space of dimension k, then the stable Hurewicz map $h: \pi_k^s(X) \to H_k(X;\mathbb{Z})$ is surjective.*

In other words, the Spivak fibration of X is trivial. We will prove it later in the course of proving Theorem 3.63.

By the theorem, if X is a finite H-space of dimension k and $h: \pi_k^s(X) \to H_k(X;\mathbb{Z})$ is the stable Hurewicz map, then any element $\phi: \mathbb{S}^{N+k} \to \Sigma^N X$ in $\pi_k^s(X)$ of the preimage of the fundamental class $[X] \in H_k(X;\mathbb{Z})$ induces a degree one normal map given by the transverse inverse image $\phi^{-1}(X) \to X$.

Recall that the *semicharacteristic of M* from Section 1.4 is defined to be the sum of the mod 2 Betti numbers below the middle dimension of M. In other words, if M^{2k+1} is an odd-dimensional manifold, then

$$\chi_{1/2}(M^{2k+1}) = \sum_{i=0}^{k} \operatorname{rank}_{\mathbb{Z}_2} H_i(M;\mathbb{Z}_2) \mod 2.$$

Theorem 3.58. *(Kervaire [350]) A stably parallelizable manifold M^m is parallelizable iff*

1. $\dim M = 1, 3, \text{or } 7$;

2. *the Euler characteristic* $\chi(M) = 0$ *if the dimension of* M *is even,*
3. *the mod 2 semicharacteristic* $\chi_{1/2}(M) = 0 \bmod 2$ *if the dimension of* M *is odd.*

Browder's work in simply connected surgery theory was partly motivated by the following theorem, which shows that, in the case when a finite H-space M is simply connected, the structure set $\mathcal{S}^{PL}(M)$ is nonempty. Therefore if we arrange for X to be homotopy equivalent to a stably parallelizable manifold, one only needs to check that $\chi_{1/2}(M; \mathbb{F}_2) = 0 \bmod 2$. In fact, in many cases, the structure set $\mathcal{S}^{Diff}(M)$ is also nonempty.

Note that, if X is a Poincaré space and if its rationalization $X_{\mathbb{Q}}$ is defined, then $\operatorname{sig}(X) = \operatorname{sig}(X_{\mathbb{Q}})$.

Theorem 3.59. *(Browder [86]) Suppose that* X *is a simply connected finite* H*-space.*

1. *If* $\dim X \not\equiv 2 \bmod 4$*, then* X *is homotopy equivalent to a closed Diff manifold.*
2. *If* $\dim X \equiv 2 \bmod 4$*, then* X *is homotopy equivalent to a closed PL manifold.*

Proof. We sketch the general outline of the proof.

1. Low-dimensional examples are few in number and can be computed by hand. Recall that there is a degree one normal map $\phi\colon M \to X$ guaranteed by Proposition 3.56, where M is a stably parallelizable Diff manifold. If $k = \dim X$ is odd, then its surgery obstruction lies in the trivial group $L_k(\mathbb{Z}[e])$. If $k \equiv 0 \bmod 4$, the surgery obstruction is detected by the difference $\frac{1}{8}(\operatorname{sig}(X) - \operatorname{sig}(M))$ of signatures.

   We remind the reader that the Hirzebruch signature theorem (see Theorem 2.43) states that the signature of a Diff $4n$-manifold M can be written as
   $$\operatorname{sig}(M) = \langle L_n(p_1(M), \ldots, p_n(M)), [M]\rangle,$$
   where $L_n(p_1(M), \ldots, p_n(M))$ is the L-class on the Pontrjagin classes $p_1(M)$, $\ldots$, $p_n(M)$ of the tangent bundle. Therefore, if a Diff $4n$-manifold M is stably parallelizable, then it has signature zero.

   Since X is rationally a product of odd-dimensional spheres (an application of the Milnor-Moore theorem on Hopf algebras [460]), the signature of X is also zero. Therefore in all dimensions $n \not\equiv 2 \bmod 4$, we can perform surgery on f, producing a stably parallelizable manifold N that is homotopy equivalent to X. We will not continue to check parallelizability.

2. In dimension $n \equiv 2 \bmod 4$, we start with the degree one normal map $\phi\colon M \to X$ as before, where M is stably parallelizable. If the surgery obstruction in $L_n(\mathbb{Z}[e]) \cong \mathbb{Z}_2$ is zero, then we are done. Otherwise, we can consider the simply connected Kervaire manifold K^n and the map $\psi\colon K^n \to \mathbb{S}^n$ with nontrivial surgery obstruction. Then the manifold structure $f = \phi \# \psi\colon M \# K^n \to$

$X\#\mathbb{S}^n = X$ has zero surgery obstruction (cf. Remark 3.17). Since the Kervaire manifold is a PL manifold, we conclude that $\mathcal{S}^{PL}(X)$ is nonempty. □

At the writing of this text, the situation with Cat = Diff above remains unsolved for a finite H-space X with $\dim X \equiv 2 \bmod 4$, but there have been several results that solve the problem under more restricted conditions. Here is a statement that holds in all dimensions.

Theorem 3.60. *(Cappell-Weinberger [129, 686]) If the simply connected finite H-space Y is a product of Lie groups at the prime 2, then Y is homotopy equivalent to a closed parallelizable Diff manifold.*

Proof. This result follows from the Zabrodsky mixing and propagation idea. Note that $\mathbb{S}^7$ is the two-fold cover of $\mathbb{RP}^7$ and every Lie group is the two-fold cover of a closed manifold: merely mod out by $\mathbb{Z}_2$ in a maximal torus. The space Y is then a two-fold cover of a Poincaré complex X. In fact, the space $Y_{(2)}$ is the two-fold cover of $X_{(2)}$.

Since the homological action is trivial, it follows that the map $X_{(2)} \to X_{(0)}$ is a rational equivalence. Then the composition $Y_{(2)} \to X_{(2)} \to X_{(0)} \to Y_{(0)}$ is just localization. Let Z be the homotopy pullback:

$$\begin{array}{ccc} Z & \longrightarrow & Y[1/2] \\ \downarrow & & \downarrow \\ X_{(2)} & \longrightarrow & Y_{(0)} \end{array}$$

We observe that (1) Z is a finite Poincaré complex whose two-fold cover is homotopy equivalent to Y; (2) there is a map $Z \to BO$ lifting the Spivak normal fibration such that $Y \to Z \to BO$ is null-homotopic.

Consider a degree one normal map $f: M \to Z$ from a Diff manifold M associated to the lifting $Z \to BO$, and the two-fold cover $\widetilde{f}: \widetilde{M} \to Y$:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & Y \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & Z \end{array}$$

Now $\sigma(f)$ is a torsion element in $\widetilde{L}^h_{2k}(\mathbb{Z}[\mathbb{Z}_2])$, but the transfer map

$$\mathrm{tr}: \widetilde{L}^h_{2k}(\mathbb{Z}[\mathbb{Z}_2]) \to \widetilde{L}^h_{2k}(\mathbb{Z}[e])$$

is zero on torsion elements. The two-fold normal invariant map $\widetilde{f}: \widetilde{M} \to W \simeq Y$ then has surgery obstruction $\sigma(\widetilde{f}) = \mathrm{tr}\,\sigma(f) = 0$, as required.

We proceed as in the proof of Theorem 3.59 to see that the surgery obstruction is zero, so that surgery on $\widetilde{f}$ yields a homotopy equivalence $g : N \to Y$ from a closed parallelizable manifold N, as required. $\square$

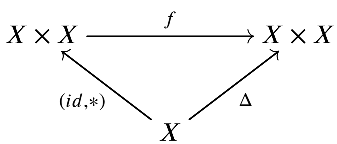
Lemma 3.61. *Let X be a finite H-space with infinite fundamental group. Then X is homotopic to a space of the form $Y \times \mathbb{S}^1$.*

Proof. Suppose that X is an H-space with infinite fundamental group. The group is clearly abelian and has an infinite cyclic subgroup $H = \langle g \rangle$. Consider the action g by translations on the $\mathbb{Z}$-cover $X_{\mathbb{Z}}$ of X. Then X is homotopy equivalent to the mapping torus M_{ϕ_g} of ϕ_g. By Browder [78] any cover of an H-space is also an H-space, so $X_{\mathbb{Z}}$ is an H-space. The covering translation $\phi_g : X_{\mathbb{Z}} \to X_{\mathbb{Z}}$ is homotopic to the identity map on $X_{\mathbb{Z}}$, since the map is translation by g, and g can be connected to the identity. The mapping torus of the identity map $id : X_{\mathbb{Z}} \to \mathbb{Z}$ is $M_{id} = X_{\mathbb{Z}} \times \mathbb{S}^1$, so X is homotopy equivalent to $X_{\mathbb{Z}} \times \mathbb{S}^1$. $\square$

Remark 3.62. *All of our spaces are finitely dominated, so the fundamental groups are always finitely generated. As a result, all simply connected results imply the non-simply connected case if the fundamental group is torsion-free. We also see that, if the fundamental group is infinite, then the Wall finiteness obstruction vanishes. The result above, in the loopspace situation, directly deals with all of these K- and L-theoretic obstructions, even for finite groups.*

Theorem 3.63. *(Weinberger [686]) Let P be a finite set of primes. If X is a finitely dominated P-local H-space, i.e. $X_{(P)}$ is finitely dominated, then X has the P-local homotopy type of a closed Top manifold. In other words, there is a closed Top manifold M such that $\pi_i(M) \otimes \mathbb{Z}_{(P)} \cong \pi_i(X) \otimes \mathbb{Z}_{(P)}$ for all i.*

Proof. (Sketch) First suppose that $\pi_1(X)$ is finite. The Wall obstruction vanishes because X, when mixed with a product of spheres away from P, is a localization of a finite dominated complex, and because the map $\widetilde{K}_0(\mathbb{Z}[\pi]) \to \widetilde{K}_0(\mathbb{Z}_{(P)}[\pi])$ vanishes. Special low-dimensional considerations deal with the case $n \leq 4$. We first show that X has a normal invariant, which is done exactly as in the Browder-Spanier Theorem 3.59. If $\mu : X \times X \to X$ is the multiplication map, then consider the map $f : X \times X \to X \times X$ given by $f(x, x') = (x, \mu(x, x'))$. The induced map $f_* : \pi_n(X \times X) \to \pi_n(X \times X)$ is given by matrix multiplication by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and is therefore an isomorphism. By the Whitehead theorem, we know that f is a homotopy equivalence, so f preserves Spivak bundles. Denote by E_X the Spivak bundle over X and by $\Delta : X \times X \to X$ the diagonal map. The commutative diagram



shows that $E_X = \Delta^*(E_{X\times X}) = E_X \oplus E_X$; i.e. the Spivak bundle is trivial. Hence X has a normal invariant. Since X is rationally a product of spheres, we have $\sigma(f) = 0$. If $\pi_1(X)$ is infinite, a straightforward argument shows that one can split off a local torus; i.e. there is a homotopy equivalence $X \simeq X' \times \mathbb{T}^k_{(P)}$, where X' is the cover corresponding to ker $(\pi_1(X) \to H_1(X; \mathbb{Q}))$. Since X' is a local manifold, then so is X. □

For quite some time, all of the known finite H-spaces were standard at the prime 2, i.e. a product of Lie groups and spheres. Dwyer and Wilkerson (see [208] and [209]) advanced the theory on the homotopy-theoretic side when they gave an example of a space $X = BDI(4)$ which is exotic at the prime 2. In particular, this space has the special property that it admits a classifying space; i.e. there is a homotopy equivalence $X \simeq \Omega BX$. The loop structure endows it with the structure of an H-space. Finite loopspaces which were exotic at odd primes were previously known. The Dwyer-Wilkerson example fits into a general theory of finite loopspaces that often have the features of compact Lie groups, such as maximal tori and Weyl groups. See Grodal [268].

We will close this section with the theorem by Bauer, Kitchloo, Notbohm, and Pedersen. The π-π theorem is the only surgical tool required. We however only give a brief hint of the proof, as the homotopical foundations are much more serious. We recommend the original paper for all the details.

Theorem 3.64. *(Bauer, Kitchloo, Notbohm, and Pedersen [49]) If X is a finite loopspace, then X is homotopy equivalent to a closed parallelizable manifold.*

The surgery part of the proof is similar to the method in Theorem 3.60. Rather than making X a two-fold cover, it is shown that X is a circle bundle over another finite Poincaré complex, such that the circle is nullhomotopic, i.e. so that the projection map makes $X \to X/\mathbb{S}^1$ into a π-π Poincaré pair. It also kills the Wall finiteness obstruction. This process is the analogue of modding out by the left action of a circle in the maximal torus of a Lie group. More precisely, if G is compact, and not a torus, there is a circle in its maximal torus that is trivial in the fundamental group of G. We sketch the proof of the theorem.

Proof. We offer an outline of the proof.

1. Let $X = \Omega Y$ be a finite connected loopspace. In other words, suppose that $H_*(X; \mathbb{Z}) = \bigoplus_i H_i(X; \mathbb{Z})$ is a finitely generated abelian group. Since $H^2(Y; \mathbb{Z})$ is free abelian, there is a map $Y \to K(\mathbb{Z}^2, 2) = B\mathbb{T}^2$ inducing an isomorphism on $H^2(\,\cdot\,; \mathbb{Z})$. If Y' is the homotopy fiber, we have a fibration $\Omega Y' \to X \to \mathbb{T}^r$, so that X is homotopy equivalent to $\Omega Y' \times \mathbb{T}^r$. Since $\mathbb{T}^r$ is a smooth parallelizable manifold, we can assume that X has finite fundamental group.
2. Clark's theorem [165] gives $H^3(X; \mathbb{Q}) \neq 0$. Assume that X has finite fundamental group. The only instances of quasifinite loopspaces of dimension at most 4 are rational homology 3-spheres. Therefore X is homotopy equivalent to $\mathbb{S}^3$

or $SO(3)$, which are smooth parallelizable manifolds. We can then assume that the dimension of X is at least 5.

3. Recall that the Euler characteristic is defined by considering the chains of the universal cover of X as a $\mathbb{Z}[\pi]$-module chain complex, which is chain homotopy equivalent to a finite-length chain complex of finitely generated projective $\mathbb{Z}[\pi]$-modules. The Euler characteristic can then be placed in $\widetilde{K}_0(\mathbb{Z}[\pi])$. If it vanishes, then X is the homotopy type of a finite complex. The authors prove that there is an orientable fibration $\mathbb{S}^1 \to X \to Y$ of finite simple spaces that implies the vanishing of the Wall obstruction in a way analogous to product formulas. (Note: A space X is *nilpotent* if X is homotopy equivalent to a CW complex and if $\pi_1(X)$ acts nilpotently on the homotopy group $\pi_*(X)$. If this action is trivial on higher homotopy, then X is called *simple*.)
4. Surgery obstructions are managed by the construction of double 1-tori or special 1-tori. Both constructions rely on the theory of p-compact groups and on arithmetic square arguments. Again the idea is to find a normal invariant for the Poincaré pair $(X/\mathbb{S}^1, X)$ and execute surgery using the π-π theorem. □

3.6 PL TORI

The Mostow rigidity theorem states that a homotopy equivalence between closed hyperbolic manifolds of dimension at least 3 is homotopic to an isometry, in particular, to a homeomorphism. For flat manifolds, rigidity is not as strong, but one still has that the fundamental group determines the homeomorphism type. In a 1953 letter to Serre, Borel asked the question whether two aspherical manifolds with isomorphic fundamental groups are homeomorphic. The Borel conjecture, stated below, is a topological analogue of Mostow rigidity, weakening the hypothesis from hyperbolic manifolds to aspherical manifolds, and similarly weakening the conclusion from an isometry to a homeomorphism.

Definition 3.65. *A topological space M is called aspherical if the universal cover $\widetilde{M}$ is contractible; i.e. M is an Eilenberg Maclane space of the form $K(\pi, 1)$, where $\pi = \pi_1(M)$.*

Conjecture 3.66. *(Borel) Let M and N be closed aspherical n-manifolds. If M is homotopy equivalent to N, then M is Top homeomorphic to N. In other words, the manifold M is topologically rigid in the sense of Definition 1.27.*

Proposition 3.67. *(Milnor [452]) In dimension 3, the Borel conjecture implies the Poincaré conjecture.*

Proof. If Σ^3 is a homotopy 3-sphere, then the Borel conjecture implies that the manifolds $\mathbb{T}^3\#\Sigma^3$ and $\mathbb{T}^3\#\mathbb{S}^3$ are Top homeomorphic. According to Bing and Moïse, three-dimensional manifolds are triangulable. Lifting to the universal cover, one could obtain a PL 2-sphere in $\mathbb{R}^3$ which does not bound a ball, contradicting a classical theorem of Alexander. □

The Borel conjecture and related themes will be highlighted in Chapter 5.

In this section we will follow Hsiang-Shaneson [321] and Wall [669] and calculate $\mathcal{S}^{PL}(\mathbb{T}^n)$. We will prove that $\mathcal{S}^{Top}(\mathbb{T}^n) = 0$, so that every homotopy torus of dimension at least 5 is homeomorphic to the n-torus. This PL calculation is of enormous significance for a number of reasons, in particular as the final missing ingredient of Kirby's proof of the annulus conjecture [358]. The topological result is the first affirmation of the Borel conjecture above dimension 2.

Our main tools will be the analysis of F/PL and the calculation of the L-groups $L_*(\mathbb{Z}[\mathbb{Z}^n])$. We will see that the surgery obstruction map

$$[\mathbb{T}^n \times \mathbb{D}^k \text{ rel } \partial : F/PL] \to L_{n+k}(\mathbb{Z}[\mathbb{Z}^k])$$

is injective, but not necessarily surjective. The cokernel for $k = 1$ gives $\mathcal{S}^{PL}(\mathbb{T}^n) \cong \Lambda^3(\mathbb{Z}^n) \otimes \mathbb{F}_2$.

In the next section, we will see how this discussion leads directly to the Kirby-Siebenmann triangulation obstruction.

3.6.1 Torus calculations

Let M be a compact n-dimensional Cat manifold without boundary. If $\pi = \pi_1(M)$, then we have the surgery exact sequence given as follows:

$$\cdots \to \mathcal{S}^{Cat}(M \times \mathbb{D}^{r+1})_{\text{rel}} \xrightarrow{\eta_{M\times\mathbb{D}^{r+1}}} \mathcal{N}^{Cat}(M \times \mathbb{D}^{r+1})_{\text{rel}} \xrightarrow{\sigma_{M\times\mathbb{D}^{r+1}}} L_{n+r}(\mathbb{Z}[\pi])$$

$$\to \cdots \to L_{n+1}(\mathbb{Z}[\pi]) \to \mathcal{S}^{Cat}(M) \xrightarrow{\eta_M} \mathcal{N}^{Cat}(M) \xrightarrow{\sigma_M} L_n(\mathbb{Z}[\pi]).$$

We are principally interested in computing $\mathcal{S}^{PL}(\mathbb{T}^n \times \mathbb{D}^r)_{\text{rel}}$ for $r \geq 0$ and $n+r \geq 5$. For simplicity we will refer to the pair $(\mathbb{T}^n \times \mathbb{D}^r \text{ rel } \mathbb{T}^n \times \mathbb{S}^{r-1})$ as $\mathbb{T}^{n,r}$ with $\mathbb{T}^{n,0} \equiv \mathbb{T}^n$ and the map $\eta_{\mathbb{T}^{n,r}}$ as $\eta_{n,r} : \mathcal{S}^{PL}(\mathbb{T}^{n,r})_{\text{rel}} \to \mathcal{N}^{Cat}(\mathbb{T}^{n,r})_{\text{rel}}$, with the relative data implied. Similarly we will denote the relevant surgery map by $\sigma_{n,r}$.

Lemma 3.68. *The suspension $\Sigma\mathbb{T}^n$ of the n-torus has the homotopy type of a wedge of spheres.*

Proof. The $n = 1$ case is obvious. In the $n = 2$ case, we first embed $\mathbb{T}^2$ into $\mathbb{S}^3$. The Thom collapse map gives a map $\mathbb{S}^3 \to \Sigma\mathbb{T}^2$, inducing an isomorphism $H^3(\Sigma\mathbb{T}^2; \mathbb{Z}) \to H^3(\mathbb{S}^3; \mathbb{Z})$. There are two embeddings $\mathbb{S}^1 \to \mathbb{T}^2$ given by the generating circles, inducing maps $\mathbb{S}^2 \vee \mathbb{S}^2 = \Sigma\mathbb{S}^1 \vee \Sigma\mathbb{S}^1 \to \Sigma\mathbb{T}^2$. The map $\mathbb{S}^3 \vee \mathbb{S}^2 \vee \mathbb{S}^2 \to \Sigma\mathbb{T}^2$ is a homology equivalence between simply connected spaces, so is a homotopy equivalence. The rest follows by induction. □

Proposition 3.69. *Suppose that $n + r \geq 5$. Then the map $\eta_{n,r} : \mathcal{S}^{PL}(\mathbb{T}^{n,r})_{\text{rel}} \to [\mathbb{T}^{n,r} : F/PL]_{\text{rel}}$ is the zero map.*

Proof. It suffices to show that the surgery map $\sigma_{n,r} : [\mathbb{T}^{n,r} : F/PL] \to L_{n+r}(\mathbb{Z}[\mathbb{Z}^n])$ is injective. Now F/PL is a loop space, so there is a space Y for which $F/PL = \Omega Y$. Therefore $[\mathbb{T}^{n,r} : F/PL] = [\Sigma^r \mathbb{T}^n : \Omega Y] = [\Sigma^{r+1}\mathbb{T}^n : Y]$, where the first equality holds because the boundary is sent to a point. Since $\Sigma^{r+1}\mathbb{T}^n$ has the homotopy type of a wedge of spheres, we have $[\Sigma^{r+1}\mathbb{T}^n : Y] = \left[\bigvee_i \binom{n}{i}\mathbb{S}^{i+r+1} : Y\right] = \bigoplus_i \binom{n}{i}\pi_{i+r+1}(Y) = \bigoplus_i \binom{n}{i}\pi_{i+r}(F/PL)$. But we also have $L_{n+r}(\mathbb{Z}[\mathbb{Z}^n]) = \bigoplus_i \binom{n}{i} L_{i+r}(\mathbb{Z}[e])$ by Corollary 2.107. In addition we have the following diagram:

$$\begin{array}{ccc}
[\Sigma^r\mathbb{T}^n : F/PL] & \xrightarrow{\sigma_{n,r}} & L_{n+r}(\mathbb{Z}[\mathbb{Z}^n]) \\
\downarrow \cong & & \downarrow \cong \\
\bigoplus_i \binom{n}{i}\pi_{i+r}(F/PL) & \xrightarrow[\oplus\sigma_{\mathbb{S}^{i+r}}]{} & \bigoplus_i \binom{n}{i} L_{i+r}(\mathbb{Z}[e])
\end{array}$$

Commutativity follows directly from the method of calculation of $L_*(\mathbb{Z}[\mathbb{Z}^n])$. But every $\sigma_{\mathbb{S}^{i+r}}$ is injective (in fact, bijective except for $n = 4$), so the map $\sigma_{n,r}$ is injective, as required. □

Corollary 3.70. *Every PL manifold M^n of dimension $n \geq 5$ that is homotopy equivalent to the torus $\mathbb{T}^n$ is (a) stably parallelizable and (b) smoothable.*

Proof. Let M be a PL manifold of dimension at least 5.

1. The composite $\mathcal{S}^{PL}(\mathbb{T}^n) \to [\mathbb{T}^n : F/PL] \to [\mathbb{T}^n : BPL]$ carries a manifold structure $M^n \to \mathbb{T}^n$ to the classifying map of the stable normal bundle of M^n, viewed as a PL lifting of the Spivak normal fibration. However, this composite is trivial. Since all the stable normal bundles are the same as for the ordinary torus, they are therefore stably trivial. So M is stably parallelizable by definition.
2. The map $\partial : L_{n+1}(\mathbb{Z}[\mathbb{Z}^n]) \to \mathcal{S}^{PL}(\mathbb{T}^n)$ is surjective and factors through $\mathcal{S}^{Diff}(\mathbb{T}^n)$. Therefore the surgery obstructions have smooth realizations. □

Remark 3.71. *In fact, by smoothing theory, (1) implies (2), but the direct proof above seems better.*

Corollary 3.72. *If $n+r \geq 5$ and $\mathrm{Stab}(\mathrm{id}_{\mathbb{T}^{n,r}})$ denotes the stabilizer of the identity map $\mathrm{id} : \mathbb{T}^{n,r} \to \mathbb{T}^{n,r}$ in the action of $L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$ on $\mathcal{S}^{PL}(\mathbb{T}^{n,r})_{\mathrm{rel}}$, then*

$$\mathcal{S}^{PL}(\mathbb{T}^{n,r})_{\mathrm{rel}} \cong L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])/\mathrm{Stab}(\mathrm{id}_{\mathbb{T}^{n,r}}).$$

To complete the analysis of PL homotopy tori, we continue the surgery exact sequence to the left:

$$\cdots \to [\Sigma^{k+1}\mathbb{T}^n : F/PL] \to L_{n+k+1}(\mathbb{Z}[\mathbb{Z}^n]) \to \mathcal{S}^{PL}(\mathbb{T}^{n,k}).$$

We have seen that $L_{n+k+1}(\mathbb{Z}[\mathbb{Z}^n]) = \bigoplus L_{n+k+1-\#S}(\mathbb{Z}[e])$ where S ranges over the subsets of $\{1, 2, \dots, n\}$. Similarly, we have

$$[\Sigma^{k+1}\mathbb{T}^n : F/PL] = \bigoplus \pi_{n+k+1-\#S}(F/PL).$$

These isomorphisms are proved inductively on n in exactly the same way (see for example Hsiang-Shaneson [322] for details). As a result, the surgery obstruction map is given by the commutative diagram:

$$\begin{array}{ccc} [\Sigma^{k+1}\mathbb{T}^n : F/PL] & \longrightarrow & L_{n+k+1}(\mathbb{Z}[\mathbb{Z}^n]) \\ \downarrow & & \downarrow \\ \oplus\pi_{n+k+1-\#S}(F/PL) & \longrightarrow & \oplus L_{n+k+1-\#S}(\mathbb{Z}[e]) \end{array}$$

The left-hand side is more canonically given by $\oplus H_j(\Sigma^{k+1}\mathbb{T}^n; \pi(F/PL))$. However, the map

$$\bigoplus \pi_{n+k+1-\#S}(F/PL) \to \bigoplus L_{n+k+1-\#S}(\mathbb{Z}[e])$$

is an isomorphism except when the subset S satisfies $n+k+1-\#S = 4$. In these cases, the map is multiplication by 2. Therefore we obtain the following.

Theorem 3.73. *(Hsiang-Shaneson [322], Wall [669]) The PL structure set of $\mathbb{T}^{n,k}$ is given by $\mathcal{S}^{PL}(\mathbb{T}^{n,k}) = \bigoplus \mathbb{Z}_2$, where the sum is taken over subsets of $\{1, 2, \dots, n\}$ of cardinality $n+k-3$. More canonically, we have*

$$\mathcal{S}^{PL}(\mathbb{T}^{n,k}) = H_3(\Sigma^k\mathbb{T}^n; \mathbb{Z}_2) = \Lambda^{3-k}(\mathbb{Z}[\mathbb{Z}^n]) \otimes \mathbb{Z}_2.$$

Remark 3.74. *It is worthwhile to understand the construction of fake tori.*[1] *From the manifold $\mathbb{T}^a \times \mathbb{D}^{3-a}$ one typically takes a Wall realization of the generator $1 \in L_4(\mathbb{Z}[e])$ and crosses it with a dual torus. However, the action by 1 on $\mathcal{S}^{PL}(\mathbb{T}^a \times \mathbb{D}^{3-a})$ cannot be trivial. If it were, one would glue the ends of the realization together and obtain a*

[1] Wall and others used the word "fake" to describe tori, projective space, and lens spaces that are homotopy equivalent to but not Cat isomorphic to their more standard counterpart. The word "exotic" seems to be reserved for spheres. In this book, we will mostly use the phrase "homotopy torus" or "homotopy lens space" as an alternative.

closed spin 4-manifold of signature 8, contradicting Rokhlin's Theorem. Note that, if instead one chose the element $2 \in L_4(\mathbb{Z}[e])$, *then the action of this element would be indeed trivial, because we could connect sum with some number of Kummer surfaces to obtain a different PL normal cobordism with vanishing surgery obstruction.*

Remark 3.75. *Freedman showed that all homology 3-spheres bound contractible manifolds. With this result, Freedman constructed an almost parallelizable simply connected 4-manifold of index 8 in the Top category. The construction is similar to that for Milnor manifolds, but instead of coning off the homology spherical boundary, one glues on a contractible topological manifold that bounds it. The Freedman* E_8*-manifold can be used as a substitute for the Kummer surface in the above discussion, yielding the result that all of these PL manifolds are topologically homeomorphic. In the next section we will explain a more precise result that* $\pi_i(F/Top) \to L_i(\mathbb{Z}[e])$ *is an isomorphism for all* i*. Then repetition of the calculation for* $\mathcal{S}^{PL}(\mathbb{T}^n)$ *gives* $\mathcal{S}^{Top}(\mathbb{T}^n) = 0$.

We now return to provide the details of the argument explicitly, essentially redoing the proof of the Shaneson formula in this context. To understand the right side of the previous correspondence, it is important to find representatives of $L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$ and then determine the elements which stabilize the identity manifold structure in $\mathcal{S}^{PL}(\mathbb{T}^{n,r})$. In the following constructions we will be relying heavily on the splitting theorem 1.77 and the Wall realization theorem 1.41.

For the rest of the section, we provide details of the proof for the classification. It is the most notation-heavy part of this book, and perhaps should not be scrutinized too closely on a first reading.

Construction 3.76. *If* $J \subseteq \{1, \dots, n\} = R$, *let* $|J|$ *denote its cardinality and* J^c *its complement. Let* $T(J)$ *denote the corresponding* $|J|$*-dimensional subtorus of* $\mathbb{T}^n$*. For all* J *with* $|J| + r$ *odd, we will construct an element* $\xi(J) \in L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$.

Denote by $I_s = I^{|J|+r+s}$ *and* ε_s *the trivial bundle* $\varepsilon^{|J|+r+s}$*. If* $|J| + r \geq 5$ *and* ζ *is a generator of* $L_{|J|+r+1}(\mathbb{Z}[e])$, *we consider the identity map* $id : I_1 \to I_1$ *as an element of* $\mathcal{S}^{PL}(I_1)_{\text{rel}}$ *and use the Wall realization theorem to choose a degree one PL normal map* $F : N \to I_1 \times I$ *with boundary maps* $id : I_1 \to I_1$ *and* $h : M \to I_1$ *such that* $\sigma_{I_1}(N, F) = \zeta$*. Let* $P_J = T(J) \times I^r \times I = T(J) \times \mathbb{D}^r \times I$*, and construct* K_J *by taking the connected sum* $P_J \# M$ *along* $T(J) \times I^r \times \{1\}$*. We then have a degree one PL normal map* $f_J : K_J \to P_J$ *of* $(|J| + r + 1)$*-manifolds given by*

$$K_J = P_J \# M \xrightarrow{f_J = id \# h} P_J \# I_1 = P_J.$$

Then $\sigma_{P_J}(P_J \# M, f_J) = i_*(\sigma_{I_1}(M, h))$ *in* $L_{|J|+r+1}(\mathbb{Z}[\mathbb{Z}^{|J|}])$, *where*

$$i_* : L_{|J|+r+1}(\mathbb{Z}[e]) \to L_{|J|+r+1}(\mathbb{Z}[\mathbb{Z}^{|J|}])$$

is induced by the inclusion map $i : \{e\} \hookrightarrow \mathbb{Z}^{|J|}$*. Consider now the map* $g_J = f_J \times id : K_J \times T(J^c) \to P_J \times T(J^c) = \mathbb{T}^n \times \mathbb{D}^r \times I = \mathbb{T}^{n,r} \times I$ *of* $(n + r + 1)$*-manifolds,* *which is also a degree one PL normal map. Define*

$$\xi(J) = \sigma_{\mathbb{T}^{n,r}\times I}(g_J) \in L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n]).$$

If $|J|+r = 3$*, consider the generator* $1 \in L_8(\mathbb{Z}[e]) \cong \mathbb{Z}$ *and the identity map* $id\colon \mathbb{D}^8 \to \mathbb{D}^8$*. The Wall realization theorem gives a degree one PL normal map* $h\colon M \to \mathbb{D}^8 \times I$ *over id with* $\sigma_{\mathbb{D}^8\times I}(M,h) = 1$*. Let* $P_J = T(J) \times \mathbb{D}^r \times \mathbb{CP}^2 \times I$ *and* $K_J = P_J \# M$*, equipped with a degree one PL normal map* $f_J = id\#h\colon K_J \to P_J$ *of 8-dimensional manifolds. Consider now the degree one PL normal map given by* $g_J = f_J \times id\colon K_J \times T(J^c) \to P_J \times T(J^c) = \mathbb{T}^{n,r} \times \mathbb{CP}^2 \times I$ *of* $(n+r+5)$*-dimensional manifolds. Define* $\xi(J) = \sigma_{\mathbb{T}^{n,r}\times\mathbb{CP}^2\times I}(K_J \times T(J^c), g_J) \in L_{n+r+5}(\mathbb{Z}[\mathbb{Z}^n]) \cong L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$.

If $|J| + r = 1$*, consider the generator* $1 \in L_2(\mathbb{Z}[e]) \cong \mathbb{Z}_2$*. Let* $h\colon \mathbb{T}^2 \to \mathbb{S}^2$ *be a degree one normal map with Arf invariant 1. Define* $P_J = T(J) \times \mathbb{D}^r \times I$ *and define* $K_J = P_J \# \mathbb{T}^2$*. There is then a degree one PL normal map* $f_J = id\#h\colon K_J \to P_J \# \mathbb{S}^2 = P_J$ *of 2-manifolds framed in the obvious way. Then* $\sigma_{P_J}(P_J\#\mathbb{T}^2, f_J) = i_*(\sigma_{\mathbb{S}^2}(\mathbb{T}^2,h))$ *in* $L_2(\mathbb{Z}[\mathbb{Z}^{|J|}])$*, where* $i_*\colon L_2(\mathbb{Z}[e]) \to L_2(\mathbb{Z}[\mathbb{Z}^{|J|}])$ *is induced by inclusion* $i\colon \{e\} \hookrightarrow \mathbb{Z}^{|J|}$*. Consider now the degree one PL normal map given by* $g_J = f_J \times id\colon K_J \times T(J^c) \to P_J \times T(J^c) = \mathbb{T}^{n,r} \times I$ *of* $(n+r+1)$*-dimensional manifolds. Define* $\xi(J) = \sigma_{\mathbb{T}^{n,r}\times I}(K_J \times T(J^c), g_J) \in L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$.

Construction 3.77. *Suppose that* $J \subseteq H \subseteq \{1,\dots,n\} = R$ *with* $|J| + r \equiv 1 \bmod 2$ *with* $n \geq 5$*. Define* $\xi(H,J) \in L_{|H|+r+1}(\mathbb{Z}[\mathbb{Z}^{|H|}])$ *as before except that we cross with* $T(H\backslash J)$ *instead of with* $T(J^c) = T(R\backslash J)$*. Note that* $\xi(J) \equiv \xi(R,J)$ *and* $\xi(J,J)$ *is the chosen generator of* $L_{|J|+r+1}(\mathbb{Z}[\mathbb{Z}^{|J|}])$.

Construction 3.78. *Suppose that* $|J| = |H| - 1$*. For* $|J| \geq 6$ *we will now define a map*

$$\alpha(J,H)\colon L_{|H|+r+1}(\mathbb{Z}[\mathbb{Z}^{|H|}]) \to L_{|J|+r+1}(\mathbb{Z}[\mathbb{Z}^{|J|}]).$$

Let $\zeta \in L_{|H|+r+5}(\mathbb{Z}[\mathbb{Z}^{|H|}])$*. By the Wall realization theorem, there is a degree one PL normal map* $f\colon N \to T(H) \times \mathbb{D}^r \times \mathbb{CP}^2 \times I$ *such that* $\sigma_{T(H)\times\mathbb{D}^r\times\mathbb{CP}^2\times I}(f) = \zeta$*. By the Splitting Theorem, we may assume that* $f|_{\partial N}$ *is transverse to both* $T(H)\times\mathbb{D}^r\times\mathbb{CP}^2\times\{0\}$ *and* $T(H)\times\mathbb{D}^r\times\mathbb{CP}^2\times\{1\}$*. Then we have a degree one PL normal map* $h_J\colon N \times \mathbb{S}^1 \to T(J) \times \mathbb{D}^r \times \mathbb{CP}^2 \times I$*. Define* $\alpha(J,H)\colon L_{|H|+r+5}(\mathbb{Z}[\mathbb{Z}^{|H|}]) \to L_{|J|+r+5}(\mathbb{Z}[\mathbb{Z}^{|H|}])$ *by* $\alpha(J,H)(\zeta) = \sigma_{T(J)\times\mathbb{D}^r\times\mathbb{CP}^2\times I}(h_J)$*. Then we use periodicity to arrive at a map*

$$\alpha(J,H)\colon L_{|H|+r+1}(\mathbb{Z}[\mathbb{Z}^{|H|}]) \to L_{|J|+r+1}(\mathbb{Z}[\mathbb{Z}^{|J|}]).$$

If $J \subset H$ *but* $|H| - |J| > 1$*, then we can define* $\alpha(J,H)\colon L_{|H|+r+1}(\mathbb{Z}[\mathbb{Z}^{|H|}]) \to L_{|J|+r+1}(\mathbb{Z}[\mathbb{Z}^{|J|}])$ *by composing. Construct the unique sequence* $J = J_0 \subseteq \cdots \subseteq J_s = H$ *such that* $|J_{i+1}| = |J_i| + 1$ *and* $\max\{J_i\backslash J\} < \max\{J_{i+1}\backslash J\}$ *for all* ℓ*. Let* $\alpha(J,H) = \alpha(J_0,J_1)\circ\cdots\circ\alpha(J_{s-1},J_s)$*, with the convention that* $\alpha(J,J)$ *is the identity and* $\alpha(H) \equiv \alpha(H,R)$.

If $|J| \leq 5$ *then consider* $\alpha(T(J) \times \mathbb{CP}^2)$ *and appeal to periodicity.*

The following two results follow directly from the definitions.

Lemma 3.79. *Let* $J \subseteq K \subseteq H$ *with* $|J| + r \equiv 1 \bmod 2$. *Then* $\alpha(K,H)\xi(H,J) = \xi(K,J) \in L_{|K|+r+1}(\mathbb{Z}[\mathbb{Z}^{|K|}])$.

Proof. Recall that, if ζ is a generator of $L_{|J|+r+1}(\mathbb{Z}[e])$, then there is $h: M \to I_1$ such that $\sigma_{I_1}(h) = \zeta$. If we define

$$g_J : ((T(J) \times \mathbb{D}^r \times I)\#M) \times T(J^c) \to T(J) \times \mathbb{D}^r \times I \times T(J^c) = \mathbb{T}^{n,r} \times I$$

and also define $\xi(J) = \sigma_{\mathbb{T}^{n,r}\times I}(g_J) \in L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$. If $J \subseteq H$, then we define

$$g_{J,H} : ((T(J) \times \mathbb{D}^r \times I)\#M) \times T(H\backslash J) \to T(J) \times \mathbb{D}^r \times I \times T(H\backslash J) = \mathbb{T}^{|H|,r} \times I,$$

then we define $\xi(H,J) = \sigma_{\mathbb{T}^{|H|,r}\times I}(g_{J,H}) \in L_{|H|+r+1}(\mathbb{Z}[\mathbb{Z}^{|H|}])$. If $|H| = |J| + 1$ and ζ is a generator of $L_{|H|+r+1}(\mathbb{Z}[\mathbb{Z}^{|H|}]$, then there is $f : N \to T(H) \times \mathbb{D}^r \times I$ such that $\sigma_{T(H)\times\mathbb{D}^r\times I}(f) = \zeta$. Construct $h: N \times \mathbb{S}^1 \to T(J) \times \mathbb{D}^r \times I$ and define

$$\alpha(J,H)\zeta = \alpha(J,H)\sigma_{T(H)\times\mathbb{D}^r\times I}(f) = \sigma_{T(J)\times\mathbb{D}^r\times I}(h) \in L_{|J|+r+1}(\mathbb{Z}[\mathbb{Z}^{|J|}]).$$

So $J \subseteq K \subseteq H$ yields

$$\begin{aligned}\alpha(K,H)\xi(H,J) &= \alpha(K,H)\sigma_{\mathbb{T}^{|H|,r}\times I}(h_J)\\ &= \sigma_{T(J)\times\mathbb{D}^r\times I}(h_{J,H} \times id_{T(H\backslash K)})\\ &= \sigma_{T(J)\times\mathbb{D}^r\times I}(h_{J,K})\\ &= \xi(K,J).\end{aligned}$$

□

Lemma 3.80. *Let* $J, K \subseteq H$ *but* $J \not\subseteq K$ *with* $|J|+r \equiv 1 \bmod 2$. *Then* $\alpha(K,H)\xi(H,J)$ *vanishes in* $L_{|K|+r+1}(\mathbb{Z}[\mathbb{Z}^{|K|}])$.

Proof. Construct the unique sequence $K = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_k = H$ such that $|K_{\ell+1}| = |K_\ell| + 1$ and $\max\{K_\ell\backslash K\} < \max\{K_{\ell+1}\backslash K\}$. Let i be the smallest integer such that $J \subseteq K_{i+1}$. Notice that

$$\begin{aligned}\alpha(K,H) &= \alpha(K,K_i)\circ\alpha(K_i,K_{i+1})\circ\alpha(K_{i+1},H)\xi(H,J)\\ &= \alpha(K,H) = \alpha(K,K_i)\circ\alpha(K_i,K_{i+1})\xi(K_{i+1},J)\end{aligned}$$

by the previous lemma. Let $\{j_0\} = K_{i+1}\backslash K_i$. Let us relabel $K = K_i$ and $H = K_{i+1}$. It suffices then to show that $\alpha(K,H)\xi(H,J) = 0$.

Let $J' = J\backslash\{j_0\}$. We briefly recall the definition of $\xi(H,J)$. Choose a degree one PL normal map $f : M \to I_5$ such that $\sigma(M,f)$ is the chosen generator of $L_{|J|+r+5}(\mathbb{Z}[e])$.

Then there is an induced map

$$h\colon\ A = ((T(J)\times \mathbb{D}^r \times \mathbb{CP}^2 \times I)\# M)\times T(H\backslash J) \to \mathbb{T}^{|H|}\times \mathbb{D}^r \times \mathbb{CP}^2 \times I,$$

and we define $\xi(H,J) = \sigma_{T^{|H|}\times\mathbb{D}^r\times I}(A,h)$ in $L_{|H|+r+1}(\mathbb{Z}[\mathbb{Z}^{|H|}])$. The boundary connected sum with M can be taken along a disk that misses $B = T(J')\times\mathbb{D}^r\times I\times\mathbb{CP}^2$. The Splitting Theorem states that h can be made transverse to B such that $h^{-1}(B) = B$ and $h|_B = id_B$. In particular, the map h restricts to a PL homeomorphism on $T(J')\times\mathbb{D}^n\times\mathbb{CP}^2\times I\times T(H\backslash J)$. Notice that $J'\cap(H\backslash K) = (J\backslash\{j_0\})\cap\{j_0\}$ is empty. The surgery obstruction of such a restriction is precisely $\alpha(K,H)\xi(H,J)$ by the definition of $\alpha(K,H)$. Hence $\alpha(K,L)\xi(H,J) = 0$. □

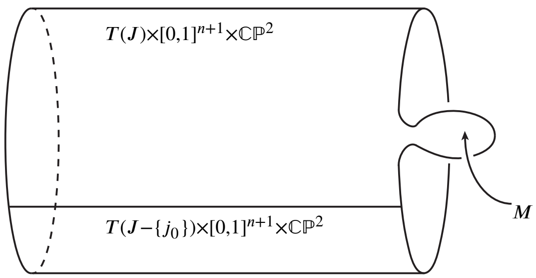


Figure 3.2: Attaching M to a cobordism

Definition 3.81. *For any $Q\subseteq R$, let*

$$w(Q)\colon\ L_{|Q|+r+1}(\mathbb{Z}[\mathbb{Z}^{|Q|}]) \to L_{|Q|+r+1}(\mathbb{Z}[e])$$

be the natural projection. For $J\subseteq H$, we define

$$\delta(H,J) = \begin{cases} 0 & \text{if } H\neq J,\\ 0 & \text{if } H = J \text{ and } |J|+r\equiv 0 \mod 2,\\ 1\in\mathbb{Z}_2 & \text{if } H = J \text{ and } |J|+r\equiv 1 \mod 4,\\ 1\in\mathbb{Z} & \text{if } H = J \text{ and } |J|+r\equiv 3 \mod 4.\end{cases}$$

For the following proposition, recall that, if $f\colon M^n\to X$ is a PL normal invariant from a Cat manifold M^n to a Poincaré complex X, and if N is the boundary of a PL $(n+1)$-manifold W, then $f\times id_N\colon M^n\times N\to X\times N$ is bordant to the zero surgery problem, and therefore the surgery obstruction of $f\times id_N$ vanishes.

Proposition 3.82. *If $J,H\subseteq R$ and $|J|+r\equiv 1 \mod 2$, then $w(H)\alpha(H,R)\xi(R,J) = \delta(H,J)$ in $L_{|H|+r+1}(\mathbb{Z}[e])$.*

Proof. There are three cases: $J = H$, $J\not\subset H$, and $J\subsetneq H$. If $J = H$, then $\alpha(H,R)\xi(R,J) = \alpha(H,J) = \alpha(H,H)$, which is the identity map. Then $w(H)\alpha(J,J)$

is the generator of $L_{|J|+r+1}(\mathbb{Z}[e])$, which is 1 if $|J|+r$ is odd and is 0 otherwise. If $J \not\subset H$, then both sides are zero by Lemma 3.80 above. If J is properly contained in H, then $\alpha(H,R)\xi(R,J) = \xi(H,J)$ by Lemma 3.79 above. Now $\xi(H,J)$ is obtained by crossing a degree one PL normal map with a torus. Since the torus is the boundary of a manifold, it follows that the surgery obstruction of the product is then zero. □

Proposition 3.83. *Every element of $L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$ has a unique expression $\sum_J b(J)\xi(J)$, where the sum is taken over all nonempty subsets J of $\{1,\dots,n\}$ with $|J|+r$ odd, with $b(J)\in\mathbb{Z}$ if $|J|+r\equiv 3 \bmod 4$ and $b(J)\in\mathbb{Z}_2$ if $|J|+r\equiv 1 \bmod 4$.*

Proof. Let A be the abelian group generated by subsets $J \subseteq \{1,\dots,n\} = R$ with $|J|+r$ odd, with the requirement that $2J = 0$ when $|J|+r \equiv 1 \bmod 4$. For each $\xi \in L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$, note that $w(J)\alpha(J)\xi$ is an element of $L_{|J|+r+1}(\mathbb{Z}[e])$. Define a map $\rho\colon L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n]) \to A$ given by $\rho(\xi) = \sum_J (w(J)\alpha(J)\xi)J$. To see surjectivity, notice that

$$\rho\left(\sum_J b(J)\xi(J)\right) = \sum_J b(J)\rho(\xi(J)) = \sum_J b(J)\sum_H (w(H)\alpha(H)\xi(J))H$$
$$= \sum_J b(J)\sum_H \delta(H,J)H = \sum_J b(J)J.$$

However we know that there is a decomposition

$$L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n]) \cong \bigoplus_{0\le k\le n+r+1} \binom{n+r+1}{k} L_{n+r+1-k}(\mathbb{Z}[e]).$$

The right-hand side has the same form as A, so $L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$ and A are abstractly isomorphic as groups. Then ρ is an isomorphism, and the elements of $L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$ have the given form. □

Proposition 3.84. *In the action of $L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$ on $\mathcal{S}^{PL}(\mathbb{T}^{n,r})_{\mathrm{rel}}$, the elements of the form*

1. *$\xi(J)$, where $|J|+r\ne 3$,*
2. *$2\xi(J)$, where $|J|+r=3$,*

stabilize the class $[id_{\mathbb{T}^{n,r}}]$ of the identity map.

Proof. We separate the proof into three cases. Suppose that $|J|+r \ge 5$. If ζ is a generator of $L_{|J|+r+2}(\mathbb{Z}[e])$, then the action of ζ on the identity map $id\colon I_1 \to I_1$ gives rise to a homotopy equivalence $h\colon M \to I_1$. Let $P_J = T(J)\times\mathbb{D}^r\times I$ and let $f_J = id\#h\colon P_J\#M \to P_J\#I_1 = P_J$. We then construct $\xi(J)$ by crossing with $T(J^c)$ and taking the associated surgery obstruction. Then the action of $\xi(J)$ on $[id_{\mathbb{T}^{n,r}}]$ gives a manifold structure

$$(id\#\partial h)\times id:\ ((T(J)\times\mathbb{D}^r\times\{1\})\#\partial M)\times T(J^c)\\ \to ((T(J)\times\mathbb{D}^r\times\{1\})\#\partial I_1)\times T(J^c).$$

Recall that ∂I_1 is merely a sphere. But this map is a PL homeomorphism, so belongs to the same class as $id_{\mathbb{T}^{n,r}}$.

Suppose that $|J|+r=1$. Since the connected sum $(T(J)\times\mathbb{D}^r\times I)\#\mathbb{T}^2$ is taken in the interior, it follows that the boundary map is a PL homeomorphism on the boundary $(T(J)\times\mathbb{D}^r\times\{1\})\times T(J^c)$.

Lastly, suppose that $|J|+r=3$. Recall that the Kummer surface K_3 discussed in Section 3.2 is a four-dimensional spin, simply connected PL manifold with signature 16. Its existence shows that the surgery map $\sigma:\ \mathbb{Z}=\pi_4(F/PL)=[\mathbb{S}^4:\ F/PL]\to L_4(\mathbb{Z}[e])=\mathbb{Z}$ is given by multiplication by 2. The Wall realization theorem then yields a degree one PL normal map $h:\ W\to\mathbb{D}^3\times I=\mathbb{D}^4$ such that $\sigma_{\mathbb{D}^4}(W,h)=2$. Once again the periodicity of surgery obstructions shows that we could have defined $2\xi(J)$ starting with this normal map, taking the boundary connected sum with $T(J)\times\mathbb{D}^r\times I$, crossing with $T(J^c)$, and evaluating the surgery obstruction of the resulting degree one normal map. Hence the same reasoning as for $|J|+r\geq 5$ implies that $2\xi[id_{\mathbb{T}^{k,r}}]=[id_{\mathbb{T}^{k,r}}]$. □

For any $Q\subseteq R$, let

$$w(Q):\ L_{|Q|+r+1}(\mathbb{Z}[\mathbb{Z}^{|Q|}])\to L_{|Q|+r+1}(\mathbb{Z}[e])$$

be the natural projection.

Proposition 3.85. *(Hsiang-Shanson [322] §6) Let $\xi=\sum\beta(J)\xi(J)\in L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$. If ξ acts trivially on $[id_{\mathbb{T}^n\times\mathbb{D}^r}]$ and $|J|+r=3$, then $\beta(J)$ is even.*

Proof. Let $\xi=\sum\beta(J)\xi(J)\in L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$ be given. Suppose that H is a subset of $\{1,\dots,n\}$ with $|H|+r=3$ and $\beta(H)$ is odd. Since even multiples of $\xi(H)$ act trivially, we can assume that $\beta(H)=1$. By the Wall realization theorem, there is a degree one PL normal map

$$\phi:\ (W,\partial_0W,\partial_1W)\to(\mathbb{T}^n\times\mathbb{D}^r\times I,(\mathbb{T}^n\times\mathbb{D}^r\times\{0\})\cup(\mathbb{T}^n\times\mathbb{S}^{r-1}\times[0,1]),\mathbb{T}^n\times\mathbb{D}^r\times\{1\})$$

with $\phi|_{\partial_0W}$ a PL homeomorphism. Since ξ acts trivially, we can assume that ϕ_{∂_1W} is a PL homeomorphism, so that $\phi|_{\partial W}$ is also a PL homeomorphism. There is $\psi\simeq\phi\,\mathrm{rel}\,\partial W$ such that ψ is transverse to $T(H)\times\mathbb{D}^r\times[0,1]$, and there is a degree one PL normal map $f:\ P\to T(H)\times\mathbb{D}^r\times I$ such that $f|_{\partial P}:\ \partial P\to\partial(T(H)\times\mathbb{D}^r\times I)$ is a PL homeomorphism. Now $\sigma(P\times\mathbb{CP}^2)=\alpha(H)\xi$ by Lemma 3.79. Since $w(H)$ projects the surgery obstruction to $\mathbb{Z}$, we have $\mathrm{sig}(P\times\mathbb{CP}^2)=w(H)\sigma(P\times\mathbb{CP}^2)=w(H)\alpha(H)\xi=\beta(H)=1$. Therefore $\mathrm{sig}(P)=1$. Using surgery, we may assume that f induces an isomorphism on π_1.

In the case $r = 0$, let W be the space obtained by glueing two copies of $\mathbb{T}^2 \times \mathbb{D}^2$ to the two boundary components of P. Notice also that two copies of $\mathbb{T}^2 \times \mathbb{D}^2$ can be glued to the boundary components of $\mathbb{T}^3 \times [0, 1]$ to form $\mathbb{T}^2 \times \mathbb{S}^2$. Then we have a map $f' : W \to \mathbb{T}^2 \times \mathbb{S}^2$ with $\sigma_{\mathbb{T}^2 \times \mathbb{S}^2}(W) = 1$. It can be shown that $\mathrm{sig}(W) = 8$ and W is spin (see Hsiang-Shaneson [322]). But this result contradicts Rokhlin's Theorem. □

Theorem 3.86. *There is a bijection between* $\Lambda^{3-r}(\mathbb{Z}^n) \otimes \mathbb{Z}_2$ *and* $\mathcal{S}^{PL}(\mathbb{T}^{n,r})_{\mathrm{rel}}$. *Here the notation* $\Lambda^k(\mathbb{Z}^n)$ *means the k-th exterior power of the ring* $\mathbb{Z}^n$ *as a* $\mathbb{Z}$*-module.*

Proof. Recall that in Corollary 3.72 we showed that

$$\mathcal{S}^{PL}(\mathbb{T}^{n,r})_{\mathrm{rel}} \cong L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])/\mathrm{Stab}(\mathrm{id}_{\mathbb{T}^{n,r}}).$$

Each element of $L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$ is of the form $\sum_J \beta(J)\xi(J)$, where $\beta(J) \in \mathbb{Z}$ if $|J|+r \equiv 3 \bmod 4$ and $\beta(J) \in \mathbb{Z}_2$ if $|J|+r \equiv 1 \bmod 4$. The subgroup of $L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$ generated by

$$\{\xi(J) : |J| + r \neq 3\} \cup \{2\xi(J) : |J| + r = 3\}$$

is precisely the kernel of ∂. Note that, in the quotient $L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])/\ker \partial$, all terms with $|J| + r \neq 3$ are trivial. There are $\binom{n}{3-r}$ possible choices of J with $|J| + r = 3$, and the coefficient can be one of two possibilities. Note that, when $r \geq 4$, then $\binom{n}{3-r}$ is interpreted to mean 0, and $\Lambda^{3-r}(\mathbb{Z}^n) \otimes \mathbb{Z}_2$ is interpreted to mean the set of one element. □

Note that $\Lambda^{3-r}(\mathbb{Z}^n) \otimes \mathbb{Z}_2$ is equinumerous with $H^{3-r}(\mathbb{T}^n; \mathbb{Z}_2)$. A direct bijection between $\mathcal{S}^{PL}(\mathbb{T}^{n,r})$ and $H^{3-r}(\mathbb{T}^n; \mathbb{Z}_2)$ can be described in the following. Consider a basis $t_1, \ldots, t_n$ of $H^1(\mathbb{T}^n; \mathbb{Z}_2)$. For each subset $J = \{i_1, \ldots, i_{|J|} : i_1 < \cdots < i_{|J|}\} \subseteq \{1, \ldots, n\}$ with $|J| + r = 3$, let $t_J = t_1 \wedge \cdots \wedge t_{|J|}$. Then the (t_J) form a basis of $H^{3-r}(\mathbb{T}^{n,r}; \mathbb{Z}_2)$. Define $\lambda_J(\xi) = w(J)\alpha(J)\xi \bmod 2$.

Theorem 3.87. *For each* $M \in \mathcal{S}^{PL}(\mathbb{T}^{n,r})_{\mathrm{rel}}$, *let* $\xi \in L_{n+r+1}(\mathbb{Z}[\mathbb{Z}^n])$ *be chosen such that* $\xi.[id_{\mathbb{T}^{n,r}}] = M$. *Then the map* $\lambda^* : \mathcal{S}^{PL}(\mathbb{T}^{n,r}) \to H^{3-r}(\mathbb{T}^n; \mathbb{Z}_2)$ *given by*

$$\lambda^*(M) = \sum_{|J|+r=3} \lambda_J(\xi) t_J$$

is a well-defined bijection.

Remark 3.88. *More importantly, all of these PL structures become trivial on taking the* 2^n*-fold cover that "unwraps" all directions twice. This statement was required by Kirby in the proof of the annulus conjecture, which is discussed in Appendix B.3.*

3.7 THE KIRBY-SIEBENMANN INVARIANT

At this point in the text, we understand the structure of F/PL very well, having determined its homotopy type using a combination of special low-dimensional arguments and surgery combined with the Poincaré conjecture. Kirby used $\mathcal{S}^{PL}(\mathbb{T}^n)$ to show that one can establish Top handlebody theory and transversality to obtain a surgery exact sequence for $\mathcal{S}^{Top}(M)$ with F/Top replacing F/PL. We discuss these ideas in somewhat more detail in Appendix B.3.

This section presents the result that the Top and PL manifold categories, which certainly differ, are only slightly different and the difference is very concrete. The analysis will use machinery that indicates the general importance of control methods in surgery theory. We will discuss control theory in the last chapter. Unlike in PL where the image of the map $\pi_4(F/Top) \to L_0(\mathbb{Z}[e])$ is of index 2, the maps $\pi_n(F/Top) \to L_n(\mathbb{Z}[e])$ will be isomorphisms for all n. This result implies our first substantial result concerning the Borel conjecture. In particular, we will show that the Top surgery set $\mathcal{S}^{Top}(\mathbb{T}^n)$ vanishes for $n \geq 5$.

To handle low-dimensional homotopy groups, Kirby and Siebenmann use the following important theorem (see Černavskiĭ [147] and Edwards-Kirby [211]).

Theorem 3.89. *If M is a topological manifold, then the space* Homeo(M) *of homeomorphisms of M is locally contractible. In other words, for M compact, for each h and $\varepsilon > 0$, there is a δ-neighborhood of h that is contractible within the ε-ball about h.*

First we make the following observation.

Proposition 3.90. *The fiber Top/PL of the fibration $F/PL \to F/Top$ is not contractible.*

Proof. Recall that Rokhlin's Theorem states that, if M is a closed spin four-dimensional PL manifold, then the signature of M is divisible by 16. If Top/PL were contractible, then every high-dimensional Top manifold can be given a PL structure. However, Freedman constructed a closed spin four-dimensional Top manifold whose signature is 8. This manifold does not even have a PL structure after crossing with a torus. □

Theorem 3.91. *The space Top/PL is homotopy equivalent to the Eilenberg-Maclane space $K(\mathbb{Z}_2, 3)$. In other words the homotopy groups $\pi_n(Top/PL)$ all vanish except that $\pi_3(Top/PL) \cong \mathbb{Z}_2$.*

Proof. We use the theorem above on the local contractibility of Homeo(M). Let $h : \mathbb{T}^k \times \mathbb{D}^{n-1} \to \mathbb{T}^k \times \mathbb{D}^{n-1}$ be a self-homeomorphism which is homotopic rel boundary to the identity. This h can be considered a structure on $\mathbb{T}^k \times \mathbb{D}^{n-1}$. Then there is a lift

$\widetilde{h}\colon \mathbb{T}^k \times \mathbb{D}^{n-1} \to \mathbb{T}^k \times \mathbb{D}^{n-1}$ with a commutative diagram

$$\begin{array}{ccc} \mathbb{T}^k \times \mathbb{D}^{n-1} & \xrightarrow{\widetilde{h}} & \mathbb{T}^k \times \mathbb{D}^{n-1} \\ \downarrow & & \downarrow \\ \mathbb{T}^k \times \mathbb{D}^{n-1} & \xrightarrow{h} & \mathbb{T}^k \times \mathbb{D}^{n-1} \end{array}$$

such that $\widetilde{h}$ is isotopic rel boundary to the identity. If we pass to a sufficiently large cover, and identify the lifted torus with the original torus subject to a homothety, then the homeomorphism can be made arbitrarily close to the identity, and therefore isotopic to the identity by a small isotopy. Any homotopy of the identity to itself near the identity is homotopic to a homeomorphism. Therefore, for all sufficiently large liftings, the transfer

$$tr\colon \mathcal{S}^{Top}(\mathbb{T}^k \times \mathbb{D}^n) \to \mathcal{S}^{Top}(\mathbb{T}^k \times \mathbb{D}^n)$$

is the zero map.

This result immediately implies that the image of the map

$$\mathcal{S}^{Top}(\mathbb{T}^k \times \mathbb{D}^n) \to [\mathbb{T}^k \times \mathbb{D}^n \text{ rel } \partial\colon F/Top]$$

is trivial. Therefore $\mathcal{S}^{PL}(\mathbb{T}^k \times \mathbb{D}^n) \to \mathcal{S}^{Top}(\mathbb{T}^k \times \mathbb{D}^n)$ is surjective and $\pi_i(F/Top) \to L_i(\mathbb{Z}[e])$ is injective. Since Top/PL is not contractible, it must be $K(\mathbb{Z}_2, 3)$; i.e. the image of $\pi_4(F/PL)$ in $\pi_4(F/Top)$ is index 2. □

Remark 3.92. *As mentioned earlier, controlled topology has entered our discussions. It will reemerge when we discuss the α-approximation theorem and yet again in the final chapter.*

Remark 3.93. *Kirby and Siebenmann did not have Freedman's E_8-manifold, so they used a variant of the above argument to obtain a surjection $\pi_4(F/Top) \to L_0(\mathbb{Z}[e])$. The replacement for the E_8 manifold is the homotopy of a large odd-folder cover of a homotopy torus to a homeomorphism.*

Theorem 3.94. *The map $\pi_4(F/Top) \to L_n(\mathbb{Z}[e])$ is an isomorphism in all dimensions.*

Proof. The claim is obvious for all $n \neq 4$. For $n = 4$ we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_4(F/PL) & \xrightarrow{\cdot 2} & \pi_4(F/Top) & \longrightarrow & \pi_3(Top/PL) & \longrightarrow & 0 \\ & & {\scriptstyle \cdot 2}\downarrow & & \downarrow{\scriptstyle \sigma} & & & & \\ & & L_4(\mathbb{Z}[e]) & \xrightarrow[\cong]{} & L_4(\mathbb{Z}[e]) & & & & \end{array}$$

where all groups are isomorphic to $\mathbb{Z}$ except for $\pi_3(Top/PL) \cong \mathbb{Z}_2$. □

Theorem 3.95. *The Top structure set for the torus $\mathbb{T}^n$ is trivial if $n \geq 5$; these tori are topologically rigid.*

Proof. The topological rigidity of the torus can been seen if we stack the PL surgery exact sequence on top of Top surgery exact sequence.

$$\begin{array}{ccccccccc}
[\Sigma\mathbb{T}^n : F/PL] & \longrightarrow & L_{n+1}(\mathbb{Z}[\mathbb{Z}^n]) & \longrightarrow & \mathcal{S}^{PL}(\mathbb{T}^n) & \longrightarrow & [\mathbb{T}^n : F/PL] & \longrightarrow & L_n(\mathbb{Z}[\mathbb{Z}^n]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[\Sigma\mathbb{T}^n : F/Top] & \longrightarrow & L_{n+1}(\mathbb{Z}[\mathbb{Z}^n]) & \longrightarrow & \mathcal{S}^{Top}(\mathbb{T}^n) & \longrightarrow & [\mathbb{T}^n : F/Top] & \longrightarrow & L_n(\mathbb{Z}[\mathbb{Z}^n])
\end{array}$$

An analysis of each term and the maps between give the following:

$$\begin{array}{ccccccccc}
\bigoplus_i \binom{n}{i}\pi_{i+1}(F/PL) & \xhookrightarrow{\times 2} & \bigoplus_i \binom{n}{i} L_{i+1}(\mathbb{Z}[e]) & \longrightarrow & \mathcal{S}^{PL}(\mathbb{T}^n) & \xrightarrow{0\text{ map}} & \bigoplus_i \binom{n}{i}\pi_i(F/PL) & \xhookrightarrow{\times 2} & \bigoplus_i \binom{n}{i} L_i(\mathbb{Z}[e]) \\
\downarrow{\scriptstyle\times 2} & & \| & & \downarrow & & \downarrow{\scriptstyle\times 2} & & \| \\
\bigoplus_i \binom{n}{i}\pi_{i+1}(F/Top) & \longrightarrow & \bigoplus_i \binom{n}{i} L_{i+1}(\mathbb{Z}[e]) & \longrightarrow & \mathcal{S}^{Top}(\mathbb{T}^n) & \xrightarrow{0\text{ map}} & \bigoplus_i \binom{n}{i}\pi_i(F/Top) & \longrightarrow & \bigoplus_i \binom{n}{i} L_i(\mathbb{Z}[e])
\end{array}$$

In the Top category, we have maps $\bigoplus_i \binom{n}{i}\pi_{i+1}(F/Top) \to \bigoplus_i \binom{n}{i} L_{i+1}(\mathbb{Z}[e])$ and $\bigoplus_i \binom{n}{i}\pi_i(F/Top) \to \bigoplus_i \binom{n}{i} L_i(\mathbb{Z}[e])$ which are isomorphisms in all degrees. In the diagram the symbol $\times 2$ means that multiplication by 2 takes place in degree 4, but that the map is an isomorphism in all other degrees. Therefore the map $\bigoplus_i \binom{n}{i} L_i(\mathbb{Z}[e]) \to \mathcal{S}^{Top}(\mathbb{T}^n)$ is the zero map by exactness. Since $\mathcal{S}^{Top}(\mathbb{T}^n) \to \bigoplus_i \binom{n}{i}\pi_i(F/Top)$ is also the zero map, we conclude that $\mathcal{S}^{Top}(\mathbb{T}^n)$ is trivial. □

Since Top/PL is an Eilenberg-Maclane concentrated in dimension 3 with homotopy group $\mathbb{Z}_2$, we can describe the obstruction properties of the *Kirby-Siebenmann invariant*.

Theorem 3.96. *(Kirby-Siebenmann) Let $n \geq 5$. There is an invariant $ks(M^n) \in H^4(M; \mathbb{Z}_2)$ which vanishes iff M has a PL structure. Concordance classes of such structures are classified in a bijective fashion by the relative obstruction to triangulating $M \times [0,1]$, which lies in $H^3(M; \mathbb{Z}_2)$.*

Equivalently, this statement states that the obstruction to triangulating M as a PL manifold is the same as solving a lifting problem

$$\begin{array}{ccc}
 & & BPL \\
 & & \downarrow \\
M & \longrightarrow & BTop
\end{array}$$

and that the fiber of the vertical map Top/PL is homotopy equivalent to the Eilenberg-Maclane space $K(\mathbb{Z}_2, 3)$.

The homotopy type of F/Top can be computed just as for F/PL. We record the result of the calculations. Note that the structure of F/Top is beautifully transparent.

Theorem 3.97. *The space F/Top can be described as the pullback:*

$$\begin{array}{ccc} F/Top & \longrightarrow & \prod K(L_n(\mathbb{Z}[e]),\, n)_{(2)} \\ \downarrow & & \downarrow \\ BO[1/2] & \longrightarrow & \prod K(\mathbb{Q}, 4n)_{(2)} \end{array}$$

Explicitly, we have $F/Top[1/2] \simeq BO[1/2]$ and

$$F/Top_{(2)} \simeq \prod_{n=1}^{\infty} K(\mathbb{Z}_2, 4n-2) \times \prod_{n=1}^{\infty} K(\mathbb{Z}_{(2)}, 4n).$$

Corollary 3.98. *There is a homotopy equivalence between $\Omega^4(F/Top)$ and $\mathbb{Z}\times F/Top$.*

Here is a simple application. The reader can find other examples in the same way.

Example 3.99. *Let $n \geq 2$. Consider $\mathbb{S}^3 \times \mathbb{S}^n$. Using the fact that $Top/PL \simeq K(\mathbb{Z}_2, 3)$, we can deduce that $[\mathbb{S}^3 \times \mathbb{S}^n : Top/PL] = H^3(\mathbb{S}^3 \times \mathbb{S}^n; \mathbb{Z}_2)$, which is nonzero. Therefore there is a non-trivial homotopy PL manifold structure on $\mathbb{S}^3 \times \mathbb{S}^n$.*

3.8 NONRIGIDITY OF NONUNIFORM ARITHMETIC MANIFOLDS

The Borel conjecture asserts that closed aspherical manifolds are topologically rigid; i.e. if M is closed and aspherical and $f : M' \to M$ is a homotopy equivalence, then f is homotopic to a homeomorphism, previously stated in Conjecture 3.66. We have shown in Section 3.6 that the Borel conjecture is true for the n-torus $\mathbb{T}^n$. In fact, as of the writing of this book, it has been verified for locally symmetric manifolds by Farrell-Jones [233] (aside from dimension 4 cases) and for many non-classical aspherical manifolds by Bartels-Lück [42].

As noted, the Borel conjecture is a topological analogue of the Mostow rigidity theorem, which states that, if M and N are closed hyperbolic n-manifolds with $n \geq 3$ with isomorphic fundamental groups, then the isomorphism is induced by a unique isometry $f : M \to N$. According to Prasad [512], the theorem also holds for complete hyperbolic manifolds of finite volume in dimensions at least 3. It is then natural to inquire whether there is a proper analogue to the Borel conjecture, motivated by the differential geometric result.

In this section we will present a proper version of the Borel conjecture for noncompact manifolds, and prove that it fails systematically for various so-called *arithmetic manifolds*. This section requires more geometric background than anything discussed thus far. Our goal is merely to illustrate the power of the machinery developed until this point.

We will discuss proper surgery in the last section. Here we will treat it in an ad hoc fashion by comparing it to a manifold with boundary.

We first give some definitions.

Definition 3.100. *We say that a noncompact topological manifold M without boundary is properly rigid if, whenever N is another manifold of the same dimension with a proper homotopy equivalence $h : N \to M$, then h is properly homotopic to a homeomorphism. As in the compact case, we can define the proper topological structure set $\mathcal{S}_p^{Top}(M)$ of M.*

A natural extension is a proper version of the Borel conjecture for noncompact aspherical Top manifolds.

CONJECTURE 3.1. *(Proper Borel) Suppose that M is a noncompact aspherical topological manifold. Then M is properly rigid; i.e. $\mathcal{S}_p^{Top}(M)$ is trivial.*

We shall soon see that this conjecture is systematically false.

Definition 3.101. *Let G be a connected (real) Lie group. Let K be its maximal compact subgroup, which is unique up to conjugacy. A lattice in G is a discrete subgroup Γ, so that $\Gamma\backslash G$ has finite volume with respect to the natural Haar measure on G. We can then construct the double coset space $\Gamma\backslash G/K$.*

Example 3.102. *As an example, we can take $G = \mathrm{SL}_n(\mathbb{R})$ and $K = \mathrm{SO}_n(\mathbb{R})$. If we take the lattice $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, then the double coset space $K\backslash G/\Gamma$ is a locally symmetric orbifold. It is not a manifold because $\mathrm{SL}_n(\mathbb{Z})$ has elements of finite order. However, one can take a cover of the form $\Gamma'\backslash G/K$ to obtain a manifold.*

Definition 3.103. *An arithmetic manifold is a double coset space $\Gamma\backslash G_{\mathbb{R}}/K$, where G is a semisimple algebraic subgroup of GL_n defined over $\mathbb{Q}$, the subgroup K is maximal compact in the real points $G_{\mathbb{R}}$, and Γ is a torsion-free arithmetic subgroup of its rational points $G_{\mathbb{Q}}$. If $\Gamma\backslash G/K$ is compact, then Γ is called uniform. Otherwise it is called nonuniform.*

Remark 3.104. *Note that, in order for $\Gamma\backslash G/K$ to be a manifold, the lattice Γ must be torsion-free. The Margulis arithmeticity theorem states that every irreducible lattice in G is arithmetic if $\operatorname{rank} G \geq 2$. Later work has given more information about the arithmeticity in rank one situation.*

Definition 3.105. *A closed, connected subgroup T of $\mathrm{GL}_n(\mathbb{C})$ is a torus if it is diagonalizable over $\mathbb{C}$; i.e. there is $g \in \mathrm{GL}_n(\mathbb{C})$ such that $g^{-1}Tg$ consists entirely of diagonal* *matrices. A torus T in $\mathrm{SL}_n(\mathbb{R})$ is $\mathbb{Q}$-split if T is defined over $\mathbb{Q}$ and T is diagonalizable over $\mathbb{Q}$. If G is a $\mathbb{Q}$-subgroup of $\mathrm{SL}_n(\mathbb{R})$ and Γ is commensurable with the set of integer points $G_{\mathbb{Z}}$, then the rational rank $\mathrm{rank}^G_{\mathbb{Q}}(\Gamma)$ of G is the dimension of any maximal $\mathbb{Q}$-split torus of G. If G is clear, then it may be omitted from the notation.*

Remark 3.106. *Because all maximal $\mathbb{Q}$-split tori are conjugate under $G_{\mathbb{Q}}$, this definition is independent of the particular torus chosen.*

In fact, for other cases of small $\mathbb{Q}$-rank, proper topological rigidity for all arithmetic manifolds has been established. When $\mathrm{rank}_{\mathbb{Q}}(\Gamma) \leq 1$, then the proper rigidity of $\Gamma\backslash G/K$ can be deduced from the work of Farrell-Jones [233], Farrell-Hsiang [229], and Gromov on the structure of cusps [271]. The $\mathrm{rank}_{\mathbb{Q}}(\Gamma) = 2$ case has been resolved through the work of Bartels-Lück-Reich-Rüping [44].

In this section we present a result for the $\mathrm{rank}_{\mathbb{Q}}(\Gamma) \geq 3$ case that shows that Mostow's rigidity theorem, while it holds for noncompact hyperbolic manifolds of finite volume, fails spectacularly for general symmetric manifolds, and cannot be weakened to provide a proper version of Borel's conjecture for manifolds of noncompact type.

Theorem 3.107. *(Chang-Weinberger [154]) Let $M = \Gamma\backslash G/K$ be a noncompact arithmetic manifold for which $\mathrm{rank}_{\mathbb{Q}}(\Gamma) \geq 3$. Then M has a finite-sheeted cover N whose proper structure set $S_p^{Top}(N)$ is non-trivial; i.e. there is a manifold M with a proper homotopy equivalence $g : M \to N$ that is not properly homotopic to a homeomorphism.*

The proof of the above theorem combines a number of well-known but deep results: the theorems of Sullivan and Wall from classical surgery theory [682] that we have already explained, the Borel-Serre compactification of arithmetic manifolds [67], Každan's property (T), and a consequence by Lubotzky [404] of Weisfeiler's strong approximation for linear groups [696].

3.8.1 Group-theoretic background

Let Γ be a group and $\mathcal{H}$ be a Hilbert space. We say that a unitary representation $\pi : \Gamma \to U(\mathcal{H})$ *contains invariant vectors* if there is a nonzero $\xi \in \mathcal{H}$ such that $\pi(\gamma)\xi = \xi$ for all $\gamma \in \Gamma$. The representation *contains almost invariant vectors* if, for each $F \subseteq \Gamma$ and $\varepsilon > 0$, there is $\xi \in \mathcal{H}$ such that $||\pi(\gamma)\xi - \xi|| < \varepsilon||\xi||$ for all $\gamma \in F$.

Let G be a locally compact group. We say that G has *Každan property (T)* if any unitary representation $\pi : G \to U(\mathcal{H})$ of G on a Hilbert space $\mathcal{H}$ which almost has invariant vectors actually has non-trivial invariant vectors. Každan [348] proves that, if G is a connected semisimple Lie group with finite center, each of whose factors has real rank at least two, then G, as well as any lattice subgroup of G, has Každan property (T). This property stands opposite the condition of amenability in the sense that, if G is amenable, then G has Každan property T iff G is compact. From this result it is easy to show that, if $\phi : G \to H$ is a homomorphism where G has Každan property (T) and

H is amenable, then the closure of $\phi(G)$ is compact.

As examples, the real Lie groups $\mathrm{Sp}(n,1)$ and $F_4^{(-20)}$ have property (T) but $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$ do not. See Bekka-Valette [52] for more on property (T). Weisfeiler's strong approximation result for general linear groups [696] states that, if Γ is a Zariski-dense subgroup in an algebraic group G, then Γ is virtually dense in G with respect to the congruence topology; i.e. the closure of Γ is of finite index in the profinite completion $\widehat{G}$. The congruence topology of $\mathrm{SL}_n(\mathbb{Z})$, for example, is the topology for which the groups $\Gamma(m) = \ker(\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}_m))$ serve as a system of neighborhoods of the identity and its completion is $\Pi_p \, \mathrm{SL}_n(\widehat{\mathbb{Z}}_p)$. The theorem implies, in particular, that a finitely generated linear group is either solvable or has a finite index subgroup with infinitely many different finite simple quotients. Lubotzky uses Weisfeiler's result to prove the following.

Theorem 3.108. *(Lubotzky [404] Theorem A) Let $\mathbb{F}$ be a field of characteristic different from 2 or 3, and let Γ be a finitely generated infinite subgroup of $\mathrm{GL}_n(\mathbb{F})$. For all $d \in \mathbb{Z}_{\geq 1}$, there is a finite index subgroup of Γ whose index in Γ is divisible by d.*

Wehrfritz gave a different proof [683] of the above result for $d = 2$ that is also valid in characteristics 2 and 3, which is all we need for our main result. It is worth noting that Theorem A is equivalent to the assertion that, for any prime q, the q-Sylow subgroup of the profinite completion $\widehat{\Gamma}$ of Γ is infinite. In our discussion it will be convenient to use the following strengthening of Theorem A.

Theorem 3.109. *(Lubotzky [404] Theorem B) Suppose that Γ satisfies the hypotheses of Theorem A and is not solvable-by-finite; i.e. Γ has no solvable subgroup of finite index. Then for every prime q, the q-Sylow subgroup of $\widehat{\Gamma}$ is infinitely generated.*

Lemma 3.110. *Let G be a semisimple Lie group and Γ an arithmetic subgroup of G. There is a normal subgroup $\Gamma'' \lhd \Gamma$ of finite even index.*

Proof. As a consequence of property (T), such an arithmetic subgroup Γ is an infinite, finitely generated subgroup of $\mathrm{SL}_n(\mathbb{C})$ for some integer n. By Theorem A, the group Γ contains a subgroup Γ' of even index. Let $k = [\Gamma : \Gamma']$ and let Γ act standardly on the coset space Γ/Γ'. Hence we have a homomorphism $\phi : \Gamma \to S_k$ whose kernel Γ'' is contained in Γ'. Clearly since $[\Gamma : \Gamma''] = [\Gamma : \Gamma'][\Gamma' : \Gamma'']$, it follows that Γ'' is normal of finite even index in Γ. □

Lemma 3.111. *The group Γ has a subgroup of even index iff Γ contains a subgroup with a surjective homomorphism onto $\mathbb{Z}_2$.*

Proof. ($\Leftarrow$) Suppose that there is a group $\Gamma' \leq \Gamma$ with a surjection $\phi : \Gamma' \to \mathbb{Z}_2$. Then $\Gamma'/\ker\phi \cong \mathbb{Z}_2$, so $[\Gamma : \ker\phi] = [\Gamma : \Gamma'][\Gamma' : \ker\phi]$ is even. ($\Rightarrow$) Suppose that there is $\Gamma' \leq \Gamma$ of even index. Hence there is a subgroup $\Gamma'' \leq \Gamma'$ normal in Γ such that $[\Gamma : \Gamma''] = 2n$ for some $n \in \mathbb{Z}_{\geq 1}$. Therefore Γ/Γ'' has even order, so there is a subgroup

$\widetilde{\Gamma}/\Gamma'' \leq \Gamma/\Gamma''$ that is isomorphic to $\mathbb{Z}_2$. Define $\phi : \widetilde{\Gamma} \to \mathbb{Z}_2$ by

$$\phi(a) = \begin{cases} 0 & \text{if } a \in \Gamma'', \\ 1 & \text{if } a \notin \Gamma''. \end{cases}$$

This map is a surjective homomorphism. □

The proof of the following proposition requires the notion of superrigidity. There is more than one result that uses this terminology. We consider one such statement. Let G be a simply connected semisimple real algebraic group in GL_n, such that the Lie group of its real points has real rank at least 2 and no compact factors. Suppose Γ is an irreducible lattice in G. For a local field $\mathbb{F}$ and ρ a linear representation of the lattice Γ of the Lie group into $\mathrm{GL}_n(\mathbb{F})$, assume the image $\rho(\Gamma)$ is not relatively compact (in the topology arising from $\mathbb{F}$) and such that its closure in the Zariski topology is connected. Then $\mathbb{F}$ is the real numbers or the complex numbers, and there is a rational representation of G giving rise to ρ by restriction.

Proposition 3.112. *Let G be a semisimple Lie group with trivial center and* $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$*. Let $M = \Gamma\backslash G/K$ be an arithmetic manifold with an irreducible lattice Γ of G. Then M has a finite cover N for which $H_1(N;\mathbb{Z})$ contains 2-torsion.*

Proof. The two lemmas above imply the existence of a subgroup $\Gamma' \leq \Gamma$ equipped with an epimorphism $\phi : \Gamma' \to \mathbb{Z}_2$. Let $H = \ker\phi$ so that $\Gamma'/H \cong \mathbb{Z}_2$. Let $J = [\Gamma' : \Gamma']$ be the commutator subgroup of Γ'. Observe that both H and J are normal in Γ'. Now consider the quotient homomorphism $\rho : \Gamma' \to \Gamma'/J$. First assume that G is simple. By the condition on the real rank of G, it follows that Γ' has property (T). Since Γ'/J is amenable, the image $\rho(\Gamma') = \Gamma'/J$ must be compact. By the discreteness of Γ, the index $[\Gamma' : J]$ is finite. If $a, b \in \Gamma'$ with $aba^{-1}b^{-1} \in J$, then $\phi(aba^{-1}b^{-1}) = \phi(a) + \phi(b) - \phi(a) - \phi(b) = 0$ in $\mathbb{Z}_2$, so $aba^{-1}b^{-1} \in H$. Therefore $J \leq H$, implying that $[\Gamma' : J] = [\Gamma' : H][H : J]$ is even, so Γ'/J has 2-torsion. Let N be the cover of M with respect to the subgroup $\Gamma' \leq \Gamma$. Then $\pi_1(N) = \Gamma'$ and $H_1(N;\mathbb{Z}) = \Gamma'/J$ contains 2-torsion.

In the case that G is not simple, the superrigidity of Γ' follows from Margulis (see Zimmer [724]), implying that the first Betti number of N vanishes; i.e. the abelianization of Γ' and therefore the first integral homology of N is finite. The remainder of the proof proceeds as in the simple case. □

Corollary 3.113. *Let N be given as above. Then the group $H^2(N;\mathbb{Z}_2)$ is nonzero.*

Proof. By the above proposition, the homology group $H_1(N;\mathbb{Z})$ contains 2-torsion. We then conclude that $\mathrm{Ext}(H_1(N;\mathbb{Z}), \mathbb{Z}_2)$ is non-trivial, since $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{(m,n)}$ for any $m, n \in \mathbb{Z}_{\geq 1}$. By the universal coefficient theorem, the map $\mathrm{Ext}(H_1(N;\mathbb{Z}), \mathbb{Z}_2) \to H^2(N;\mathbb{Z}_2)$ is injective, so $H^2(N;\mathbb{Z}_2)$ is non-trivial as well. □

Theorem 3.114. *Let $M = \Gamma\backslash G/K$ be a noncompact arithmetic manifold whose $\mathbb{Q}$-rank is at least 3. Then M has a finite-sheeted cover N whose proper structure set is non-trivial; i.e. the manifold M is virtually properly rigid.*

Proof. Let Γ' be a normal subgroup of Γ of finite even index and let N be the cover of M corresponding to Γ'. Then $\pi_1(N) = \Gamma'$ and $H^2(N;\mathbb{Z}_2)$ is non-trivial. This N can be compactified to a π-π manifold $\overline{N}$ with boundary since the $\mathbb{Q}$-rank is greater than 2. By the Borel-Serre compactification and its homotopy type established in this paper, this result follows from the identification of the homotopy type of the universal cover of the boundary with a wedge of $(q-1)$-spheres using the Solomon-Tits theorem [67]. The nonarithmetic case follows from Margulis's result that such a manifold will virtually split and Gromov's results about the structure at infinity of pinched negatively curved manifolds of finite volume. See Block-Weinberger [59] for details. According to Siebenmann's thesis, any manifold that is properly homotopy equivalent to M will have the same property. Using the h-cobordism theorem, any such manifold has a unique compactification so that the extension of the proper homotopy equivalence to the compactification is a simple homotopy equivalence. We can then identify $\mathcal{S}^{Top}(\overline{N})$ with the proper structure set $\mathcal{S}_p^{Top}(N)$. By the π-π theorem of Wall, the structure set $\mathcal{S}(\overline{N})$ of $\overline{N}$ is isomorphic to $[\overline{N} : F/Top]$. Since $F/Top \simeq Z \times K(\mathbb{Z}_2,2)\times K(\mathbb{Z}_2,6)\times K(\mathbb{Z}_2,10)\times\cdots$ for some space Z, we then have

$$\begin{aligned}\mathcal{S}_p^{Top}(N) &= \mathcal{S}^{Top}(\overline{N}) = [\overline{N} : F/Top] = [N : F/Top]\\ &= [N : Z\times K(\mathbb{Z}_2,2)\times K(\mathbb{Z}_2,6)\times K(\mathbb{Z}_2,10)\times\cdots]\\ &= [N : Z]\times[N : K(\mathbb{Z}_2,2)]\times[N : K(\mathbb{Z}_2,6)]\times\cdots\end{aligned}$$

which by the previous corollary is non-trivial since $[N : K(\mathbb{Z}_2,2)] = H^2(N;\mathbb{Z}_2)$. Note that the first equality follows again from Siebenmann's thesis, since there is no end obstruction in the π-π situation. □

Theorem 3.115. *Let $M = \Gamma\backslash G/K$ be an arithmetic manifold with Γ irreducible. If* $\operatorname{rank}_{\mathbb{Q}}\Gamma \geq 3$, *then M has finite-sheeted covers N whose proper structures are arbitrarily large finite groups.*

Proof. By the proof of Lubotzky's Theorem B the profinite completion $\widehat{\Gamma}$ of Γ contains an infinitely generated elementary abelian 2-group $\mathbb{Z}_2^\infty$. If r is a given positive integer, then there is a cover N_r of $\Gamma\backslash G/K$ with a surjection $\pi_1(N_r)\to\mathbb{Z}_2^r$. By property (T), the homology group $H_1(N;\mathbb{Z})$ contains a summand R which is a direct sum of r non-trivial even-order cyclic groups. Then $H^2(N_r;\mathbb{Z})$ contains $\operatorname{Ext}(\mathbb{Z}_2,R)\cong\mathbb{Z}_2^r$. Therefore the proper structure set $\mathcal{S}_p^{Top}(N_r)$ contains at least 2^r elements. □

Remark 3.116. *Indeed it is possible to perform more refined calculations. For lattices commensurate to* $\mathrm{SL}_n(\mathbb{Z})$ *and $n\geq 5$ one can construct examples detected by rational Pontrjagin classes using "modular symbols." See Chang-Weinberger [155].*

Chapter Four

Topological surgery and surgery spaces

In this chapter, we shall discuss the fact that the Top structure set is actually an abelian group and that surgery theory is functorial in the topological category. In doing so, we will need the surgery spaces, introduced by Casson [143] and Quinn [516]. The theory also has a kind of periodicity, due to Siebenmann, that is analogous to Bott periodicity. Later, in Chapter 8, we will discuss how periodicity takes a more perfect form with a more general class of space replacing Top manifolds, and that functoriality is then also improved.

4.1 BEGINNING THE SPACIFICATION

Many times one has a long exact sequence

$$C_n \to B_n \to A_n \to C_{n-1} \to B_{n-1} \to A_{n-1} \to \cdots$$
$$\to C_1 \to B_1 \to A_1 \to C_0 \to B_0 \to A_0.$$

If there are spaces $\boldsymbol{C}$, $\boldsymbol{B}$, and A together with a fibration $\boldsymbol{C} \to \boldsymbol{B} \to A$ whose long exact sequence of homotopy groups is the original sequence, then we say that the long exact sequence has been *spacified* and $\boldsymbol{C} \to \boldsymbol{B} \to A$ is a *spacification*. Obviously, one can spacify any sequence of abelian groups by merely taking the corresponding product of Eilenberg-MacLane spaces. But it is too trivial to be useful, and we often want other spacifications that are more interesting.

In this section, we will begin the spacification of the surgery exact sequence, which was initiated by Casson in the simply connected case, and by Quinn [516] in general. The theory is explained in Siebenmann's essay in [361], and Nicas's thesis [482]. The spacification is very natural when surgery is developed in appropriate generality. Wall uses the theory of *n-ads*, which naturally adapts to n-simplices in a simplicial space. Moreover, Casson and Quinn have shown that n-ads arise in the classification of block bundles (see also Burghelea-Lashof-Rothenberg [106]). These ideas will reappear in Chapter 8 when we discuss stratified spaces.

We will see that the spacified version of Top surgery satisfies a sort of periodicity: it is (almost) its own fourth loop space. As a consequence, the homotopy group $\pi_0(\mathcal{S}^{Top}(M))$ is an abelian group. The existence of a group structure on topological structure sets is a deep and critical feature of the Top category that we will exploit throughout the rest of this book. Even more profoundly, we will see that the structure set $\mathcal{S}^{Top}(M)$ is itself a sort of L-group.

The context of the construction involves the study of complexes along with their subcomplexes. In particular, we may concern ourselves with a complex and finitely many distinguished subcomplexes of it. The notion of an n-ad arrives from our wish to encode these subobjects and their intersections. For example, if $f_1 : M_1 \to X$ and $f_2 : M_2 \to X$ are elements of the normal invariant set $\mathcal{N}^{Cat}(X)$, then they are equivalent if there is a normal cobordism W between them. Typically we imagine W to be a Cat manifold with two boundary components M_1 and M_2. However, if M_1 and M_2 are both Cat manifolds with boundary, then W becomes a *manifold with corners*, which has a somewhat more complicated combinatorial structure than the previous one. A cobordism between manifolds with corners will have additional combinatorics. The structure of an n-ad will allow us to keep track of various subcomplexes within a space and their relationship to one another. Once we have cast our surgery theory in the context of n-ads, it is a small step to build a space from them and study such spaces.

Much of the discussion below is adapted from Nicas [482], which expands Wall [672]. This section is largely a catalogue of definitions and theorems without extensive comments or proofs.

Definition 4.1. *Denote by* 2^{n-1} *the category whose objects are subsets of*

$$P_{n-1} = \{0, 1, \dots, n-2\}$$

and whose morphisms are the inclusion maps between such objects. Let C *be a subcategory of the category of sets and maps. An* n*-ad in* C *is an intersection-preserving functor* $X : 2^{n-1} \to C$*. In other words, it satisfies the condition* $X(A \cap B) = X(A) \cap X(B)$ *for all* $A, B \in 2^{n-1}$*. If* A *is a subset of* P_n*, denote by* $||A||$ *the number of elements in* A*.*

In this section we will usually be taking C to be the category of CW complexes, topological spaces, or compact Cat manifolds. In these cases, a functor $X : 2^{n-1} \to C$ will be called a *CW n-ad*, a *topological n-ad*, or a *Cat manifold n-ad*.

Definition 4.2. *Let* $X : 2^{n-1} \to C$ *be an* n*-ad.*

1. *Define* $|X| = X(P_{n-1}) \in C$*.*
2. *If* A *is a subset of* P_{n-1}*, then define the* $(n - ||A||)$*-ad* $\partial_A X : 2^{n-||A||-1} \to C$ *by*

$$(\partial_A X)(B) = X(P_{n-||A||-1} - B)$$

for all $B \subseteq P_{n-||A||-1}$*. Also define* $|\partial_A X| = X(P_{n-||A||-1})$*.*

3. *If* $A = \{j\}$ *is a singleton we simply write* $\partial_j X$ *and* $|\partial_j X|$*.*

4. *A morphism* $f : X \to Y$ *of n-ads is a map* $|f| : |X| \to |Y|$ *which for every subset* $A \subseteq P_{n-1}$ *restricts to a map* $|f|_A : |\partial_A X| \to |\partial_A Y|$.

Definition 4.3. *We can consider an n-ad to be a set with* $n-1$ *preferred subsets* $\partial_j X$, *called the* j*-th preferred subset (even though it is a functor). The n-ad X can be notated* $(X; \partial_0 X, \partial_1 X, \dots, \partial_{n-2} X)$. *This collection of data determines all* $\partial_A X$ *by the formula* $(\partial_A X)(B) = \bigcap_{j \in A} (\partial_j X)(B)$.

Example 4.4. *For example, the 5-ad* $X(\{1,2,3,4\})$ *has the preferred subsets* $X(\{1,2,3\})$, $X(\{1,2,4\})$, $X(\{1,3,4\})$, *and* $X(\{2,3,4\})$.

Definition 4.5. *We use this notation to define the Cartesian product of an m-ad X and an n-ad Y to form the* $(m+n-1)$*-ad*

$$(X \times Y; X \times \partial_0 N, \dots, X \times \partial_{n-2} Y, \partial_0 X \times Y, \dots, \partial_{m-2} X \times Y).$$

Example 4.6. *For example, one can use the framework of n-ads to construct the smooth cobordism spectrum* MO_n *quite geometrically. It is a complex made in the following way. A vertex in* MO_n *is a smooth n-manifold. An edge between two vertices is the cobordism between two smooth n-manifolds. In general, a k-simplex in* MO_n *is a smooth manifold k-ad. We will see that* $\pi_k(MO_n)$ *is an* $(n+k)$*-dimensional bordism. (Note that smoothness is quite inessential to the definition, and one can, for example, build* MPL *the same way.)*

Similarly $MO_n(X)$ would be made of manifold k-ads mapping to X. We now give standard notions and properties of n-dimensional simplicial complexes.

Definition 4.7. *Let* $V = \{v_0, \dots, v_n\}$ *be a collection of* $n+1$ *points in some Euclidean space such that the vectors* $v_1 - v_0, \dots, v_n - v_0$ *are linearly independent. The convex set spanned by these points is called the (geometric) n-simplex spanned by* $v_0, v_1, \dots, v_n$. *The points* v_i *are vertices of it and the convex set spanned by a nonempty subset A of V is called a face of A.*

Definition 4.8. *A (geometric) simplicial complex X in* $\mathbb{R}^n$ *is a collection* $\mathcal{A}$ *of simplices in* $\mathbb{R}^n$, *perhaps of varying dimensions, such that (1) every face of a simplex of X belongs to* $\mathcal{A}$; *(2) the intersection of any two simplices of X is a face of each of them. Any simplex in* $\mathcal{A}$ *can be expressed as* $[v_{i_0}, \dots, v_{i_k}]$ *in terms of its vertices.*

Definition 4.9. *An abstract simplicial complex is a collection X of sets* $\{X_i\}_{i \in \mathbb{Z}_{\geq 0}}$ *such that (1) for all* $k \in \mathbb{Z}_{\geq 0}$ *the set* X_k *consists of subsets of* X_0 *of cardinality* $k+1$; *(2) any* $(j+1)$*-element subset of an element of* X_k *is an element of* X_j. *The elements of* X_0 *are called vertices and* X_0 *is usually regarded as a collection of points.*

Definition 4.10. *In either the geometric or abstract setting above, if we submit V to a total ordering, we can make X an ordered simplicial complex with the condition that* $[v_{i_0}, \dots, v_{i_k}]$ *represents a simplex iff* $v_{i_0} < \cdots < v_{i_k}$ *and* $\{v_{i_0}, \dots, v_{i_k}\}$ *is an element of* X_k. *The notion of an ordered n-simplex should be clear.*

Definition 4.11. *Let X and Y be geometric simplicial complexes with vertex sets X_0 and Y_0, respectively. Then a map $f : X \to Y$ is a simplicial map if it satisfies the following conditions: (1) $f(X_0) \subseteq Y_0$; (2) if $[v_{i_0}, \dots, v_{i_k}]$ is a simplex of X, then $f(v_{i_0}), \dots, f(v_{i_k})$ are vertices (not necessarily distinct) of some simplex in Y; (3) f is determined by linear interpolation on each simplex; i.e. if $x \in [v_0, \dots, v_n]$ and $x = \sum_{j=1}^n t_j v_j$, then $f(x) = \sum_{j=1}^n t_j f(v_j)$.*

Definition 4.12. *If X is an ordered geometric simplicial complex, then we define the face maps $d_0, \dots, d_n : X_n \to X_{n-1}$ for each $n \geq 1$ given by*

$$d_j[v_{i_0}, \dots, v_{i_n}] = [v_{i_0}, \dots, \hat{v}_{i_j}, \dots, v_{i_n}]$$

for all n-simplices $[v_{i_0}, \dots, v_{i_n}]$ of X_n. Here the notation $\hat{v}_{i_j}$ means that the term v_i is omitted. Compositions of face maps are called generalized face maps.

Remark 4.13. *Note that there are certain relations satisfied by the face maps. In particular, if $i < j$, then $d_i \circ d_j = d_{j-1} \circ d_i$.*

We generalize this notion in the following definition.

Definition 4.14. *A Delta set (or Δ-set) X is a sequence $X = \{X_i\}_{i=0}^{\infty}$ of sets such that, for all $n \geq 0$ and $i \in \{0, \dots, n+1\}$, there are maps $d_i : X_{n+1} \to X_n$ satisfying $d_i \circ d_j = d_{j-1} \circ d_i$ whenever $i < j$. We will refer to such X as a Δ-set.*

Definition 4.15. *If X is a geometric simplicial complex, then a degenerate simplex has the form $[v_{i_0}, \dots, v_{i_k}]$ where $i_0 \leq \cdots \leq i_k$, where at least two adjacent indices i_j and i_{j+1} are the same. Given a degenerate or nondegenerate simplex $[v_{i_0}, \dots, v_{i_n}]$, we define the degeneracy maps by*

$$s_j[v_{i_0}, \dots, v_{i_n}] = [v_{i_0}, \dots, v_{i_j}, v_{i_j}, \dots, v_{i_n}]$$

for all $j \leq n$.

Definition 4.16. *A simplicial set is a sequence $X = \{X_i\}_{i \in \mathbb{Z}_{\geq 0}}$ together with, for each $n \geq 0$, functions $d_0, \dots, d_n : X_n \to X_{n-1}$ and $s_0, \dots, s_n : X_n \to X_{n-1}$ satisfying the following conditions:*

1. $d_i \circ d_j = d_{j-1} \circ d_i$ *if* $i < j$,
2. $d_i \circ s_j = s_{j-1} \circ d_i$ *if* $i < j$,
3. $d_j \circ s_j = d_{j+1} \circ s_j = id$,
4. $d_i \circ s_j = s_j \circ d_{i-1}$ *if* $i > j + 1$,
5. $s_i \circ s_j = s_{j+1} \circ s_i$ *if* $i \leq j$.

Remark 4.17. *We can cast these notions in the language of category theory.*

1. *Let $\widehat{\Delta}$ be the category whose objects are the finite ordered sets $[n] = \{0, 1, \dots, n\}$ and whose morphisms are strictly order-preserving functions $[m] \to [n]$. Then a Delta set is a covariant functor $X : \widehat{\Delta}^{op} \to Sets$, where the morphisms are now interpreted to be generalized face maps.*
2. *Let Δ be the category whose objects are the finite ordered sets $[n] = \{0, 1, \dots, n\}$ and whose morphisms are (weakly) order-preserving functions $[m] \to [n]$. A simplicial set is a covariant functor $X : \Delta^{op} \to Sets$.*

Definition 4.18. *Let $n \in \mathbb{Z}_{\geq 0}$. Then Δ^n is a simplicial set whose i-simplices are given by*

$$\Delta^n_i = \{[v_0, v_1, \dots, v_i] : 0 \leq v_0 \leq v_1 \leq \cdots \leq v_i \leq n\}$$

with maps $d_j : \Delta^n_j \to \Delta^n_{j-1}$ and $s_j : \Delta^n_j \to \Delta^n_{j+1}$ given by

$$\begin{aligned} d_j[v_0, v_1, \dots, v_i] &= [v_0, v_1, \dots, v_{j-1}, \widehat{v}_j, v_{j+1}, \dots, v_i] \\ s_j[v_0, v_1, \dots, v_i] &= [v_0, v_1, \dots, v_{j-1}, v_j, v_j, v_{j+1}, \dots, v_i] \end{aligned}$$

for all $j \geq 1$.

Definition 4.19. *For each $n \geq 0$, define*

$$\Delta_n = \left\{ (t_0, \dots, t_n) : 0 \leq t_i \leq 1 \text{ and } \sum_{i=0}^{n} t_i = 1 \right\} \subset \mathbb{R}^{n+1}.$$

Let $X = \{X_n\}$ be a simplicial set with maps d_i and s_j defined as above. Let each X_n be endowed with the discrete topology. Let Δ^n be the geometric n-simplex with the standard topology. The realization $|X|$ of X is given by the set

$$|X| = \left(\coprod_{n=0}^{\infty} (X_n \times \Delta_n) \right) / \sim$$

where $\sim$ is the equivalence relation generated by the relations

1. $(x, D_i(p)) \sim (d_i(x), p)$ *for all $x \in X_{n+1}$ and $p \in \Delta_n$;*
2. $(x, S_i(p)) \sim (s_i(x), p)$ *for all $x \in X_{n-1}$ and $p \in \Delta_n$.*

Here $D_i : \Delta_n \to \Delta_{n+1}$ and $S_i : \Delta_{n+1} \to \Delta_n$ are defined by

$$D_i[t_0, \dots, t_n] = [t_0, \dots, 0, \dots, t_n]$$

where the 0 is inserted into the i-th slot, and

$$S_i[t_0, \dots, t_{n+1}] = [t_0, \dots, t_i, t_i, \dots, t_n].$$

Define the topology on $|X|$ to be the quotient topology.

Definition 4.20. *Let Λ^n_k be the simplicial set that is the subcomplex of Δ^n with simplices $[i_0, \dots, i_m]$ where $0 \le i_0 \le \cdots \le i_m \le n$ such that (1) not all numbers $0, \dots, n$ appear and (2) $[i_0, \dots, \widehat{i}_k, \dots, i_m]$ does not appear. This Λ^n_k is often called a horn. If $k \le n$, then $|\Lambda^n_k|$ is the subcomplex of $|\Delta^n|$ obtained by removing the interior of $|\Delta^n|$ and the interior of the face $d_k(\Delta^n)$.*

Definition 4.21. *The simplicial object X satisfies the Kan property or Kan condition if any morphism $\Lambda^n_k \to X$ of simplicial sets can be extended to a morphism $\Delta^n \to X$ of simplicial sets. In such a case, we say that X is Kan or X is a Kan complex.*

Remark 4.22. *As X is technically a functor, the horn Λ^n_k is also a functor, and the map $\Lambda^n_k \to X$ can be considered a natural transformation of functors.*

Remark 4.23. *In the language of category theory, simplicial sets satisfying the extension condition of Kan are the fibrant objects in the model category of simplicial sets. The homotopy relation on maps $K \to L$ between simplicial sets is not an equivalence relation unless the target L is a Kan complex.*

Definition 4.24. *A path in a simplicial set X is a simplicial set J whose geometric realization is homeomorphic to an interval, together with a simplicial map $J \to X$. We say that two 0-simplices a and b of a simplicial set X are in the same path component if there is a path taking the two ends of the interval J to a and b.*

Proposition 4.25. *Let X be a simplicial set. Define a relation $a \sim b$ on 0-simplices of X if a is in the same path component of b. Then $\sim$ is an equivalence relation.*

Definition 4.26. *Two simplicial maps $f, g : X \to Y$ are homotopic if there is a simplicial map $H : X \times I \to Y$ such that $H(x, 0) = g(x)$ and $H(x, 1) = f(x)$ for all $x \in X$.*

Theorem 4.27. *Homotopy of paths $X \to Y$ is an equivalence relation if Y is a Kan complex.*

Notation 4.28. *We denote by $\pi_0(X)$ the collection of path components of X.*

Definition 4.29. *If X is a simplicial set, then A is a subcomplex of X if A is a simplicial set such that $A^n \subseteq X_n$ for all n and the face and degeneracy maps d_i and s_j of A agree with those from X. We denote by (X, A) the simplicial pair. Simplicial maps $(X, A) \to (Y, B)$ of pairs are simplicial maps $f : X \to Y$ such that $f(A) \subseteq B$. If A is a subcomplex of a simplicial set X, then (X, A) is called a Kan pair if both X and A satisfy the Kan condition.*

Definition 4.30. *Suppose that $(X, *)$ is a Kan complex with basepoint $*$. If $n \ge 1$, then the n-th homotopy group $\pi_n(X, *)$ is the set of homotopy classes of maps $(\partial\Delta^{n+1}, *) \to (X, *)$. The simplicial subset of Δ^{n+1} generated by $[0]$ will be considered the basepoint of $\partial\Delta^{n+1}$. All homotopies fix the basepoint. When $n = 1$, we call $\pi_1(X, *)$ the fundamental group of $(X, *)$.*

Theorem 4.31. *If X is a Kan complex, then there is a bijective correspondence between homotopy classes of maps $f\colon (\partial\Delta^{n+1}, *) \to (X, *)$ and homotopy classes of maps $g\colon (\Delta^n, \partial\Delta^n) \to (X, *)$. In particular, the homotopy groups $\pi_n(X, *)$ can be defined. More generally, there is a bijective correspondence $[K : X] \to [K : |X|]$ between combinatorial homotopy and the homotopy of spaces.*

4.2 THE SURGERY PROBLEM

As mentioned in the previous section, if $(f, b)\colon (M, \nu_M) \to (X, \eta)$ is a normal map and X in particular has boundary, then a normal cobordism between two such maps is a manifold with corners. Wall handles this issue by extending the terms in the surgery exact sequence in terms of n-ads, enabling the definition of various useful Δ-sets which will spacify the surgery exact sequence.

Definition 4.32. *Let $\mathbb{R}^\infty$ be the set of real sequences such that $x_i = 0$ for all but finitely many i; i.e. it is the colimit of the sequence $\mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}^3 \to \cdots$. Let the geometric j-simplex Δ^j be endowed with the standard topology. Let X be a compact oriented Cat manifold r-ad of dimension n. A Cat normal map of type j over X relative to $\partial_0 X$ consists of the following data:*

1. *A compact orientable Cat manifold $(j+r+1)$-ad M of dimension $n+j$, embedded in $\Delta^j \times \mathbb{R}^s$ for some s such that M has a normal Cat bundle in $\Delta^j \times \mathbb{R}^s$. In addition we assume the compatibility condition that $M \cap (\partial_k \Delta^j \times \mathbb{R}^s) = \partial_k M$ for all $k \in \{0, \dots, j\}$ and that $M \setminus \left(\bigcup_{k=0}^{j} \partial_k M\right) \subseteq \operatorname{int}(\Delta^j \times \mathbb{R}^s)$.*
2. *A degree one map $f\colon M \to X \times \Delta^j$ of manifold $(j+r+1)$-ads such that $\partial_{j+1} f\colon \partial_{j+1} M \to \partial_{j+1}(X \times \Delta^j)$ is a Cat isomorphism of $(j+r)$-ads.*
3. *A Cat bundle ξ over X and a map $b\colon \nu_M \to \xi$ of bundles covering f, where ν_M is the normal Cat bundle of M in $\Delta^j \times \mathbb{R}^s$:*

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & X \end{array}$$

An object above will be denoted by

$$M \to X \times \Delta^j \,:\, (b, \xi).$$

Then $\mathbb{N}^{Cat}(X \operatorname{rel} \partial_0 X)$, called the Δ-set of Cat normal maps over X relative to $\partial_0 X$, is the Δ-set whose j-simplices are normal maps of type j over X relative to $\partial_0 X$ and whose face maps are induced by the $(j+r+1)$-ad structure of each simplex.

Definition 4.33. *Let X be a compact oriented Cat manifold r-ad of dimension n. A Cat simple homotopy equivalence of type j over X relative to $\partial_0 X$ consists of*

1. *A Cat manifold M as in (1) above.*
2. *A simple homotopy equivalence $f : M \to X \times \Delta^j$ of $(j+r+1)$-ads such that $\partial_{j+1} f : \partial_{j+1} M \to \partial_{j+1}(X \times \Delta^j)$ is a Cat isomorphism of $(j+r)$-ads.*

Then $\mathbb{S}^{Cat}(X \operatorname{rel} \partial_0 X)$, called the Δ-set of homotopy Cat structures on X relative to $\partial_0 X$, is the Δ-set whose j-simplices are Cat simple homotopy equivalences of type j over X relative to $\partial_0 X$ and whose face maps are the obvious ones. The identity map on X determines a basepoint for $\mathbb{S}^{Cat}(X \operatorname{rel} \partial_0 X)$.

In other words, we can construct $\mathbb{S}^{Cat}(Y)$ as a simplicial complex. An n-simplex in $\mathbb{S}^{Cat}(Y)$ is a simple homotopy equivalence $M \to Y \times \Delta^n$ of n-ads from a Cat manifold n-ad M. The faces of the simplex are the restrictions of the simple homotopy equivalence to faces of the n-ads. In particular, a 0-simplex is a homotopy equivalence $M \to Y \times \Delta^0 = Y$, whose equivalence classes form the usual structure set $S^{Cat}(Y)$ of Y. Since the equivalence is described by the two ends of 1-simplices, i.e. homotopy equivalences $W \to Y \times \Delta^1$ of 1-ads, the usual structure set of Y can be identified with the set $\pi_0(\mathbb{S}(Y))$ of connected components of $\mathbb{S}(Y)$.

Theorem 4.34. *Both $\mathbb{N}^{Cat}(X \operatorname{rel} \partial_0 X)$ and $\mathbb{S}^{Cat}(X \operatorname{rel} \partial_0 X)$ are Kan complexes. In addition, there is a homotopy equivalence between $\mathbb{N}^{Cat}(X \operatorname{rel} \partial_0 X)$ and $(F/Cat)^{X/\partial X_0}$ as n-ads, i.e. the function space $(X, \partial X_0) \to (F/Cat, *)$.*

The proof is an n-ad version of the π_0-version of this statement. The crucial point is that, as Kan complexes, the homotopy groups of these n-adic spaces can be defined by Theorem 4.31. Consult Quinn's thesis [516] and Nicas [482] for details.

4.2.1 The Wall groups

We will now start to construct the L-groups in this general framework.

Definition 4.35. *Let B be a Top r-ad and $q, j \in \mathbb{Z}_{\geq 0}$. A Cat surgery problem of type (q, j) over B consists of the following objects.*

1. *Compact orientable Cat manifold $(j+r+2)$-ads M and X of dimension $q+j$, embedded in $\Delta^j \times \mathbb{R}^s$ for some s such that M has a normal Cat bundle in $\Delta^j \times \mathbb{R}^s$. In addition we assume the compatibility condition that $M \cap (\partial_k \Delta^j \times \mathbb{R}^s) = \partial_k M$ for all $k \in \{0, \dots, j\}$ and that $M \setminus \left(\bigcup_{k=0}^{j} \partial_k M \right) \subseteq \operatorname{int}(\Delta^j \times \mathbb{R}^s)$.*
2. *A degree one map $f : M \to X$ of $(j+r+2)$-ads such that $\partial_{j+1} f : \partial_{j+1} M \to \partial_{j+1} X$ is a simple homotopy equivalence of $(j+r+1)$-ads.*
3. *A Cat bundle ξ over X and a map $b : \nu_M \to \xi$ of bundles covering f, where ν_M*

is the normal Cat bundle of M in $\Delta^j \times \mathbb{R}^s$:

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & X \end{array}$$

4. *A continuous map $h: X \to B \times \Delta^j$ of $(j+r+2)$-ads, called the reference map, such that*

$$\begin{array}{ccc} X & \xrightarrow{i} & \Delta^j \times \mathbb{R}^\infty \\ {\scriptstyle h}\downarrow & & \downarrow \\ B \times \Delta^j & \longrightarrow & \Delta^j \end{array}$$

commutes. Here the space $B \times \Delta^j$ is given the n-ad product structure discussed in Definition 4.3.

Definition 4.36. *Let M^{n+j} and X^{n+j} be $(j+r+2)$-ads and B be an r-ad. The notation*

$$M \xrightarrow{f} X \xrightarrow{h} B \times \Delta^j \ : (b, \xi)$$

is used for a Cat surgery problem of type $(q, j-1)$ over B with the given b and ξ. If x is such a Cat surgery problem of type $(q, j-1)$, then for all $k \in \{0, \dots, j\}$ the notation $\partial_k x$ will denote the Cat surgery problem of type $(q-1, j)$ over b given by restriction

$$\partial_k M \xrightarrow{\partial_k f} \partial_k X \xrightarrow{\partial_k h} B \times \partial_k \Delta^j \ : (b|_{\partial_k M}, \xi|_{\partial_k X})$$

over the k-th face.

Definition 4.37. *If B is a Top CW r-ad, then the surgery space $\mathbb{L}_q(B)$ is the pointed Δ-set whose j-th space $\mathbb{L}_q^j(B)$ is the set of all Cat surgery problems of type (q, j) over B. The basepoint ϕ_j is the empty surgery problem over B. The face maps are given by the ∂_k with $k \in \{0, 1, \dots, j\}$, as defined previously.*

If $f: X_1 \to X_2$ is a map of CW r-ads, we can obtain a map $\mathbb{L}_q(f): \mathbb{L}_q(X_1) \to \mathbb{L}_q(X_2)$ of pointed Δ-spaces by composing f with the reference map of each simplex of $\mathbb{L}_q(X_1)$, i.e.

$$X \xrightarrow{h} X_1 \times \Delta^j \xrightarrow{f} X_2 \times \Delta^j.$$

Orientation reversal gives a basepoint-preserving map $i: \mathbb{L}_q(B) \to \mathbb{L}_q(B)$. We denote by $-x$ the Cat surgery problem of type (q, j) over B obtained by reversing the orientations of M and X. The map $j: \mathbb{L}_q(B) \to \mathbb{L}_q(B)$ given by $x \mapsto -x$ on all vertices and faces will be called the *inverse map*. The maps ∂_k of Definition 4.36 define a basepoint-preserving map $\partial_k: \mathbb{L}_q(B) \to \mathbb{L}_{q-1}(\partial_k B)$. Disjoint union endows $\mathbb{L}_q(B)$

with an H-space structure. This construction makes $\mathbb{L}_q$ a covariant functor from the category of CW r-ads to the category of H-spaces and homomorphisms.

Some important properties of these L-groups are listed below. Recall that it is possible to find the homotopy groups $\pi_n(X, *)$ for a basepointed Kan complex $(X, *)$. The following is actually quite straightforward and not so different from the Wall Chapter 9 definition of L-groups.

Theorem 4.38. *Let B be a CW r-ad and let $\emptyset$ be the zero surgery problem. Then we have the following.*

1. *The space $\mathbb{L}_q(B)$ is Kan complex for all $q \geq 0$.*
2. *The map $e_q : \mathbb{L}_q(B) \to \Omega\mathbb{L}_{q-1}(B)$ is a homotopy equivalence for all $q \geq 1$, where $\Omega\mathbb{L}_{q-1}(B)$ is the loop space of $\mathbb{L}_{q-1}(B)$.*
3. *If $i : \mathbb{L}_q(B) \to \mathbb{L}_q(B)$ is the map that reverses the orientation on the domain and range, then the induced $i_* : \pi_j(\mathbb{L}_q(B), \emptyset) \to \pi_j(\mathbb{L}_q(B), \emptyset)$ coincides with the group inverse map for all $j \geq 0$.*
4. *Suppose that $j+q-r \geq 5$. Then there is a functorial isomorphism $\pi_j(\mathbb{L}_q(B), \emptyset) \cong L_{q+j}(\mathbb{Z}[\pi_1(B)])$, where the latter is the Wall L-group of the group r-ad $\pi_1(B)$.*
5. *For $q - r \geq 5$, there is a natural homotopy equivalence $\theta_k : \mathbb{L}_q(B) \to \mathbb{L}_{q+4k}(B)$ defined by taking products with the k-fold Cartesian product of $\mathbb{CP}^2$.*

If X is an n-ad, then $\partial_j X$ is naturally an $(n-1)$-ad whose faces are the intersections of $\partial_j X$ with the other faces of X.

4.2.2 The surgery exact sequence

We review our ingredients. Let X be a Poincaré n-ad of dimension m such that $\partial_0 X$ is a Cat manifold $(n-1)$-ad. If X is an n-ad, let $\gamma_0 X$ be the $(r-1)$-ad obtained from X by deleting the first face $\partial_0 X$.

1. Let $\mathbb{S}^{Cat}(X \text{ rel } \partial_0 X)$ be the Δ-set whose k-simplices are homotopy equivalences $M \to X \times \Delta^k$ of $(n+k+2)$-ads from a Cat manifold M, where $\partial_{k+3} M \to \partial_0 X \times \Delta^k$ is a Cat isomorphism.
2. Under the same conditions, define $\mathbb{N}^{Cat}(X \text{ rel } \partial_0 X)$ to be the Δ-set whose k-simplices are degree one normal maps of $(n+k+2)$-ads $M \to X \times \Delta^k$ where $\partial_{k+3}(M) \to \partial_0 X \times \Delta^k$.
3. Since a homotopy equivalence is a degree one normal map, there is a natural forgetful map $\mathbb{S}^{Cat}(X \text{ rel } \partial_0 X) \to \mathbb{N}^{Cat}(X \text{ rel } \partial_0 X)$.
4. Since a Cat isomorphism is a (simple) homotopy equivalence, it follows that there is a natural map $\mathbb{N}^{Cat}(X \text{ rel } \partial_0 X) \to \mathbb{L}_m(\gamma_0 X)$.

For more information, see Quinn [516].

We can assemble all of this information and arrive at the *surgery exact sequence for Poincaré n-ads.*

Theorem 4.39. *Let X be a Poincaré n-ad satisfying the conditions above. When* $n+m \geq 4$*, the sequence*

$$\mathbb{L}_{m+1}(\gamma_0 X) \to \mathbb{S}^{Cat}(X \text{ rel } \partial_0 X) \to \mathbb{N}^{Cat}(X \text{ rel } \partial_0 X) \to \mathbb{L}_m(\gamma_0 X)$$

is a series of homotopy fibrations; i.e. any three consecutive terms form a fibration. So the map $\mathbb{S}^{Cat}(X \text{ rel } \partial_0 X) \to \mathbb{N}^{Cat}(X \text{ rel } \partial_0 X)$ *is a homotopy principal* $\Omega(\mathbb{L}(\gamma_0 X))$*-fibration. To prove that it is a fibration, one can construct the action of* $\mathbb{L}_{m+1}(\gamma_0 X) \simeq \Omega(\mathbb{L}_m(\gamma_0 X))$ *on* $\mathbb{S}^{Cat}(X \text{ rel } \partial_0 X)$*.*

The same formal considerations prove that $\mathbb{S}^{Cat}(X \text{ rel } \partial_0 X) \to \mathbb{N}^{Cat}(X \text{ rel } \partial_0 X)$ is a homotopy principal $\mathbb{L}_{m+1}(\gamma_0 X)$ fibration.

In fact, the space $\mathbb{S}^{Cat}(X \text{ rel } \partial_0 X)$ can be identified with the homotopy fiber over the 0-component of the geometrically defined surgery obstruction map $F : \mathbb{N}^{Cat}(X \text{ rel } \partial_0 X) \to \mathbb{L}_m(\gamma_0 X)$.

Remark 4.40. *The n-ads* $\mathbb{S}^{Cat}(X)$*,* $\mathbb{N}^{Cat}(X)$*, and* $\mathbb{L}_m(\gamma_0 X)$ *are the spacifications of* $\mathcal{S}^{Cat}(X)$*,* $\mathcal{N}^{Cat}(X)$*, and* $L_m(\mathbb{Z}[\pi_1(X)])$ *which we mentioned in the introduction to the chapter. These spacifications are critical to a modern view of surgery.*

4.3 BLOCKED SURGERY

4.3.1 Bundles and block bundles

The ideas of this section are due to Casson [144] and Quinn [516].

Definition 4.41. *A block bundle on a simplicial complex P with fiber Y is a space E, and an assignment of a subspace* $E_\sigma \subset E$ *to each simplex* σ *of P, such that*

1. $E = \bigcup_{\sigma \in P} E_\sigma$ *and* $E_\sigma \cap E_\tau = E_{\sigma \cap \tau}$*;*
2. *for each* σ*, there is a Cat homeomorphism* $\phi_\sigma : E_\sigma \to \sigma \times Y$ *such that* $\phi_\sigma(E_\tau) = \tau \times Y$ *for all faces* τ *of* σ*.*

The first condition means that the *total space* E is the union of *blocks* E_σ, and that the blocks are related in the same way as the simplices of P. The condition implies that $E_\tau \subset E_\sigma$ for all faces τ of σ. The second condition means that each block E_σ has a *trivialization chart* ϕ_σ respecting the faces of σ. In particular, the restriction of ϕ_σ to a face τ gives a homeomorphism $\phi_\sigma|_\tau : E_\tau \to \tau \times Y$. Comparing this map with the chart of the smaller block $\phi_\tau : E_\tau \to \tau \times Y$, we obtain a homeomorphism

$$\phi_{\tau\subset\sigma} = \phi_\sigma \circ \phi_\tau^{-1} : \tau \times Y \to \tau \times Y.$$

The homeomorphisms are similar to the transition maps between charts of fiber bundles, and can also be regarded as *glueing maps* for reconstructing the total space E from the blocks E_σ.

If Y is a Cat manifold and the glueing maps are also in the Cat category, we have block bundles in the smooth, piecewise linear, or topological category.

Remark 4.42. *Suppose that $p : E \to P$ is a fiber bundle. Then for each simplex σ of P, the restriction of the bundle $p_\sigma : E_\sigma = p^{-1}(\sigma) \to \sigma$ is a trivial bundle. In other words, a bundle is a block bundle. However, not all block bundles are fiber bundles; we do not require all points to have copies of F above them. Consequently, the map $\phi_{\tau\subseteq\sigma}$ is not necessarily a fiberwise homeomorphism. Still, we will denote block bundles as $E \to P$, with the understanding that $\to$ may only mean "over" instead of "map." Note however that there is a canonical homotopy class of maps from E to P.*

The block bundle is a simplicial version of the usual notion of the bundle. Naturally we would like to have a similar bundle theory such as pullbacks, homotopy lifting properties, and classifying spaces. These ideas are developed in Rourke-Sanderson [557–559]. In Appendix B, we point out their centrality in PL topology. The ingredients of block surgery are as follows.

1. As expected, there are notions of block bundle maps, isomorphic block bundles, pullbacks, and the homotopy uniqueness of pullbacks.
2. There are classifying spaces for block bundles with respect to a particular fiber, and the block bundle over a given base is classified by maps to the classifying space.

In summary, a block bundle E over P means

1. a triangulation of P,
2. a decomposition of P into blocks E_σ corresponding to simplices in P such that $E_\sigma = \sigma \times Y$ for all simplices σ in the triangulation of P, and
3. the condition that intersections of blocks and the isomorphisms of blocks with $\sigma \times Y$ are all compatible with the intersection of underlying simplices.

There are subdivision operations that allow us to pass from one triangulation of P to a finer one and maintain the same base. We now define the classifying space $B\widetilde{Aut}(Y)$ for block bundles with fiber Y.

Definition 4.43. *Let Y be a manifold. Then the classifying space for block bundles with fiber Y is a Δ-complex $B\widetilde{Aut}(Y)$ in which an n-simplex is a space P modelled* *like the standard simplex Δ^n, i.e. an n-ad, together with a stratified homeomorphism $P \to Y \times \Delta^n$, preserving all the face structures. The total space EY of the classifying block bundle is the union of all these spaces P.*

From a simplicial map $Z \to B\widetilde{Aut}(Y)$, one assembles the image simplices to be a block bundle over Z, and conversely a block bundle over Z can be considered a map $Z \to B\widetilde{Aut}(Y)$.

4.3.2 Block structures

Definition 4.44. *Let E and E' be block bundles over P. A blockwise simple homotopy equivalence is a map $f : E' \to E$ that preserves the blocks for which the restrictions $f_\sigma : E'_\sigma \to E_\sigma$ to the blocks σ are simple homotopy equivalences. Two blockwise homotopy equivalences are homotopic if they can be extended to become the two ends of a blockwise homotopy equivalence $E' \times [0,1] \to E \times [0,1]$ over $P \times [0,1]$.*

Suppose that both block bundles $E \to P$ and $E' \to P$ have a Cat manifold fiber Y. If σ is a simplex of P, then the chart $\phi_\sigma : E_\sigma \to \sigma \times Y$ gives a homotopy equivalence $\phi_\sigma \circ f_\sigma : E'_\sigma \to \sigma \times Y$ that compatibly restricts to the faces of σ.

Definition 4.45. *Suppose that $E \to P$ is a block bundle with manifold fiber. The blocked structure set $S^{blk}(E \to P)$ is the equivalence classes of blockwise simple homotopy equivalences $E' \to E$ from block bundles $E' \to P$ with manifold fiber. Here $E'_1 \to E$ and $E'_2 \to E$ are equivalent if there is a blockwise homeomorphism $E'_1 \to E'_2$, i.e. a homeomorphism of the total space preserving the blocks, such that $E'_1 \to E$ and $E'_1 \to E'_2 \to E$ are homotopic.*

In the following, we let the surgery space $\mathbb{S}(Y)$ be as defined in Definition 4.33. Again, we suppress the superscript Cat since the discussion holds for all manifold categories.

We can analyze the blocked structure set in the special case when $E \to P$ is a trivial block bundle $E \cong P \times Y$. Then we have charts ϕ_σ such that all the glueing maps are identity maps. For a blockwise homotopy equivalence $f : E' \to E$, we have the map $P \to \mathbb{S}(Y)$ sending $\sigma \in P$ to the simplex $[\phi_\sigma \circ f_\sigma : E'_\sigma \to \sigma \times Y] \in \mathbb{S}(Y)$. The triviality of the glueing map means that the image $[\phi_\tau \circ f_\tau]$ of a face τ of σ is the τ-face of $[\phi_\sigma \circ f_\sigma]$. The composition $\phi_\sigma \circ f_\sigma$ is a simplex of the same dimension as σ in the structure space $\mathbb{S}(Y)$. Therefore the map $P \to \mathbb{S}(Y)$ is a simplicial map.

Conversely, a map $P \to \mathbb{S}(Y)$ is a compatible collection $g_\sigma : E'_\sigma \to \sigma \times Y$ of simplices in $\mathbb{S}(Y)$ indexed by simplices of P. The compatibility means that the blocks can be glued together to obtain the total space $E' = \bigcup_{\sigma \in P} E'_\sigma$ of a potential block bundle $E' \to P$. The only problem is to show that the blocks E'_σ are homeomorphic to $\sigma \times Y'$ and respect the faces of σ. This issue is solved precisely by the higher version of the s-cobordism theorem. In other words, if we insist all the homotopy equivalences are simple homotopy equivalences, then we can find homeomorphisms $E'_\sigma \cong \sigma \times Y'$

that are compatible with all faces, and we obtain a one-to-one correspondence between blockwise simple homotopy equivalences $E' \to E$ and maps $P \to \mathbb{S}(Y)$, with the simple homotopy version of the structure space $\mathbb{S}(Y)$. Moreover, the equivalence between two blockwise homotopy equivalences can be interpreted as a homotopy equivalence between two maps $P \to \mathbb{S}(Y)$. Therefore the blockwise structure set can be identified with the homotopy classes of maps from P to the structure space of the fiber:

$$\mathcal{S}^{blk}(P \times Y \to P) = [P : \mathbb{S}(Y)].$$

Note that the terms of usual surgery exact sequence appear neatly in the second place of each term above, since $\mathbb{N}^{Cat}(B \times Y) \cong \text{Maps}(B : \mathbb{N}(Y)$.

In the event that a map $p : E \to P$ is not the trivial bundle, we can still discuss blocked surgery for E. We may construct an associated "block bundle" $E(\mathbb{S}(Y))$ over B. For each simplex σ in B, the corresponding block $E(\mathbb{S}(Y))_\sigma$ is simply the structure space $\mathcal{S}(E_\sigma)$. For each pair $\tau \subseteq \sigma$ of faces, we have the map $\mathcal{S}(E_\sigma) \to \mathcal{S}(E_\tau)$ that restricts the n-ad structure to the corresponding face. The total space $E(\mathbb{S}(Y))$ is then obtained by assembling all the blocks $E(\mathbb{S}(Y))_\sigma$ together. The total space can be considered as a space over P, except that we do not have the simultaneous trivialization of the blocks $E(\mathbb{S}(Y))_\sigma$, i.e. an identification with the product $\mathbb{S}(Y) \times \sigma$. Still, we have the space of sections $\text{Sect}(E(\mathbb{S}(Y)) \to P)$ and the bijective correspondence

$$\mathcal{S}^{blk}(E \to P) \longrightarrow \text{Sect}(E(\mathbb{S}(Y)) \to P).$$

4.3.3 Blocked surgery exact sequence

Suppose that P is a polyhedron and $p : E \to P$ is a blocked Poincaré complex over P in the sense that the inverse image of vertices are Poincaré complexes, the inverse images of edges are cobordisms of Poincaré complexes, and in general the inverse images of k-simplices are n-ads of Poincaré complexes. Then E would be a blocked Poincaré complex over P.

Once the structure set $\mathcal{S}^{blk}(E \to P)$ is identified with $[P : \mathbb{S}(Y)]$ for trivial block bundles, the homotopy theory can be applied. As usual we can define $\mathbb{L}(Y)$ by surgery problems

$$\begin{array}{ccccc} \partial M & \longrightarrow & \partial N & & \\ \downarrow & & \downarrow & & \\ M & \longrightarrow & N & \longrightarrow & Y \end{array}$$

with reference to Y such that the map $\partial M \to \partial W$ is already a simple homotopy equivalence. Any two such classes are identified if there is cobordism of the form

$$\begin{array}{ccccc} \partial V & \longrightarrow & \partial W & & \\ \downarrow & & \downarrow & & \\ V & \longrightarrow & W & \longrightarrow & Y \times I \end{array}$$

between the two, where V and W are appropriate manifolds with corners.

Theorem 4.46. *(Blocked surgery for general block bundles) Suppose that P is a polyhedron and $p\colon\ E \to P$ is a blocked Poincaré complex over P with k-dimensional fiber. Then there is a blocked surgery L-spectrum $\mathbb{L}_n^{blk}(E \to P)$ such that*

$$\mathbb{L}_{k+1}^{blk}(E \to P) \to \mathbb{S}^{blk}(E \to P) \to \mathbb{N}^{blk}(E \to P) \to \mathbb{L}_k^{blk}(E \to P)$$

is an exact sequence of pointed spaces; i.e. any three consecutive spaces form a fibration. In fact, if there is $E \to Y$ such that all fibers $Z \to Y$ induce an isomorphism on fundamental group, then the result also holds. Additionally $S^{blk}(E \to P) =$ $\mathrm{Sect}(P, S^{fiber}(E))$, where $S^{fiber}(E)$ fits into the sequence $\mathbb{S}(Y) \to S^{fiber}(E) \to P$. Analogously we have $\mathbb{L}^{blk}(E \to P) = \mathrm{Sect}(P, L^{fiber}(E))$ and quite trivially $\mathbb{N}^{blk}(E \to P) = \mathbb{N}^{Cat}(E) = \mathrm{Sect}(P, N^{fiber}(E))$.

Theorem 4.47. *(Blocked surgery sequence for trivial block bundles) Let Y^k be a k-dimensional Cat manifold, and suppose that P is a polyhedron and $p\colon\ P \times Y \to P$ is a trivial block bundle. There the identifications*

$$\mathbb{L}_n^{blk}(P \times Y \to P) = [P:\ \mathbb{L}_n(Y)]$$

and

$$\mathbb{N}^{blk}(P \times Y \to P) = [P \times Y:\ F/Cat] = [P:\ [Y:\ F/Cat]].$$

The blocked surgery exact sequence then becomes

$$\cdots \to [P:\ \mathbb{L}_{k+1}(Y)] \to [P:\ \mathbb{S}(Y)] \to [P:\ [Y:\ F/Cat]] \to [P:\ \mathbb{L}_k(Y)].$$

It is the induced map on function spaces of the fibration $\mathbb{S}(Y) \to [Y:\ F/Cat] \to \mathbb{L}_k(Y)$.

4.4 SURGERY SPECTRA

4.4.1 The surgery exact sequence in spacified form

Since F/Top is a component of $\Omega^4(F/Top)$, the space $[X:\ F/Top]$ obtains an additive structure that is quite different from the one coming from Whitney sum. It can be shown that with this H-space structure the surgery obstruction map $[X:\ F/Top] \to L_n(\mathbb{Z}[\pi])$

is a homomorphism. In fact, the surgery set $\mathcal{S}^{Top}(X)$ can be given an abelian group structure, and the surgery exact sequence a sequence of groups and homomorphisms.

This process requires us to extend the definition of $\mathcal{S}^{Top}(X)$ outside the setting of manifolds, so that one has a series of functors $\mathcal{S}^{Top}(X)$, with $\mathcal{S}^{Top}_{n+1}(X) = \mathcal{S}^{Top}(X)$ when X is a closed oriented n-manifold. For simplicity we concentrate on the oriented case.

Now $\mathcal{S}_n$ becomes covariantly functorial, and the normal invariants become homological. In other words, when X is an oriented n-manifold, there is a Poincaré duality isomorphism $[X : F/Top] \to H_n(X; \mathbb{L}_\bullet)$ for some homology theory.

Definition 4.48. *If $q \in \mathbb{Z}$, an Ω-spectrum $\mathbb{E}_\bullet$ is q-connective if $\pi_n(\mathbb{E}_\bullet) = 0$ for all $n \le q-1$. A q-connective cover of an Ω-spectrum $\mathbb{E}_\bullet$ is a q-connective Ω-spectrum $\mathbb{E}\langle q\rangle_\bullet$ with a map $\mathbb{E}\langle q\rangle_\bullet \to \mathbb{E}_\bullet$ inducing isomorphisms $\pi_n(\mathbb{E}\langle q\rangle_\bullet) \cong \pi_n(\mathbb{E}_\bullet)$ for all $n \ge q$. One can obtain $\mathbb{E}\langle q\rangle_\bullet$ by killing the homotopy groups $\pi_n(\mathbb{E}_\bullet)$ for $n \le q-1$ using Postnikov decompositions and Eilenberg-MacLane spectra.*

The most desirable homology theory to use is $L_*(\mathbb{Z}[e])$, but it is not quite right because that spectrum is periodic, and F/Top is only one component of $\Omega^4(F/Top)$. For the classification of topological manifolds, one has to use $\mathbb{L}^{\langle 1\rangle}_\bullet(\mathbb{Z}[e])$, the 1-connective version of the spectrum $\mathbb{L}_\bullet(\mathbb{Z}[e])$. We will denote them by $\mathbb{L}^{\langle 1\rangle}_\bullet$ and $\mathbb{L}_\bullet$.

In this subsection we will describe the surgery exact sequence in spacified form. Afterwards, we will give a geometric approach to the periodicity which gives an abelian group structure to $\mathcal{S}^{Top}(X)$. In fact, this group structure can be used to define the surgery exact sequence. We will also offer a more direct approach to the sequence that directly gives functoriality and an abelian group structure without using periodicity.

The spectrum $\mathbb{L}_\bullet = \mathbb{L}_\bullet(\mathbb{Z}[e])$ is often called the *periodic spectrum* and $\mathbb{L}^{\langle 1\rangle}_\bullet = \mathbb{L}^{\langle 1\rangle}_\bullet(\mathbb{Z}[e])$ is called the *connective spectrum*. There is a map $j : \mathbb{L}^{\langle 1\rangle}_\bullet \to \mathbb{L}_\bullet$ that induces an isomorphism on π_i when $i \ge 1$ and $\pi_i(\mathbb{L}_\bullet) = 0$ otherwise. As a result we have a composition

$$A : H_n(X; \mathbb{L}^{\langle 1\rangle}_\bullet) \xrightarrow{j_*} H_n(X; \mathbb{L}_\bullet) \to L_n(\mathbb{Z}[\pi]).$$

This A is called the *assembly map for X*, which we will discuss further in the next section. The usual surgery map $\sigma : [M : F/Top] \to L_n(\mathbb{Z}[\pi])$ can be factored by

$$\sigma : [M : F/Top] = H^0(M; \mathbb{L}^{\langle 1\rangle}_\bullet) = H_n(M; \mathbb{L}^{\langle 1\rangle}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi]).$$

If X is a Poincaré complex of dimension n, the map $H_k(X; \mathbb{L}^{\langle 1\rangle}_\bullet) \to H_k(X; \mathbb{L}_\bullet)$ is bijective for $k \ge n+1$ and injective for $k = n$. In addition there is a spectrum $\mathbb{H}_\bullet(X; \mathbb{L}^{\langle 1\rangle}_\bullet)$ whose n-th homotopy group is $H_n(X; \mathbb{L}_\bullet)$. If π is the fundamental group of X, then the homotopy equivalence produces a spectrum-level assembly map

$$\mathbb{A}_\bullet : \mathbb{H}_\bullet(X; \mathbb{L}^{\langle 1\rangle}_\bullet) \to \mathbb{L}_\bullet(\mathbb{Z}[\pi]).$$

Definition 4.49. *Let $\mathbb{A}_\bullet$ be the above map. Denote by $\mathbb{S}_\bullet^{Top}(X)$ the fiber of $\mathbb{A}_\bullet$. We call it the n-th Top structure space (spectrum) of X. Let the homotopy group $\pi_n(\mathbb{S}_\bullet^{Top}(M))$ be denoted by $\mathcal{S}_{n+1}^{Top}(M)$, called the algebraic structure set.*[1]

Theorem 4.50. *(Surgery exact sequence for Top manifolds) The 1-connective spectrum $\mathbb{L}_\bullet^{\langle 1\rangle}$ is appropriate for topological manifolds. Let X be a Poincaré complex with dimension $n \geq 5$ and fundamental group π. The spectrum exact sequence*

$$\mathbb{S}_\bullet^{Top}(X) \to \mathbb{H}_\bullet(X;\mathbb{L}_\bullet^{\langle 1\rangle}) \to \mathbb{L}_\bullet(\mathbb{Z}[\pi])$$

gives rise to a long exact sequence

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi]) \to \mathcal{S}_{n+1}^{Top}(X) \to H_n(X;\mathbb{L}_\bullet^{\langle 1\rangle}) \to L_n(\mathbb{Z}[\pi]),$$

where $\mathcal{S}_{n+1}^{Top}(X)$ is the topological structure set of X if it is nonempty.

Remark 4.51. *All of these results hold regardless of decoration s or h.*

Theorem 4.52. *(see Ranicki [544]) Let $n \geq 5$ and consider a closed Top n-manifold M^n with fundamental group π. The Sullivan-Wall geometric surgery exact sequence of M is isomorphic to the algebraic surgery exact sequence:*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & L_{n+1}(\mathbb{Z}[\pi]) & \longrightarrow & \mathcal{S}^{Top}(M) & \longrightarrow & [M : F/Top] & \longrightarrow & L_n(\mathbb{Z}[\pi]) \\
 & & \| & & {\scriptstyle s}\downarrow\cong & & {\scriptstyle t}\downarrow\cong & & \| \\
\cdots & \longrightarrow & L_{n+1}(\mathbb{Z}[\pi]) & \longrightarrow & \mathcal{S}_{n+1}^{Top}(M) & \longrightarrow & H_n(M;\mathbb{L}_\bullet^{\langle 1\rangle}) & \longrightarrow & L_n(\mathbb{Z}[\pi])
\end{array}$$

Furthermore, for all $i \geq 0$ we have the identifications

1. $\mathcal{S}^{Top}(M \times \mathbb{D}^i, M \times \mathbb{S}^{i-1}) = \mathcal{S}_{n+i+1}^{Top}(M)$,
2. $[(M \times \mathbb{D}^i, M \times \mathbb{S}^{i-1}) : (F/Top, *)] = H^{-i}(M;\mathbb{L}_\bullet^{\langle 1\rangle}) = H_{n+i}(M;\mathbb{L}_\bullet^{\langle 1\rangle})$.

Therefore $H_n(M;\mathbb{L}_\bullet^{\langle 1\rangle}) = [M : F/Top]$ is the bordism group of normal maps $(f,b) : N \to M$ of closed Top n-manifolds N to M.

Remark 4.53. *We would like to emphasize this very important point. The conceptual significance of replacing the normal invariant term $[M : F/Top]$ with with the homology term $H_n(M;\mathbb{L}_\bullet^{\langle 1\rangle})$ is the following. The usual surgery exact sequence $\mathcal{S}^{Top}(X) \to [X : F/Top] \to L_*(X)$ has a functoriality problem. The functors $\mathcal{S}^{Top}(\cdot)$ and $L_*(\cdot)$ are*

[1] Different conventions exist in the literature about the indexing of structure sets. One of the authors prefers indexing $\pi_n(\mathbb{S}^{Top}(M))$ as $\mathcal{S}_n^{Top}(M)$.

covariant with respect to codimension 0 inclusions, at least while $[\cdot : F/Top]$ *is contravariant. The correspondence* $[X : F/Top] = H^0(X; \mathbb{L}_\bullet^{\langle 1\rangle}) = H_n(X; \mathbb{L}_\bullet^{\langle 1\rangle})$ *described above for Top manifolds* M *gives a sequence*

$$\mathcal{S}^{Top}(X) \longrightarrow H_n(X; \mathbb{L}_\bullet^{\langle 1\rangle}) \longrightarrow L_n(X)$$

all of whose terms are covariant.

Theorem 4.54. *(Ranicki [536], see Kühl-Macko-Mole [376]) Let* X^n *be a (simple) Poincaré complex with* $n \geq 6$ *and fundamental group* π*. There is an algebraic surgery sequence given by the following:*

$$\cdots \to H_{n+1}(X; \mathbb{L}_\bullet^{\langle 1\rangle}) \to L_{n+1}(\mathbb{Z}[\pi]) \to \mathcal{S}_{n+1}^{Top}(X) \to H_n(X; \mathbb{L}_\bullet^{\langle 1\rangle}) \to \cdots$$

with the following properties:

1. *A Poincaré duality space* X *has a total surgery obstruction* $\mathcal{O}(X)$ *in* $\mathcal{S}_{n+1}^{Top}(X)$ *whose image in* $H_n(X; \mathbb{L}_\bullet^{\langle 1\rangle})$ *vanishes iff the Spivak fibration* $X \to B\pi$ *can be lifted to* $BTop$*.*
2. *The total surgery obstruction* $\mathcal{O}(X) \in \mathcal{S}_{n+1}^{Top}(X)$ *vanishes iff* X *is homotopy equivalent to compact Top n-manifold.*

One can think of the structure groups $\mathcal{S}_*^{Top}(X)$ as measuring the extent to which the surgery obstruction groups $L_*(\mathbb{Z}[\pi])$ fail to be a generalized homology theory, or equivalently the extent to which the algebraic L-theory assembly maps A fail to be isomorphisms.

Remark 4.55. *The space* $F/Top \times \mathbb{Z}$ *satisfies a 4-fold periodicity: there is a homotopy equivalence given by*

$$\Omega^4(F/Top \times \mathbb{Z}) \simeq F/Top \times \mathbb{Z}.$$

This periodicity can be seen as one halfway between real and complex Bott periodicity. The assembly map $A : H_n(X; \mathbb{L}_\bullet^{\langle 1\rangle}) \to L_n(\mathbb{Z}[\pi])$ *is 4-periodic. By naturality, it factors through* $H_n(B\pi; \mathbb{L}_\bullet^{\langle 1\rangle})$*.*

Remark 4.56. *In Section 8.6, we define a homology structure set* $\mathcal{S}^H(M)$ *for an n-dimensional closed homology manifold* M *with fundamental group* π *that fits into the exact sequence*

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi]) \to \mathcal{S}^H(M) \to H_n(M; \mathbb{L}_\bullet) \to L_n(\mathbb{Z}[\pi]).$$

In other words, the spectrum $\mathbb{L}_\bullet = \mathbb{L}_\bullet(\mathbb{Z}[e])$ *is appropriate for the classification of homology manifold structures. This sequence is isomorphic to its own fourth loop space.*

We close this section by summarizing the features of the L-spectrum.

1. Let $\mathbb{L}_\bullet$ be the 1-connected surgery spectrum whose zeroth space is F/Top. Quinn suggests that we write

$$[M : F/Top] = H^0(M; \mathbb{L}_\bullet^{\langle 1\rangle}) \cong H_n(M; \mathbb{L}_\bullet^{\langle 1\rangle}) \xrightarrow{A} L_n(\mathbb{Z}[\pi]).$$

2. Using the spectrum-level structure set $\mathbb{S}_\bullet$, one constructs a structure set $\mathcal{S}_*^{Top}(X)$ using quadratic Poincaré complexes in categories containing more information than just the fundamental group of X, from which one can construct the algebraic surgery exact sequence for any space X:

$$\cdots \to H_{n+1}(X; \mathbb{L}_\bullet^{\langle 1\rangle}) \xrightarrow{A} L_{n+1}(\mathbb{Z}[\pi]) \to \mathcal{S}_{n+1}^{Top}(X) \to H_n(X; \mathbb{L}_\bullet^{\langle 1\rangle}) \to \cdots$$

There is however a bijection $s : \mathcal{S}^{Top}(M) \to \mathcal{S}_{n+1}^{Top}(M)$ for n-manifolds with $n \geq 5$.

4.4.2 The "loc" notation

Suppose that M^n is a nonorientable manifold with $n \geq 5$ and fundamental group π. Let w be its orientation character. Then $\mathcal{S}^{Cat}(M)$ still fits into an ordinary surgery exact sequence

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi], w) \to \mathcal{S}^{Cat}(M) \to [M : F/Cat] \to L_n(\mathbb{Z}[\pi], w).$$

In the oriented case when Cat = Top, it made better functorial sense to Poincaré dualize and replace the normal invariants by a homological term $[M : F/Cat] \cong H_n(M; \mathbb{L}_\bullet^{\langle 1\rangle})$. Of course, Poincaré duality does not hold in its most straightforward form when M is not orientable. We correct it by twisting by the orientation local coefficient system, i.e. cohomology is dual to homology with local coefficients, e.g. $H^i(M; \mathbb{Z}) \cong H_{n-i}(M; H_n(M, M\backslash\{m\}; \mathbb{Z}))$.

Note that the group $H_n(M, M\backslash\{m\}; \mathbb{Z})$ is $\mathbb{Z}$ for any point $m \in M$. However, it is naturally a coefficient system that twists according to the orientation of M which near a point p is a generator of $H_n^{lf}(\mathcal{O})$, where $\mathcal{O}$ is any neighborhood of p homeomorphic to $\mathbb{R}^n$. If M' is the 2-fold oriented cover of M, then this twisted homology group is the homology of the chain complex $C^*(M') \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathbb{Z}$, where $\mathbb{Z}$ is viewed as a $\mathbb{Z}[\mathbb{Z}_2]$-module through w. It might be best to call this local coefficient group loc; therefore $H^i(M; \mathbb{Z}) = H_{n-i}(M; H_n(\text{loc}))$.

In general L-theory requires an orientation. A Poincaré complex requires as part of its structure a choice of fundamental class. If $f : M \to X$ is a degree one normal map, then by changing orientation on both the domain and range, we still have a degree one normal map, but their surgery obstructions are negatives of one another.

With this notion, we have the isomorphism $[M : F/Top] \cong H_n(M; \mathbb{L}_\bullet(\mathrm{loc}))$. The notation $H_n(M; \mathbb{L}_\bullet^w)$ would also convey the necessary information, and is slightly more interactive in that it encodes that the coefficient system is essentially "flat" with structure group $\mathbb{Z}_2$; the other notation however is more useful for certain generalizations.

We note that when there is room for confusion, e.g. when we consider $\mathcal{S}^{Top}(X)$ as a homotopy functor of X (and even introduce a subscript to keep track of the dimension), then it makes sense to pad the notation further to $\mathcal{S}^{Top}(M, w)$, where w is a real line bundle over M. In that case, when M is nonorientable, the group $\mathcal{S}^{Top}(M)$ would actually be isomorphic to $\mathcal{S}^{Top}(M, w)$, where w is the orientation bundle over M. It is unfortunate that in this case there are two meanings to the symbol $\mathcal{S}^{Top}(M)$, but it need not cause any problem if one is careful.

Remark 4.57. *It is also possible to consider $\mathcal{S}^{Top}(M, w)$ for non-trivial w even if M is oriented. This notion is valuable, because, for example, the Borel conjecture suggests that $H_n(B\pi; \mathbb{L}_\bullet^w) \to L_n(\mathbb{Z}[\pi], w)$, where the left-hand side is the w-twisted L-homology, should be an isomorphism when π is torsion-free. So even when $\pi = \mathbb{Z}$, we are tacitly dealing with "a nonorientable circle."*

The notation $H_n(M; \mathbb{L}_\bullet(\mathrm{loc}))$ can also be used when M is a Top manifold with boundary ∂M. We will discuss the cases when we compute the structure set (1) relative to the boundary, and (2) not relative to the boundary.

The Top structure set of M relative to ∂M fits into the sequence

$$\mathcal{S}^{Top}(M \,\mathrm{rel}\, \partial M) \to [(M, \partial M) : (F/Top, *)] \to L_n(M) \cong L_n(\mathbb{Z}[\pi_1(M)]).$$

The normal invariant term can be seen as $H_n(M, \mathbb{L}_\bullet(\mathrm{loc\,rel}\,\partial))$ where $\mathbb{L}_\bullet(\mathrm{loc\,rel}\,\partial)$ equals $\mathbb{L}_\bullet(\mathbb{Z}[e])$ for points in the interior of M and also with loc ∂ on ∂M. The notation $\mathbb{L}_\bullet(e)$ enforces that loc is uniform throughout the manifold (it is actually shorthand for $\mathbb{L}_\bullet(\mathbb{R}^m)$). Therefore the surgery exact sequence is given by the fibration

$$\mathcal{S}^{Top}(M \,\mathrm{rel}\, \partial M) \to H_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) \to L_n(M \,\mathrm{rel}\, \partial M).$$

In the non-relative case, the spectrum term is slightly different. When $x \in \partial M$, the spectrum term collapses. At each $x \in M$, we have

$$\mathbb{L}_\bullet(\mathrm{loc}) = \begin{cases} \mathbb{L}_\bullet(\mathcal{O}_x) = \mathbb{L}_\bullet(\mathbb{Z}[e]) & \text{if} \quad x \in M \backslash \partial M, \\ \mathbb{L}_\bullet(\mathcal{O}_x, \partial_+ \mathcal{O}_x) = \mathbb{L}_\bullet(e, e) = * & \text{if} \quad x \in \partial M. \end{cases}$$

Therefore the normal term is given by $H_*(M, \partial M; \mathbb{L}_\bullet^{\langle 1 \rangle})$. The structure set for pairs can be calculated by

$$\mathcal{S}^{Top}(M, \partial M) \to H_n(M, \partial M; \mathbb{L}_\bullet^{\langle 1 \rangle}) \to L_n(M, \partial M).$$

For example, Wall's π-π theorem asserts that, if $\pi_1(M) \cong \pi_1(\partial M)$, then $L_*(M, \partial M) =$

0, therefore giving an isomorphism

$$\mathcal{S}^{Cat}(M, \partial M) \cong H_*(M, \partial M; \mathbb{L}_\bullet^{\langle 1 \rangle}).$$

The point behind the loc notation is simply a way to take into account all the possible features of M. Given a Top manifold M (with or without boundary, orientable or not), we can effectively procure a spectrum $\mathbb{L}_\bullet$(loc) for which the surgery exact sequence holds. As seen above, the value of $\mathbb{L}_\bullet$ (loc) may be uniformly the same across all points of M, or may differ from point to point, accounting for the pieces in the manifold that serve very different purposes (e.g. boundary point versus interior point). We will discuss this notion more in Chapter 8. In fact, this description indicates the manner in which the stratified case in Section 8.7 can be understood. In particular, one generalizes $\mathbb{L}_\bullet$ suitably to include such spaces and then the rest of the theory more or less follows.

4.5 PERIODICITY OF STRUCTURE SETS

One of the most striking phenomena in surgery theory is the almost 4-fold periodicity in structure sets. In this section we follow Cappell-Weinberger [130] to explain the 4-periodicity of L-groups and structure sets from a geometric point of view. We will use this perspective to develop the assembly map and the surgery exact sequence. Later we will give a somewhat less geometric description that is more natural.

Although the statements are given for Top, for simplicity we will describe the proof as if we were in the PL category machinery using embedding theory, block bundles, and transversality to subpolyhedra. All of these tools have topological analogues. For example, topological locally flat embeddings can be classified via block bundles, and geometrically they have mapping cylinder structures. There is an adequate Top transversality theory for these purposes. However, these structures in Top are not particularly transparent, and so we will prove our results using PL tools, with the hope that the reader will accept that the Top theory works similarly.

The statement of Siebenmann periodicity as corrected by Nicas is given as follows.

Theorem 4.58. *(Siebenmann periodicity) If M is a manifold of dimension at least 5, then there is an exact sequence $0 \to \mathcal{S}(M) \to \mathcal{S}(M \times \mathbb{D}^4) \to \mathbb{Z}$. If the boundary of M is nonempty, then the image of $\mathcal{S}(M \times \mathbb{D}^4) \to \mathbb{Z}$ is 0.*

Remark 4.59. *We will later see that this $\mathbb{Z}$ has a nice interpretation in terms of an obstruction of Quinn to resolve homology manifolds.*

Definition 4.60. *Let E' and W be topological spaces and let $p\colon E' \to W$ be a surjection. We say that p is a topological branched covering if there is a nowhere dense set $M \subseteq W$ such that the restriction of p on $E = E' \backslash p^{-1}(M)$ is a covering space onto $W \backslash M$ and $p|_{p^{-1}(M)}$ is a homeomorphism. Then $W \backslash M$ is called the regular set and M*

is the singular set.

Construction 4.61. *Let M be a codimension two submanifold of a Top manifold W. Suppose that (1) there is a k-fold cyclic regular cover $p\colon E' \to W\backslash M$, (2) there is circle $\mathbb{S}^1$ meridionally linking M such that $p^{-1}(\mathbb{S}^1) \to \mathbb{S}^1$ coincides with the usual covering map $z \mapsto z^k$ as a function $\mathbb{S}^1 \to \mathbb{S}^1$. These conditions give us a class $\alpha \in H^1(W\backslash M; \mathbb{Z}_k)$ describing the k-fold cyclic cover, such that $\alpha|_{[\mathbb{S}^1]} = 1$ in $H^1(\mathbb{S}^1; \mathbb{Z}_k)$. (If $\alpha|_{[\mathbb{S}^1]} = 0$ then the circle lifts to multiple copies of itself and the branching is trivial.)*

Proposition 4.62. *With respect to the cover $p\colon E \to W\backslash M$ above, there is a topological space $\widehat{W}$ and a map of pairs $p'\colon (\widehat{W}, M) \to (W, M)$ such that*

1. *$E \subseteq \widehat{W}$ and $p'|_E = p$;*
2. *the $\mathbb{Z}_k$-action on E extends to a $\mathbb{Z}_k$-action on $\widehat{W}$;*
3. *$\mathbb{Z}_k$ acts on $\widehat{W}$ and $\widehat{W}/\mathbb{Z}_k = W$;*
4. *the fixed set $\mathrm{Fix}_{\mathbb{Z}_k}(\widehat{W})$ of the $\mathbb{Z}_k$-action on $\widehat{W}$ is exactly M (in fact, we can write $\mathrm{Fix}_{\mathbb{Z}_k}(\widehat{W}, M) = (W, M)$),*

as given in the diagram

$$\begin{array}{ccc} E & \xrightarrow{1-1} & \widehat{W} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ W\backslash M & \xrightarrow[1-1]{} & W \end{array}$$

Construction 4.63. *Consider the Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^3 \xrightarrow{H} \mathbb{S}^2$. Let M be a codimension three submanifold of a Top manifold W. Suppose that we have (1) a principal $\mathbb{S}^1$-bundle $p\colon E \to W\backslash M$, and (2) a sphere $\mathbb{S}^2$ linking M such that $p^{-1}(\mathbb{S}^2) \to \mathbb{S}^2$ coincides with the Hopf map $h\colon \mathbb{S}^3 \to \mathbb{S}^2$. Such a bundle is classified by a homotopy class of map from $W\backslash M$ to $B\mathbb{S}^1 = \mathbb{CP}^\infty = K(\mathbb{Z}, 2)$, i.e. an element $\alpha \in H^2(W\backslash M; \mathbb{Z})$. Condition (2) requires that $\alpha|_{[\mathbb{S}^2]} = 1$ in $H^2(\mathbb{S}^2; \mathbb{Z})$.*

Example 4.64. *We can cone the Hopf map H to arrive at a map $cH\colon \mathbb{D}^4 \to \mathbb{D}^3$. The preimage of the origin in $\mathbb{D}^3$ is the origin in $\mathbb{D}^4$, and the preimage of any other point in $\mathbb{D}^3$ is a circle in $\mathbb{D}^4$. Therefore $cH\colon \mathbb{D}^4 \to \mathbb{D}^3$ is an $\mathbb{S}^1$-branched covering space along the origin $M = \{0\}$. Note that M is of codimension 3 inside of $\mathbb{D}^3$, which is unique up to isotopy.*

The next theorem is discussed in Wall [672]. See also Weinberger [691] for several proofs.

Theorem 4.65. *(Browder, Casson, Haefliger, Sullivan, Wall) Suppose that M is a Top or PL manifold and let $(M', f) \in \mathcal{S}^{PL}(M)$. Then the composite $M' \xrightarrow{f} M \xrightarrow{i} M \times \mathbb{D}^3$ is homotopic to an embedding $e\colon M' \to M \times \mathbb{D}^3$.*

This theorem says that, if M' is (simple) homotopy equivalent to M then a $\mathbb{D}^3$-block bundle over M is (s-) h-cobordant to one that block fibers over M'. This statement follows from blocked surgery; the obstruction groups are all trivial by the π-π theorem.

Remark 4.66. *The caveat, which is irrelevant in the topological category because of controlled surgery (see Chapter 8), is that $\mathbb{D}^3$ is too low-dimensional for blocked surgery to work. This technical issue can be managed in various ways. The most elegant is perhaps to prove directly that F_k/PL_k stabilizes at $k = 3$ as a consequence of the Zeeman knotting theorem, i.e. that there is a unique PL embedding of $\mathbb{S}^n$ in $\mathbb{S}^{n+k}$ when $k \geq 3$.*

Remark 4.67. *A PL_c-bundle over $\mathbb{S}^n$ trivialized as a spherical fibration is abstractly isomorphic to $\mathbb{S}^n \times \mathbb{D}^c$ iff the normal invariant is trivial, i.e. if it is trivial in $\pi_n(F/PL)$. One can then glue it into $\mathbb{S}^{n+\ell}$, so one would potentially obtain an embedding violating Zeeman if stability were false.*

Construction 4.68. *Let $(M', f) \in \mathcal{S}^{Top}(M)$. By the theorem above, we can regard M' as a subset $M \times \mathbb{D}^3$ of codimension 3. Consider the $\mathbb{S}^1$-branched cover*

$$p = id \times cH : M \times \mathbb{D}^4 \to M \times \mathbb{D}^3.$$

Let $W = M \times \mathbb{D}^3$. Then p restricts to two principal $\mathbb{S}^1$-fibrations, one over $W \setminus M$ and one over $W \setminus M'$. The former recompactifies to the original branched cover $p = id \times cH : M \times \mathbb{D}^4 \to M \times \mathbb{D}^3$. The latter compactifies to an $\mathbb{S}^1$-branched cover $p' : E_{M'} \to M \times \mathbb{D}^3$ such that the $\mathbb{S}^1$-action on $E_{M'}$ has fixed set M'. Since M and M' are homotopy equivalent, we know that $E_{M'}$ is homotopy equivalent to $M \times \mathbb{D}^4$ via a map p_f. In addition, there is a homeomorphism $\partial E_{M'} \to M \times \mathbb{S}^3$. The correspondence $M' \mapsto E_{M'}$ gives a well-defined map $\mathcal{S}^{Top}(M) \to \mathcal{S}^{Top}(M \times \mathbb{D}^4)_{\text{rel}}$.

The following related construction will be used to understand why it is a periodicity map.

Construction 4.69. *Let $e : M' \to M \times \mathbb{D}^3$ be an embedding of M' in $M \times \mathbb{D}^3$. Identify the image $e(M')$ with M' itself as a subset of $M \times \mathbb{D}^3$. Let L be the complement in $M \times \mathbb{D}^3$ of the interior of a regular neighborhood of M'. Then L is an s-cobordism between ∂L and $\partial(M \times \mathbb{D}^3) = M \times \mathbb{S}^2$. Therefore L has a product structure. Since ∂L has a block (sphere) bundle structure over M', we can endow $M \times \mathbb{S}^2$ with an $\mathbb{S}^2$-block bundle structure over M'. Consider the $\mathbb{D}^2$ Hopf bundle given by*

$$\mathbb{D}^2 \to \mathbb{CP}^2_0 \to \mathbb{S}^2.$$

We then replace the $\mathbb{S}^2$-fibers in the bundle $M \times \mathbb{S}^2 \to M'$ with $\mathbb{CP}^2_0$, yielding a $\mathbb{CP}^2_0$-bundle which we call $q : E(\mathbb{CP}^2_0) \to M'$. In fact it is easy to see that $E(\mathbb{CP}^2_0)$ is home-

omorphic to $M \times \mathbb{CP}^2_0$*. We now have a homotopy commutative diagram:*

$$\begin{CD} E(\mathbb{CP}^2_0) @>{\cong}>> M \times \mathbb{CP}^2_0 \\ @V{q}VV @VV{pr_1}V \\ M' @>>{h}> M \end{CD}$$

The notion of regular neighborhood, strictly speaking, is only appropriate to the PL category. However, locally flat topological submanifolds do have closed mapping cylinder neighborhoods. Details for modifying the following discussion to work in Top can be based on Rourke-Sanderson [560] and Quinn [520]. See also Hutt [332], Crowley-Macko [183], and Weinberger-Xie-Yu [695]. With this construction we will sketch the main theorem.

Theorem 4.70. *The map* $M' \mapsto E_{M'}$ *gives an injection* $\mathcal{S}^{Top}(M) \to \mathcal{S}^{Top}(M \times \mathbb{D}^4)_{\text{rel}}$*. The cokernel is at most* $\mathbb{Z}$*.*

Proof. Let $(M', f) \in \mathcal{S}^{Top}(M)$ and let and $(E_{M'}, p_f)$ be trivial in $\mathcal{S}^{Top}(M \times \mathbb{D}^4)_{\text{rel}}$. In other words, the simple homotopy equivalence $p_f : E_{M'} \to M \times \mathbb{D}^4$ is simple homotopy equivalent to a homeomorphism. The bundle $p' : E_{M'} \to M \times \mathbb{D}^3$ restricts to a $\mathbb{D}^4$ bundle $F_{M'} \to M'$ over M'.

$$\begin{CD} F_{M'} @>>> E_{M'} @>{p_f}>> M \times \mathbb{D}^4 \\ @VVV @VV{p'}V @VV{p=id \times cH}V \\ M' @>>{e}> M \times \mathbb{D}^3 @= M \times \mathbb{D}^3 \end{CD}$$

Recall that we have two $\mathbb{CP}^2_0$-bundles given by $q : E(\mathbb{CP}^2_0) \to M'$ and $pr_1 : M \times \mathbb{CP}^2_0 \to M$. We glue the $\mathbb{D}^4$ bundle $F_{M'} \to M'$ to the first and the $\mathbb{D}^4$ bundle $M \times \mathbb{D}^4 \to M$ to the second. The resulting bundles $q' : E(\mathbb{CP}^2) \to M'$ and $q : M \times \mathbb{CP}^2 \to M$ are simple homotopy equivalent via a homotopy equivalence c fitting into a homotopy commutative diagram

$$\begin{CD} E(\mathbb{CP}^2) @>{c}>> M \times \mathbb{CP}^2 \\ @V{q'}VV @VV{q''}V \\ M' @>>{h}> M \end{CD}$$

Since the homotopy equivalence $p_f : E_{M'} \to M \times \mathbb{D}^4$ is homotopic to a homeomorphism relative to its boundary, then the map $c : E(\mathbb{CP}^2) \to M \times \mathbb{CP}^2$ is also homotopic to a homeomorphism.

We can regard $M' \in \mathcal{S}^{Top}(M)$ to be the algebraic obstruction to block fibering M' over

M (or vice versa). We have shown that, after crossing M with $\mathbb{CP}^2$, we can block fiber over M', so the fact that crossing with $\mathbb{CP}^2$ preserves surgery obstructions means that the original element of $\mathcal{S}^{Top}(M)$ vanishes. The proof of surjectivity is very similar. □

Remark 4.71. *The paper Cappell-Weinberger [130] does this exercise using the language of characteristic varieties, but the above description is presumably easier to understand.*

Theorem 4.72. *Let V be a Cat manifold.*

1. *The set $\mathcal{S}^{Cat}(V \times I)_{\text{rel}}$ has a group structure.*
2. *The set $\mathcal{S}^{Cat}(V \times \mathbb{D}^2)_{\text{rel}}$ has an abelian group structure.*

Proof. For (1) simply glue the two manifold structures along one of the two boundary components of each to define multiplication. For (2) mimic the usual proof that the second homotopy group of a space is abelian. □

Corollary 4.73. *The injection $\mathcal{S}^{Top}(M) \to \mathcal{S}^{Top}(M \times \mathbb{D}^4)_{\text{rel}}$ gives $\mathcal{S}^{Top}(M)$ an abelian group structure.*

One needs to check that the map $\mathcal{S}(M \times \mathbb{D}^4) \to L_0(\mathbb{Z}[e])$ is a homomorphism, which ultimately relies on a straightforward fact about signature.

Remark 4.74. *Readers can refer to Nicas [482], who shows that PL structures and surgery sequences can also be made into groups and homomorphisms.*

Remark 4.75. *If one were to use the construction given here directly in PL, there would be a failure of periodicity because the image of $\pi_{4i}(F/PL)$ in $L_{4i}(\mathbb{Z}[e])$ (i.e. the range of some "splitting invariants") is different for $i = 1$ than for higher i. One obtains a map that can be directly compared to the Top periodicity map. We note that $\mathcal{S}^{PL}(M \times \mathbb{D}^4)_{\text{rel}} \cong \mathcal{S}^{Top}(M \times \mathbb{D}^4)_{\text{rel}}$, giving an exact sequence*

$$H^3(M;\mathbb{Z}_2) \to \mathcal{S}^{PL}(M) \to \mathcal{S}^{PL}(M \times \mathbb{D}^4)_{\text{rel}} \to \mathbb{Z}, \tag{4.76}$$

where the map $H^3(M;\mathbb{Z}_2) \to \mathcal{S}^{PL}(M)$ is the Kirby-Siebenmann map giving non-uniqueness of triangulations of Top manifolds.

4.5.1 The assembly map

One can formally develop a theory of the assembly map using periodicity. However, we now give the more traditional yet algebraic construction for the assembly map, the group structure on structure sets, and functoriality that do not depend on periodicity; it is worth seeing both approaches. In this section, we describe the idea of Quinn [527]. (See also Ranicki [541].)

Consider the cobordism definition of $L_n(\pi_1(X))$. It is based on objects which are surgery problems $N \to M$, where M is equipped with a reference map $f : M \to X$. In blocked surgery, when X is a manifold, we have objects that are decomposed over the simplices of X, giving us a map $[X : \mathbb{L}_\bullet] \to L_n(\pi_1(X))$. But if X is not a manifold, we can also try to regard surgery problems on manifolds or Poincaré complexes with a map to X that are locally subdivided and decompose into suitable pieces.

One way to formulate this idea is to insist that manifolds map into a regular neighborhood of X, and then to decompose them according to a triangulation of this regular neighborhood. They are "globally n-dimensional," which gives the empty problem on the boundary. The approach is "dual" to the picture. What does it mean on the domains of the surgery problems? In this case, we have a closed n-manifold mapping to a regular neighborhood of X. The domain is closed because we assume that we have the empty set over the inverse image of the boundary. In other words, we have an n-manifold N with a map $f : N \to X$. We have therefore produced a very indirect description of ordinary bordism of X, and we can now give a direct (if somewhat more complicated) definition of the bordism of a polyhedron X that gives a homology theory, making use of some transversality obtained from the Kan condition.

As in Wall's definition of a surgery problem, we want to construct a map from an object, so that, over each simplex Δ, there is an object decomposed according to the star of Δ. As before, we can perform addition by taking disjoint unions and inverses by reversing orientations. The assembly map then assembles the pieces over all the simplices. Alternatively, it can be regarded as a forgetful map: if one thinks of objects in the domain of the assembly map as "global objects over X that are subdivided in a particular way using the triangulation," then the map to $L_n(\pi_1(X))$ forgets this subdivision. Some people (e.g. Weinberger-Xie-Yu [695]) call these objects "PL controlled surgery problems over X," because they are surgery problems over all the pieces of a triangulation of X. The Kan condition allows us to subdivide these surgery problems to arbitrarily fine subdivisions of a given triangulation.

As we have defined, the structure spectrum $\mathbb{S}_\bullet^{Top}(X)$ is the fiber of this assembly map. The elements are concretely given by $(n+1)$-dimensional objects with boundary mapping to X, but whose boundary has PL control, i.e. a subdivided problem. With the empty surgery problem as the identity element and an addition given by the disjoint union, one obtains a group structure on $\mathcal{S}_*^{Top}(X)$; in other words, the structure set $\mathcal{S}_*^{Top}(X)$ is defined exactly the same way as L-groups. We refer the reader to Ranicki [540, 541] for the verification that this definition is appropriate for the problem.

4.5.2 Functoriality of the structure set

The functoriality of $\mathcal{S}^{Top}$ is now pretty straightforward. There are a number of ways to approach it. The abstract way is the following. Let $\mathbb{H}_\bullet(X; \mathbb{L}_\bullet^{\langle 1\rangle}) \to \mathbb{L}_\bullet(X)$ be the assembly-level map of spectra that we discussed in Section 4.4. Recall that these spectra satisfy the condition that $\pi_j(\mathbb{H}_\bullet(X; \mathbb{L}_\bullet^{\langle 1\rangle})) = H_j(X; \mathbb{L}_\bullet^{\langle 1\rangle})$ and $\pi_j(\mathbb{L}_\bullet(X)) = L_j(X)$. Now define $\mathbb{S}_{\bullet}^{Top}(X)$ to be the spectrum-level mapping fiber of $\mathbb{H}_{\bullet}(X;\mathbb{L}_{\bullet}^{\langle 1\rangle}) \to \mathbb{L}_{\bullet}(\pi_1(X))$. Since the homology and L-theory terms are covariantly functorial and have abelian fundamental groups, we have the following from purely formal considerations.

On the other hand, we can also use the definitions above directly and see that a map $f: X \to Y$ induces a map $\mathcal{S}_n^{Top}(X) \to \mathcal{S}_n^{Top}(Y)$ just as there is an induced map $L_n(X) \cong L_n(\pi_1(X)) \to L_n(Y) \cong L_n(\pi_1(Y))$. The map on homology is completely analogous to the induced map on $MSCat_n$.

Theorem 4.77. *Let X be a CW complex with fundamental group π. Let $\mathbb{S}_{\bullet}^{Top}(X)$ be the spectrum-level mapping fiber of $\mathbb{H}_{\bullet}(X;\mathbb{L}_{\bullet}^{\langle 1\rangle}) \to \mathbb{L}_{\bullet}(X)$. Then there is a fibration sequence of spectra given by $\mathbb{S}_{\bullet}^{Top}(X) \to \mathbb{H}_{\bullet}(X;\mathbb{L}_{\bullet}^{\langle 1\rangle}) \to \mathbb{L}_{\bullet}(X)$ that gives rise to the algebraic surgery exact sequence*

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi]) \to \mathcal{S}_{n+1}^{Top}(X) \to H_n(X;\mathbb{L}_{\bullet}^{\langle 1\rangle}) \to L_n(\mathbb{Z}[\pi]).$$

Here $\mathcal{S}_{n+1}^{Top}(X)$ is functorial. If X is homotopy equivalent to an n-manifold, then $\mathcal{S}_{n+1}^{Top}(X)$ is isomorphic to the usual structure set $\mathcal{S}^{Top}(X)$ when the latter is nonempty and gives it the structure of an abelian group. In other words, the whole surgery exact sequence is a functorial fibration of infinite loop spaces just as the homology and L-theory terms are both functorial; the structure set is a covariant functor $\mathcal{S}^{Top}: Complexes \to Ab\ Groups$ or even $\mathcal{S}^{Top}: Complexes \to Infinite\ Loopspaces$.

If $f: M \to N$ is a map between manifolds of the same dimension, then there is a map $f_*: \mathcal{S}^{Top}(M) \to \mathcal{S}^{Top}(N)$ by the general fact that $\mathbb{S}^{Top}(M) = \mathrm{Fib}(\mathbb{H}_{\bullet}(X;\mathbb{L}_{\bullet}^{\langle 1\rangle}) \to \mathbb{L}_{\bullet}(X))$.

Sometimes it is convenient to have a geometric interpretation of the situation. Although we have only explained periodicity $\mathcal{S}^{Top}(M) \to \mathcal{S}^{Top}(E \text{ rel } \partial)$ for trivial $\mathbb{D}^{4k}$-block bundles over M, let us take for granted that there is such a periodicity (Thom isomorphism) for any oriented $\mathbb{D}^{4k}$-block bundle over M (or "$\mathbb{D}^{4k}$-approximate fibration").

Suppose in this context that $\dim N \geq \dim M$ and $\dim N \equiv \dim M \bmod 4$, and that the manifolds have the same orientation character. This set-up avoids the issue of mapping the connective spectrum to one with a different connectivity. We can then replace M^m with the regular neighborhood $\mathfrak{N}^{m+4k}(M)$ of M in $\mathbb{R}^{n+4k}$. For simplicity, we have assumed that M is orientable; otherwise we could choose a neighborhood of M in a bundle over $\mathbb{RP}^N$. In this case $\mathcal{S}^{Top}(M) \cong \mathcal{S}^{Top}(\mathfrak{N}^{m+4k}(M) \text{ rel } \partial)$ for all sufficiently large k. Now if $f: M \to M'$ is a continuous map with $\dim M \equiv \dim M' \bmod 4$, then for suitable k and ℓ we have the picture below.

Since $N = \mathfrak{N}^{m+4k}(M)$ is a codimension 0 submanifold of $\mathfrak{N}^{n+4\ell}(M')$, we can glue on the complement $N^{n+4\ell} \backslash \mathrm{int}\, \mathfrak{N}^{m+4k}(M)$, giving a map

$$\mathcal{S}^{Top}(\mathfrak{N}^{m+4k}(M) \text{ rel } \partial) \to \mathcal{S}^{Top}(\mathfrak{N}^{n+4\ell}(M') \text{ rel } \partial)$$

which identifies the domain with $S^{Top}(M)$ by Siebenmann-Thom periodicity, and the range with $S^{Top}(N)$. These identifications describe a functoriality.

This description is motivated by the pushforward of elliptic operators from Atiyah-Singer [28]. This idea was in turn adapted from the Grothendieck Riemann-Roch theorem [66]. The idea that surgery theory can be mapped to index theory has been very valuable and much exploited since the 1990s. We will discuss one of the ramifications of it in the next chapter when we examine the Novikov conjecture.

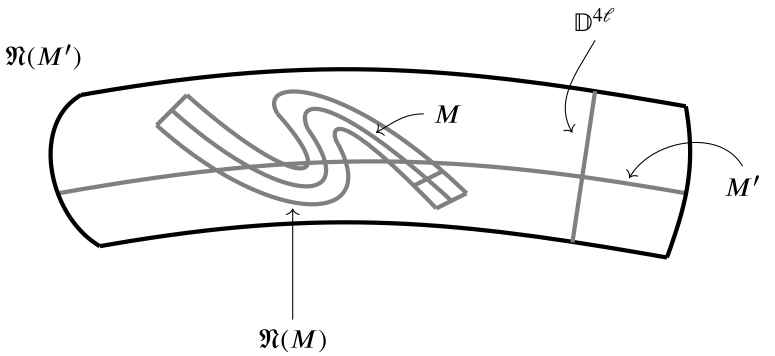


Figure 4.1: A regular neighborhood of M

Remark 4.78. *In Section 8.6 we will discuss the theory for homology manifolds. It does not differ much from the manifold case, but it has a perfect periodicity, i.e. no extra copies of $\mathbb{Z}$ entering the discussion. The relevant $\mathbf{L}$-spectrum is not connective, and the functoriality requires only that manifolds be of the same dimension mod 4 and have the same orientation character. In that setting, however, the push-forward of an ordinary manifold structure might be a nonresolvable homology manifold.*

4.6 SMOOTH STRUCTURES

Surgery was developed first in the smooth category, and its first applications were to smooth manifolds, notably the theory of Kervaire and Milnor that examines the number of differential structures on n-spheres. However, in general, smooth surgery is much more complicated than the PL and Top cases for both computational and conceptual reasons.

1. The homotopy type of the classifying space F/O is more elaborate than F/Top and F/PL. An excellent general reference is Madsen-Milgram [420]. We will explain some of this homotopy theory below.
2. Unlike the situation in the topological case, the surgery exact sequence is gener-

ally not a sequence of groups and homomorphisms. Indeed the image of the map $[M : F/O] \to L_n(\mathbb{Z}[\pi])$ is not necessarily a subgroup.

3. Despite these complications, when M is a compact and smooth, the map $\mathcal{S}^{Diff}(M) \to \mathcal{S}^{Top}(M)$ is finite-to-one and the image contains a subgroup of finite index. Therefore for some purposes, topological calculations can often give useful information about smooth manifolds.

The study of the classifying space F/O is essentially the study of the interaction of $KO(X) = [X : BO]$ and the spherical fibrations $KSph(X) = [X : BF]$. As discussed in Appendix A.2, the 8-periodic homotopy groups of BO are $\mathbb{Z}_2$, $\mathbb{Z}_2$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$, and the homotopy groups of BF are the stable homotopy groups of spheres. Quite early these groups were known to be related. The map that relates the two is defined by our forebearer George Whitehead.[2]

Definition 4.79. *The J-homomorphism $J : \pi_*(O) \to \pi_*^S$ takes an element of $\pi_k(O)$ and uses it to twist the standard framing of the normal bundle of $\mathbb{S}^k$ in $\mathbb{S}^{n+k}$, giving a new framed manifold, i.e. an element of the k-th stable stem $\pi_{n+k}(\mathbb{S}^n)$. This map is induced by $BO \to BF$ on homotopy.*

The image of the J-homomorphism was described by Adams assuming the eponymous Adams conjecture. To describe his conjecture, consider a finite CW complex X and the *Adams operations* Ψ^k defined on the topological K-theory group $KO(X)$ characterized by the following properties:

1. The Ψ^k are ring homomorphisms.
2. If ℓ is the class of a line bundle, then $\Psi^k(\ell) = \ell^k$.
3. The Ψ^k are functorial.

Calculations are possible from the case of line bundles by additivity, when the vector bundle V in $KO(X)$ is a Whitney sum of line bundles, and generally using the splitting principle.

Let $J(X)$ be the quotient of $KO(X)$ by the subgroup generated by differences $\xi - \nu$, where ξ and ν are vector bundles whose associated sphere bundles are fiber homotopy equivalent. The following is the Adams conjecture, which asserts the triviality of spherical fibrations associated to K-theory classes via the J-homomorphism.

CONJECTURE 4.1. *(Adams) If $k \in \mathbb{Z}$ and X is a finite CW complex with an element $y \in KO(X)$, then there is a non-negative integer $r = r(k, y)$ such that $k^r(y - \Psi^k y)$ lies in the kernel of $KO(X) \to J(X)$. Equivalently, the composite map*

$$BO \xrightarrow{1-\Psi^k} BO \xrightarrow{\sigma} BF \to BF[1/k]$$

[2]Whitehead advised Moore, who advised Browder, who advised Cappell, who advised Weinberger, who advised Chang. Whitehead was a student of Steenrod, who was in turn advised by Lefschetz.

is null-homotopic on finite skeleta.

Assuming the conjecture, Adams determined the image of the J-homomorphism, and in fact finds that $\operatorname{im} J$ is a direct summand of $J(X)$ with the other part $\operatorname{cok} J$. Later Quillen [513] proved the Adams conjecture by showing that this composite map is null-homotopic when p is prime, using an argument based on Brauer induction. Sullivan [626] gave another proof at around the same time. Both arguments depend on étale homology theory. Later Becker and Gottlieb [51] found a simple proof that relies on a transfer map for fiber bundles, generalizing classical transfer associated to finite covers. An account of these ideas can be found in Adams [6].

Sullivan used the results of Adams to prove that $F/O_{(p)} \cong BO_{(p)} \times BCok(J)$ at each prime p separately. The reader should be warned that the map $F/O_{(p)} \to BO_{(p)}$ is not the obvious map. A good reference is Madsen-Milgram [420].

This work gives us exquisite control over $\operatorname{im} J$. However, it also tells us that the part of stable homotopy theory that is relevant is precisely what is irrelevant for smooth surgery! This point should have been obvious; already in the work of Kervaire and Milnor on exotic spheres, the "normal invariant" that they assign to a homotopy k-sphere is an element of $\pi_k^s/\operatorname{im} J$, i.e. an element of $\operatorname{cok} J$. Note that F/O is the fiber of the map $BO \to BF$, which is an H-map under Whitney sum and is therefore an H-space. With this added structure, some calculations can be simplified. The description above of F/O does not respect the H-space structure when $p = 2$. See Boardman-Vogt [61] and May [433].

Here is a sample application to free smooth actions of the circle on an odd-dimensional sphere, and more generally to a free action by a connected compact Lie group on a simply connected manifold.

Example 4.80. *As an example, we can compare smooth free $\mathbb{S}^1$-actions on $\mathbb{S}^{2n+1}$ to its finite subgroups by studying the collection of transfer maps*

$$\mathcal{S}^{Diff}(\mathbb{CP}^n) \to \prod_k \mathcal{S}^{Diff}(L_k^{2n+1}),$$

where L_k^{2n+1} is a $(2n+1)$-dimensional Lens space with fundamental group $\mathbb{Z}_k$. The left-hand side injects into its normal invariants, but the right-hand side does not (there is also a ρ-invariant). Nevertheless, we could simply pay attention to the normal invariants $[L_k^{2n+1} : F/O]$. The obstruction theory perspective, i.e. considering the E^2-term of the Atiyah-Hirzebruch spectral sequence, would give us a series of obstructions that lie in Θ_{2i} when $2i < 2n+1$, while on the right-hand side we have $\Theta_{2i} \otimes \mathbb{Z}_k$. Obviously, by choosing sufficiently large k, we may hope to detect any particular non-trivial normal invariant.

The problem is that a priori obstructions do not exist at every stage: higher obstructions require the vanishing of the lower ones even to be defined. Various maps might then not be homomorphisms. However, the following proposition tells us that, although we do not obtain any estimates about the nature or required number of finite subgroups of $\mathbb{S}^1$, *at least the qualitative result follows.*

Proposition 4.81. *Suppose that h^* is a cohomology theory with finite coefficients; i.e. the coefficient groups $h^*(pt)$ are all finite. If $\mathbb{S}^1$ acts on a finite complex X, the map $h^*(X/\mathbb{S}^1) \to \prod h^*(X/\mathbb{Z}_k)$ is an injection.*

Proof. (Sketch) Note that in this proposition one does not require anything about the action. Let us consider the sequence of integers $k!$, so that we can form the unions of the mapping cylinders $C = X \to X/\mathbb{Z}_2 \to X/\mathbb{Z}_6 \to X/\mathbb{Z}_{24} \to \cdots \to X/\mathbb{Z}_{k!} \to \cdots$. Since (a) the coefficients are finite and (b) $\lim^1$ vanishes for sequences of finite abelian groups, we have $h^*(C) = \lim h^*(X/\mathbb{Z}_{k!})$. Our objective therefore is to show that $h^*(X/\mathbb{S}^1) \to h^*(C)$ is injective. To prove this claim, we note that there is a natural map $C \to X/\mathbb{S}^1$, and the inverse image of each point in $X/\mathbb{S}^1$ is homotopy equivalent either to a point (i.e. if it corresponds to a fixed point) or to a copy of $K(\mathbb{Q}/\mathbb{Z}, 1)$, whose homology with finite coefficients is that of a point. If $f : X \to Y$ is a continuous function such that every point inverse is acyclic from the point of view of some homology, then from the Vietoris-Begle theorem [654] the map is an isomorphism on homology. As a result, the map $C \to X/\mathbb{S}^1$ is an isomorphism on homology with finite coefficients, and therefore for all (co)homology theories by the Atiyah-Hirzebruch spectral sequence. □

In fact, this proposition can be extended to the case when G is a compact Lie group acting freely on a finite complex X.

Proposition 4.82. *If X is a finite complex, and G is a compact Lie group acting freely on X, then for any finitely generated cohomology theory h, the map $h^*(X/G) \to \prod h^*(X/G')$, where the product is taken over finite subgroups G' of G, is injective.*

Indeed the Becker-Gottlieb transfer [51], which is used for proving the Adams conjecture, reduces the claim to the case of a finite extension of a torus, which in turn reduces to the case of a torus. The result follows for the torus by an induction over orbits and the fact argued above that the map $\lim H_i(\mathbb{S}^1/\mathbb{Z}_k; A) \to H_i(\mathbb{S}^1/\mathbb{S}^1; A)$ is an isomorphism for all finite A. Therefore, for any X and homology theory h with finite coefficients, the map $\lim h_i(X/\mathbb{Z}_k^r) \to h_i(X/\mathbb{T}^k)$ is an isomorphism. If h is finitely generated, since for any finite X we know that $h(X) \to H(X; \mathbb{Z}_k)$ is injective, so to detect all elements we can use finite coefficients, and apply Proposition 4.82 to give the conclusion.

This general theorem proves the first part of the following.

Theorem 4.83. *(Weinberger [690]) Let $n \geq 3$.*

1. *A free smooth $\mathbb{S}^1$-action on $\mathbb{S}^{2n+1}$ is equivalent to the standard sphere iff it is equivalent to the standard sphere when restricted to $\mathbb{Z}_k$ for every k.*
2. *More generally, if G is a connected compact Lie group and G acts freely on a simply connected manifold M with $\dim(M/G)$ even, then the map $S^{Diff}(M/G) \to$*

$\prod S^{Diff}(M/H)$ is injective, where the product is taken over the "roots of unity" in the normalizer of a maximal torus of G.

Remark 4.84. *The first statement might seem surprising from the rational perspective. If L_k^{2n+1} denotes a $(2n+1)$-dimensional Lens space with fundamental group $\mathbb{Z}_k$, then implicit in the statement is that we can detect elements of $[\mathbb{CP}^n : F/O]$ via elements in $[L_n^{2k+1} : F/O]$ by some limit. From a rational point of view this detection is impossible, since the former is isomorphic to $\mathbb{Q}^{[(n-1)/2]}$ detected by Pontrjagin classes, and the latter is trivial since it rationally agrees with $\pi_{2n+1}(F/O) \otimes \mathbb{Q} = 0$. In other words, the rationalization $[\mathbb{CP}^n : F/O_{(0)}] \to \prod[L_k^{2n+1} : F/O_{(0)}]$ is exactly the zero map. Fortunately Theorem 4.81 tells us that finite coefficients can be exploited to prove (1). The space $F/O_{(0)}$ is precisely the kind of theory not satisfying the hypothesis of Proposition 4.82.*

Remark 4.85. *Statement (2) follows from the proof of (1) when one assumes the injectivity of $S^{Diff}(M/G) \to S^{Diff}(M/N_G(T))$, where $N_G(T)$ is the normalizer of the maximal torus. The proof of this injectivity is due to Becker-Gottlieb [51] and uses the fact that F/O is an infinite loopspace, and additionally that $G/N_G(T)$ has Euler characteristic 1 for any compact Lie group. When one takes the transfer and then the projection from M/G to M/H, the Becker-Gottlieb transfer on homology multiplies by $\chi(G/H)$, and so the transfer is injective for any homology theory.*

Remark 4.86. *The infinite loopspace structure on F/O that we just invoked above is compatible with analogous ones on F/PL and F/Top. Note however that these latter spaces have a completely different infinite loopspace structure that is implicit in the description given earlier (see Theorems 3.36 and 3.42):*

1. *$F/PL[{}^1\!/\!_2] \simeq BO[{}^1\!/\!_2]$ and*

$$F/PL_{(2)} \simeq E^4 \times \prod_{n\geq 2} K(\mathbb{Z}_{(2)}, 4n) \times K(\mathbb{Z}_2, 4n-2),$$

2. *$F/Top[{}^1\!/\!_2] \simeq BO[{}^1\!/\!_2]$ and*

$$F/Top_{(2)} \simeq \prod_{i\geq 1} K(\mathbb{Z}_2, 4i-2) \times K(\mathbb{Z}_{(2)}, 4i).$$

The group structure that arises from spacification or periodicity is this second H-space structure, as we have seen. It is associated to the equivalence $\mathbb{Z} \times F/Top \to \mathbb{L}_\bullet(e)$. However, the space F/O does not have an analogous H-space structure to this one; as we will presently see, the proof is essentially the one used by Milnor to detect the first exotic 7-sphere.

This following proposition shows that smooth surgery is more complicated than topological surgery over and beyond the difficulties posed by the fact that F/O is so much more complicated than F/Top. Other indications of the fact that smooth surgery does not have a sequence of groups and homomorphisms follow.

Proposition 4.87. *The image of the surgery map $\sigma : [\mathbb{T}^8 : F/O] \to L_8(\mathbb{Z}[\mathbb{Z}^8])$ is not a subgroup.*

Proof. We localize at the prime 7. The L-group is easily identified with $H^8(\mathbb{T}^8; \mathbb{Z}_{(7)}) \oplus H^4(\mathbb{T}^8; \mathbb{Z}_{(7)}) \oplus H^0(\mathbb{T}^8; \mathbb{Z}_{(7)})$. The surgery maps are given by signature and codimension 4 signatures, i.e. by the L-polynomials $L_8 = (7p_2 - p_1^2)/45$ and $L_4 = p_1/3$. On the other hand, since there is no 7-torsion in the homotopy of BF until dimension 10, we are working with BO, and through dimension 10, localized at 7, the map given by (p_1, p_2) adequately describes $[\mathbb{T}^8 : F/O] \to [\mathbb{T}^8 : BO] \to H^8(\mathbb{T}^8; \mathbb{Z}_{(7)}) \oplus H^4(\mathbb{T}^8; \mathbb{Z}_{(7)})$. We now have coordinates on domain and range, and can compute this nonlinear map. Notice that in the image of the surgery map one has $L_8 \equiv 9L_4^2 \bmod 7$, a condition which does not describe a subgroup. □

Example 4.88. *Crowley [181] has also shown that in general the product space $\mathbb{S}^{4j-1} \times \mathbb{S}^{4k}$ cannot be endowed with a group structure such that the smooth surgery exact sequence is a long exact sequence of groups. In particular, suppose that there is a group structure on $\mathcal{S}^{Diff}(\mathbb{S}^3 \times \mathbb{S}^4)$ such that all maps in the Diff surgery sequence are homomorphism. Then, as Crowley shows, there is a short exact sequence of pointed sets*

$$0 \to \mathbb{Z}_{28} \to \mathcal{S}^{Diff}(\mathbb{S}^3 \times \mathbb{S}^4) \to \mathbb{Z} \to 0$$

where $\mathbb{Z}_{28} = \Theta_7$ acts transitively on the fibers of η; i.e. there is one orbit under the group action and therefore the stabilizers of any two points in $\mathcal{S}^{Diff}(\mathbb{S}^3 \times \mathbb{S}^4)$ must have the same number of elements. However, he also shows that the Sylow 2-subgroup acts freely on all $\mathcal{S}^{Diff}(\mathbb{S}^3 \times \mathbb{S}^4)$ but that the Sylow 7-subgroup acts freely on $\eta^{-1}(r)$ iff r is divisible by 7. Therefore the stabilizers of points in $\mathcal{S}^{Diff}(\mathbb{S}^3 \times \mathbb{S}^4)$ are not the same size, a contradiction. With this example, one can show that the fibers of the map F in the sequence

$$[M : Top/O] \to \mathcal{S}^{Diff}(M) \xrightarrow{F} \mathcal{S}^{Top}(M)$$

are not in general equinumerous.

Remark 4.89. *Related to this discussion is the calculation of the group of isotopy classes of diffeomorphisms of $\mathbb{S}^3 \times \mathbb{S}^3$ by Krylov [375]. It is the semidirect product of $\mathrm{SL}_2(\mathbb{Z})$ with the non-trivial extension of $\mathbb{Z} \times \mathbb{Z}$ by $\mathbb{Z}_{28}$. The $\mathrm{SL}_2(\mathbb{Z})$ is the action on cohomology, whose kernel consists of maps that are homotopic to the identity. A diffeomorphism of M with a homotopy to the identity gives an element of $\mathcal{S}^{Diff}(M \times [0,1] \operatorname{rel} \partial)$, which is a group in all categories by the stacking principle; i.e. two elements can be glued to each other along a common boundary to form a new element. The group $\mathbb{Z} \times \mathbb{Z}$ comes from the first Pontrjagin class p_1 of the mapping torus, and the group $\mathcal{S}^{Diff}(\mathbb{S}^3 \times \mathbb{S}^3 \times [0,1] \operatorname{rel} \partial)$ is nonabelian because the smooth structures on the sphere are non-trivially extended; i.e. the sequence is not split.*

Despite all these complications in Diff surgery, the following proposition implies that, when M is a smooth compact manifold, or a noncompact manifold with finitely generated homology, the map $S^{Diff}(M) \to S^{Top}(M)$ has image containing a subgroup of finite index.

Proposition 4.90. *(Weinberger [690]) For any H-space structure on F/Top and any finite-dimensional complex X, there is an integer N_k depending on $k = \dim X$ such that the image of the natural map $[X : F/O] \to [X : F/Top]$ contains the multiples of N_k.*

Proof. To lift a map $X \to F/Top$ to F/O, one must provide a null-homotopy to the composite to $B(Top/O)$, a space with finite homotopy groups. The difficulty is, of course, that the map $s : F/Top \to B(Top/O)$ need not be an H-map. We show that, for each k, there is an integer N_k such that, for any k-dimensional space and map $f : X \to F/Cat$, the map $s(N_k f)$ is nullhomotopic. The result then follows from a diagram chase with the surgery exact sequence.

Clearly we can work one prime at a time since Top/O has finite homotopy groups. Consider the infinite mapping cylinder of $f : F/Top \to F/Top$ (i.e. the infinite cyclic cover of the mapping torus of this map). It is homotopy equivalent to $F/Top[{}^1/_p]$ so that any map to $B(Top/O)_{(p)}$ is nullhomotopic. We restrict to a skeleton S of F/Top. Because the homotopy groups of $B(Top/O)$ are finite, the $\lim^1$ obstructions vanish, so that

$$\lim[S : B(Top/O)_{(p)}] = [S[{}^1/_p] : B(Top/O)_{(p)}] = 0,$$

and therefore some number of multiplications by p will kill the map from S. The result now follows for arbitrary X. □

Chapter Five

Applications of the assembly map

In this chapter we will derive some simple consequences of the material of the previous chapter.

With an abelian group structure on the structure set $\mathcal{S}^{Top}(M)$ of a Top manifold M, we can consider procedures like localization and tensor products of this set. Questions about rational Pontrjagin classes can be attacked from this point of view. For example, if M is a Top manifold with finite fundamental group, then the following are equivalent:

1. There is a Top manifold N^n that is homotopy equivalent to M^n, but with different rational Pontrjagin classes than M^n.
2. There are infinitely many Top manifolds N^n satisfying the condition in (1).
3. There is some i with $0 < 4i < n$ for which $H^{4i}(M; \mathbb{Q})$ is nonzero.

Remark 5.1. *Since the first Pontrjagin class p_1 is rigid, these statements fail for $\mathbb{S}^1 \times \mathbb{S}^4$. Indeed, there is a unique manifold in this homotopy type. The Novikov conjecture gives a conjectural picture of this story.*

The Borel conjecture, which we have already discussed in the context of aspherical manifolds, is a more precise and integral statement. We shall enlarge its scope to torsion-free groups, and explain, if it is correct, how it reduces the calculation of $\mathcal{S}^{Top}(M)$ into a homological problem. In Section 5.2, we will describe a far-reaching generalization of the Borel conjecture, due to Farrell and Jones, that could give a similar but much more complicated picture for all fundamental groups.

The previous chapter demonstrated the functorial nature of surgery and also defined $\mathcal{S}^{Top}(X)$ when X is not a manifold. As a result, the critical case of $X = K(\pi, 1)$ sheds light on all manifolds with fundamental group π. This insight leads us to the Borel conjecture, the Novikov conjecture, and the much easier converse statements.

We showed in Section 3.6 that the Borel conjecture holds for free abelian groups $\mathbb{Z}^n$, or equivalently that the structure set $\mathcal{S}^{Top}(\mathbb{T}^n)$ for the n-torus is trivial. In this section, we discuss other variants of the Borel conjecture and the circle of ideas that have been associated to it for the past 50 years. In particular, we will discuss the classifying space $B\pi$ when π is a finitely presented group, how to interpret the Borel conjecture when $B\pi$ is not a manifold, and the role of the Novikov conjecture in the attempts to understand the statement. We will be using material from Section 4.4 and the formulation of the normal invariant term using spectra.

5.1 THE BOREL CONJECTURE AND RELATED QUESTIONS

We first recall the statement of the Borel conjecture.

CONJECTURE 5.1. *If M is a closed aspherical manifold and $f : M' \to M$ is a homotopy equivalence, then f is homotopic to a homeomorphism.*

To prove the Borel conjecture, every method that we have developed to distinguish manifolds from each other is a potential obstacle that needs to be overcome. In particular, one needs to understand why and how the characteristic classes of homotopy equivalent manifolds must be the same. As we saw in Section 3.8, the Borel conjecture cannot be directly extended to the noncompact setting with proper maps, because of the theorem below.

Theorem 5.2. *(Chang-Weinberger [154]) Let $M = \Gamma\backslash G/K$ be a noncompact arithmetic manifold for which $\mathrm{rank}_{\mathbb{Q}}(\Gamma) \geq 3$. Then M has a finite-sheeted cover N whose proper structure set $S_p^{Top}(N)$ is non-trivial; i.e. there is a manifold X with a proper homotopy equivalence $g : X \to N$ that is not properly homotopic to a homeomorphism.*

In the proof of this theorem, the existence of exotic noncompact arithmetic manifolds is reduced to the construction of certain nonzero elements in cohomology. However, for now, we need a reason why it does not happen in the closed case.

We recall the Hirzebruch signature theorem in Section 2.3, which states that the signature of $4k$-dimensional oriented manifolds is a homotopy invariant which can be calculated by the formula

$$\mathrm{sig}(M^{4k}) = \langle L_k(p_1, \dots, p_k), [M] \rangle,$$

where the L-genus $L_k = L_k(M)$ is a particular polynomial combination of the Pontrjagin classes $p_1, \dots, p_k$ of M. For example, the Hirzebruch theorem shows that, since all homotopy spheres Σ^{4k} have zero signature, their top Pontrjagin class $p_k(\Sigma^{4k})$ must vanish. The theorem then gives a first step to understanding how homotopy equivalent manifolds could sometimes be forced to have the same characteristic classes.

Novikov proposed the following generalization of one of the key consequences of the Hirzebruch theorem.

Conjecture 5.3. *Novikov conjecture (simplest form). Suppose that M is a closed oriented manifold of dimension $4k + i$ with fundamental group π. Let $f : M \to B\pi$ be a map and suppose that $\alpha \in H^i(B\pi; \mathbb{Q})$. Then the higher signature defined by*

$$\mathrm{sig}_\alpha(M) \equiv \langle f^*(\alpha) \cup L_k(M), [M] \rangle$$

is an oriented homotopy invariant in $\mathbb{Q}$. *Here* $L_k(M)$ *is the* L*-genus in* $H^{4k}(M;\mathbb{Q})$.

Remark 5.4. *Note that* $\alpha \mapsto f^*(\alpha)$ *is a map* $f^* : H^i(B\pi;\mathbb{Q}) \to H^i(M;\mathbb{Q})$. *Therefore we say that the map* f^* *is a higher signature map.*

Remark 5.5. *When* α *is the element 1 in* $H^0(B\pi;\mathbb{Q})$, *then* $\mathrm{sig}_\alpha(M)$ *is just the usual signature* $\mathrm{sig}(M)$. *The higher signature can often be given a geometric interpretation. If the Poincaré dual of* $f^*(\alpha)$ *of* α *can be represented by a* $4k$*-dimensional submanifold* $N \subseteq M$ *with trivial normal bundle, then the higher signature* $\mathrm{sig}_\alpha(M)$ *of* M *is just the signature* $\mathrm{sig}(N)$ *of* N. *Better yet, if* $B\pi$ *is a closed, oriented manifold and the Poincaré dual of* α *in* $B\pi$ *can be represented by a submanifold* $K \subseteq M$ *with trivial normal bundle, the higher signature* $\mathrm{sig}_\alpha(M)$ *is the signature* $\mathrm{sig}(f^{\pitchfork}(K))$ *of the transverse image* $f^{\pitchfork}(K)$ *of* K. *The Novikov conjecture implies that all such signatures are homotopy invariants.*

A useful dual formulation is the following.

Conjecture 5.6. *(Novikov conjecture) Suppose that* M *is a closed manifold with fundamental group* π *and let* $f : M \to B\pi$ *be a map. If* L_M *is the total* L*-genus of* M, *then* $f_*(L_M \cap [M])$ *is an oriented homotopy invariant in* $\bigoplus H_{n-4i}(B\pi;\mathbb{Q})$ *among manifolds with a reference map to* $B\pi$. *In other words, if* $g : N \to M$ *is an orientation-preserving homotopy equivalence between closed, oriented manifolds, then for any discrete group* π *and map* $f : M \to B\pi$ *we have*

$$(f \circ g)_*(L_N \cap [N]) = f_*(L_M \cap [M]) \in H_*(B\pi;\mathbb{Q}).$$

Proposition 5.7. *Suppose that* M *and* N *are closed aspherical manifolds with the same fundamental group* π. *Suppose that the Novikov conjecture holds for* π. *If* $g : N \to M$ *is a homotopy equivalence, then* $g^*(p_i(M)) = p_i(N)$ *in* $H^{4i}(N;\mathbb{Q})$*; i.e. the map* g *preserves the* i*-th rational Pontrjagin class for all* i.

Proof. Suppose that $g : N \to M$ is a homotopy equivalence, but $g^*(p_i(M)) \neq p_i(N)$ in $H^i(N;\mathbb{Q})$. Then $g^*(L_i(M)) \neq L_i(N)$ in $H^i(N;\mathbb{Q})$. Therefore $g^*(L_i(M)) - L_i(N)$ is nonzero. By Poincaré duality, one can find a homology class $\gamma \in H_{4k}(N;\mathbb{Q})$ such that $\langle g^*(L_i(M)) - L_i(N), \gamma\rangle \neq 0$, violating the Novikov conjecture. Indeed, since all the L_i agree, so do all the classes p_i. □

The Novikov conjecture can be cast in the language of the surgery assembly map. To understand this viewpoint, we will work with homology with spectral coefficients, which is discussed in Section 4.4. Let X be a Poincaré complex of dimension n with fundamental group π. Then in this venue, the surgery map $\sigma : [X^n : F/Top] \to L_n(\mathbb{Z}[\pi])$ is replaced with the assembly map $A : H_*(X;\mathbb{L}_\bullet^{\langle 1\rangle}) \to L_*(\mathbb{Z}[\pi])$. The Novikov conjecture has multiple incarnations with regard to this map A. It is a statement that does not require a closed aspherical manifold, merely an aspherical space.

Actually, the Novikov conjecture was one of the driving forces leading to the assembly map. We follow the original idea of Miščenko for now, although the assembly technology of the previous chapter can give more precise information, since we work rationally and it is so appealing; its main defects concern the prime 2. Even better we could work in the symmetric L-groups $L^*(\mathbb{Z}[\pi])$.

Suppose that X is a Poincaré complex with fundamental group π. Let $\Omega_*^{STop}(B\pi)$ be the oriented topological bordism of $B\pi$. Denote by Ω_* the bordism group $\Omega_*^{STop}(*)$. Following Conner and Floyd [171], we have a natural equivalence

$$\phi\colon \Omega_*^{STop}(X) \otimes_{\Omega_*} \mathbb{Q} \to H_*(X;\mathbb{Q})$$

defined by $\phi[M,f] = f_*(L_M \cap [M])$. The reader may recall that this theorem was valuable in our analysis of F/PL in Section 3.4. The theories are $\mathbb{Z}_4$-graded and the Ω_*-module structure on $\mathbb{Q}$ is provided by the signature. Suppose that E_8 is the Milnor manifold of signature 8; the map $g\colon E_8 \to \mathbb{S}^8$ has degree one. Also let $f_*\colon L_*(\mathbb{Z}[\pi_1(M)]) \to L_*(\mathbb{Z}[\pi])$ be the obvious induced map on L-theory. Define a map $\psi\colon \Omega_*^{STop}(B\pi) \to L_*(\mathbb{Z}[\pi])$ by $\psi[M,f] = f_*(\theta)$, where θ is the surgery obstruction of a normal map covering

$$1 \times g\colon M \times E_8 \to M \times \mathbb{S}^8.$$

We have an induced map

$$\psi \otimes 1\colon \Omega_*^{STop}(B\pi) \otimes_{\Omega_*} \mathbb{Q} \to L_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}.$$

Then Miščenko defines A with the diagram

$$\begin{array}{ccc}
\Omega_*^{STop}(B\pi) \otimes_{\Omega_*} \mathbb{Q} & \xrightarrow{\psi\otimes 1} & L_*(\mathbb{Z}[\pi]) \\
& \searrow^{\phi} \qquad \nearrow_{A} & \\
& H_*(B\pi;\mathbb{Q}) &
\end{array}$$

Theorem 5.8. *The Novikov conjecture holds for* π *iff* $A\colon \bigoplus_{i\in\mathbb{Z}} H_{*+4i}(B\pi;\mathbb{Q}) \to L_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$ *is injective.*

Proof. Let ϕ, ψ, and θ be given as above. Suppose that $[M,f]$ and $[N,g]$ are elements in $\Omega_*^{STop}(B\pi)$. Let $h\colon M \to N$ be an orientation-preserving homotopy equivalence such that $g\circ h \simeq f$. The map h is covered by the bundle map $H\colon \nu_M \to (h^{-1})^*(\nu_M)$. By Wall [672], we know that $8\theta[h,H] = \psi[N,g] - \psi[M,f]$. Since h is a homotopy equivalence, we have $\psi[M,f] = \psi[N,g]$, and it follows that $A\phi[M,f] = Af_*(L_M \cap [M])$ is equal to $A\phi[N,g] = Ag_*(L_N \cap [N])$. Therefore, the injectivity of A implies the Novikov conjecture for π.

Conversely, assume that the Novikov conjecture holds for π. Suppose that $Af_*(L_M \cap [M]) = 0$. Then $\psi[M,f] = 0$ in $L_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$, so $\psi[M,f]$ has finite order. By Theorem 9.4 of Wall [672], there is a bordism of normal maps over $B\pi$ from $N \times E_8 \to N \times \mathbb{S}^8$ to a normal map $W \to V$ with $\pi_1(V) = \pi$. Because the surgery obstruction in $L_*(\mathbb{Z}[\pi])$ is preserved, this surgery problem is normally cobordant to a homotopy equivalence. Using bordism invariance and the homotopy invariance provided by the assumed Novikov conjecture, we have

$$(f \circ p_1)_*(L_{N\times E_8} \cap [N \times E_8]) = (f \circ p_1)_*(L_{N\times \mathbb{S}^8} \cap [N \times \mathbb{S}^8]).$$

It follows that

$$f_*((L_N \cap [N])\sigma(E_8) = f_*((L_N \cap [N])\sigma(\mathbb{S}^8).$$

Since $\sigma(E_8) = 8$ and $\sigma(\mathbb{S}^8) = 0$, we have $f_*(L_N \cap [N]) = 0$. Since $H_*(B\pi; \mathbb{Q})$ is a rational vector space, it follows that $f_*(L_M \cap [M]) = 0$ as well. Therefore A is injective. $\square$

Remark 5.9. *Of course, this map is just a rationalization of the assembly map of the previous chapter. However, we describe it because it is direct, if "old-fashioned."*

In summary, we can cast the Novikov conjecture in the following fashion, with part (b) as clearly a kind of natural extension.

Conjecture 5.10. *Let π be a group.*

1. *(Novikov conjecture) The assembly map $A\colon H_*(B\pi; \mathbb{L}_\bullet) \to L_*(\mathbb{Z}[\pi])$ is a rational injection. In other words, the induced map*

$$A \otimes \mathbb{Q}\colon H_*(B\pi; \mathbb{L}_\bullet) \otimes \mathbb{Q} \to L_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$$

   is an injection. If so, we say that the Novikov conjecture holds for π.
2. *(Integral Novikov conjecture) The map $A\colon H_*(B\pi; \mathbb{L}_\bullet) \to L_*(\mathbb{Z}[\pi])$ is an injection.*

Remark 5.11. *When M is a nonorientable closed manifold with fundamental group π, then one can consider the orientation sheaf which gives classes in twisted homology corresponding to some twisted spectrum $\mathbb{L}_\bullet(\mathbb{Z}[\pi], w)$. Then the Novikov conjecture states that the map $H_*(B\pi; \mathbb{L}_\bullet(\mathbb{Z}[\pi], w)) \to L_*(\mathbb{Z}[\pi], w)$ is a rational injection. Similarly there is an integral version.*

Remark 5.12. *One can see that for finite groups the integral Novikov fails as stated. However, it is potentially correct for torsion-free groups.*

In summary, the relationship between the Borel conjecture and the Novikov conjecture can be stated very concisely.

Theorem 5.13. *If the Borel conjecture holds for the group π, then the integral Novikov conjecture also holds for π. Therefore, a counterexample for the Novikov conjecture will produce a counterexample to the Borel conjecture.*

Conjecture 5.14. *(Strong Borel) The assembly map $A\colon H_*(B\pi; \mathbb{L}_\bullet^{\langle 1\rangle}) \to L_*(\mathbb{Z}[\pi])$ is* *an isomorphism.*

We note that the Strong Borel conjecture, in the presence of the analogous conjecture that $\mathrm{Wh}(\pi) = 0$, means that any aspherical compact manifold W of any dimension with fundamental group π, typically with boundary, is topologically rigid relative to the boundary, i.e. $\mathcal{S}^{Top}(W) = 0$; i.e. if $B\pi$ is a finite complex or even when π is torsion-free, then $\mathrm{Wh}(\pi) = 0$ and $\mathcal{S}_n^{Top}(B\pi) = 0$ for all n. Therefore, the version for noncompact manifolds could be $\varinjlim_K \mathcal{S}^{Top}(M \text{ rel } M \backslash K)$ as K runs over compact subsets of M.

M. Davis [161] showed that any finite aspherical complex X is a retract of a closed aspherical M whose dimension can be chosen to be any $n > 2 \dim X$. Therefore we have the following.

Corollary 5.15. *If the Borel conjecture is true for all closed aspherical manifolds, then the Borel conjecture is true for groups π for which $K(\pi, 1)$ is finite.*

Proof. We use functoriality and obtain the sequence $\mathcal{S}^{Top}(X) \to \mathcal{S}^{Top}(M) \to \mathcal{S}^{Top}(X)$. If the Borel conjecture is true for M, then the middle group vanishes. Therefore $\mathcal{S}^{Top}(X)$ must vanish as well, since the composite is the identity. □

Remark 5.16. *Note that here we are using the stable theory, which has all the functoriality. Of course the stable theory of M is a factor of $\mathcal{S}^{Top}(M \times \mathbb{T}^4)$, so the Borel conjecture would require this stable group to vanish, not merely $\mathcal{S}^{Top}(M)$.*

It is an exercise to use the Shaneson $\mathbb{Z} \times \pi$ formula and the structure of F/Top to check that, if the counterexample to the Borel conjecture on M obtained did not live in the connective version of structure sets for M, there would be a "genuine" connective counterexample on $M \times \mathbb{T}^4$. Or if the counterexample to Integral Novikov Conjecture came from $H_*(M; \mathbb{L}_\bullet)$ instead of $H_*(M; \mathbb{L}_\bullet^{\langle 1 \rangle}(e))$, there would still be a counterexample to the Borel Conjecture for $M \times \mathbb{T}^4$ detected by normal invariants.

We remind the reader that in the previous chapter we defined a structure set for possibly infinite complexes, in particular for the classifying space $B\pi$ for a group π. It is 4-periodic in its index and will fit into the obvious surgery exact sequence.

Definition 5.17. *Let X be a finite complex and let $n \geq 5$. We define $\mathcal{S}_n^H(X)$ to be $\mathcal{S}^{Top}(M)$, where M is any compact oriented n-manifold with boundary that is simple homotopy equivalent to X. If no such M exists or if $n \leq 4$, then define $\mathcal{S}_n^H(X)$ to be $\mathcal{S}_{n+4k}^{Top}(X)$ where k is sufficiently large. If X is an infinite complex, define $\mathcal{S}_n^H(X)$ by taking the limit over all finite subcomplexes of X.*

Remark 5.18. *The definition is the 4-periodic version of the structure set of X. The justification for the notation will be given in the last chapter.*

Remark 5.19. *An analogous theory can be established for nonorientable manifolds. A proper theory for noncompact manifolds is also possible. In fact, additional variants* *can be formulated for both statements when M is a manifold with boundary (see Chang-Weinberger [153]).*

5.1.1 Brief history of the Novikov conjecture

Novikov originally observed the Novikov conjecture for $\pi = \mathbb{Z}$. Later Farrell-Hsiang and Novikov ascertained the free abelian case, more or less by the method that we explained using codimension one splitting. Cappell [117] took this method to its most perfect form, and Waldhausen [658, 659] did the same in K-theory.

The Novikov conjecture is a central problem in topology, as we have seen in this chapter. Its variations, the Borel and Farrell-Jones conjectures, are deeper and have yet further implications. It is possible to write endlessly about this problem. We wrote a survey years ago [153], and the second author has recently written another monograph about this circle of problems [694]. Our discussion is not as detailed as this subject deserves, but we refer instead to surveys[1] and the book by Connes [174], which positions it within the subject of noncommutative geometry. There is a parallel between the topological world of surgery and the analytic world of index theory and noncommutative geometry that has been an area of much research. The book by Higson-Roe discusses this subject [306].

It is also important to realize that the analytic approach has a much broader range of application; these methods are based on deep generalizations of the Atiyah-Singer theorem. Just as the Hirzebruch formula is one of several consequences of the index theorem, the Novikov conjecture is one of several implications of the C^*-version of the injectivity of the operator-theoretic assembly map. Of course, topological methods sometimes give more refined information about topological applications.

A. Free abelian groups

We have already discussed the topological approach to this case via the Farrell fibering theorem. In his thesis, Lusztig [414] gave an analytic approach to this case, which we now describe.

Atiyah and Singer introduced a signature operator D on an even-dimensional manifold M, whose index $\dim(\ker D) - \dim(\operatorname{coker} D)$ can be calculated by their index theorem. In this case, the index theorem specializes to the Hirzebruch theorem. They also developed an index theorem associated to families of elliptic operators, parametrized by a virtual vector bundle over P, roughly the formal difference between the kernel bundle and the cokernel bundle.

If D is the signature operator, then one can tensor it with a flat $U(k)$-representation to obtain a new operator. In the finite fundamental group case, this idea gives the multisignature as one varies over representations. For closed manifolds, the new operator does not give any new invariants; the twisted signature is k times the old signature. These

[1]Bartels [39], Cappell [116, 117], Carlsson-Pedersen [139], Chang-Ferry-Yu [151], Connes-Gromov-Moscovici [175, 176], and Connes-Moscovici [177]

bundles are flat, so they do not contribute anything to the index formula.

Lusztig's idea was to use a homomorphism to $\mathbb{Z}^k$ to obtain a family of $U(1)$-bundles over M, and check that (1) de Rham Hodge theory enables the definition of a homotopy invariant of M in this way and (2) the families index theorem identifies the Chern character of this bundle with the higher signature of M.

B. Two Generalizations

Miščenko [467] considered using infinite-dimensional flat bundles together with the index theorem. This approach requires the development of new index theorems. The KK-theory of Kasparov [346] led to the proof for discrete subgroups of real Lie groups.

A completely different idea used by Connes-Gromov-Moscovici [176] considers finite-dimensional bundles, but without the flatness condition. The idea is that, for a given homotopy equivalence, the index for D with coefficient in any bundle with sufficiently small curvature will be preserved. So one asks, what Chern characters arise from families of bundles with smaller and smaller curvature? See Connes-Gromov-Moscovici [176]. According to Connes and Moscovici, it is powerful enough to cover all cup products of one- and two-dimensional cohomology classes. See Hanke-Schick [291] and Mathai [431].

C. Noncommutative geometry

According to the spectral theorem, commutative unital C^*-algebras are exactly the continuous functions on compact Hausdorff spaces with respect to the uniform metric. It is useful to have parameter spaces to define families of operators. Infinite-dimensional representations can be regarded as projective modules over a C^*-algebra $C^*\pi$ which may be noncommutative. In these cases one might use these C^*-algebras as parameter spaces for defining families of operators. The natural language of index theory, K-theory, emerges here. Also useful is cyclic homology, a target for a generalization of the Chern character associated to algebras to detect the size of K-groups.

The core of the idea is the following. Starting from a quadratic form over $\mathbb{Z}[\mathbb{Z}^n]$, one can obtain a Hermitian form on $C[\mathbb{Z}^n]$. Completing and taking Fourier transforms, one can study forms over the continuous functions on the torus, and diagonalize the form using the spectral theorem. Spectral projections are projections over the continuous functions on the torus. Projective modules over the continuous functions are vector bundles by a well-known theorem of Swan. The Novikov conjecture is implicated by the Fourier transform in the free abelian case.

Connes's cyclic homology is rich enough for group algebras to detect all the group homology. The problem is that one cannot use the spectral theorem in this setting. At this point, it becomes relevant to identify dense non-C^*-subalgebras within $C^*\pi$; i.e. smooth approximations, roughly speaking. The approach is then to obtain invariants of the K-theory of $C^*\pi$. Although there are now many, Connes's method was the first one to establish the Novikov conjecture for Gromov hyperbolic groups. See Connes-Moscovici [177].

Cyclic homology, originally introduced as an analytic tool, is now critical in algebraic K-theory as well. Refining Connes's map to the cyclotomic trace in Bokstedt-Hsiang-Madsen [63], one can prove an algebraic K-theory analogue of the Novikov conjecture in great generality. Hesselholt and Madsen, over a series of important papers, showed that cyclic homology gives an enormous amount of information in higher K-theory, and it has become an important tool in number theory, and enters into the theory of prismatic cohomology of Bhatt-Scholze [54].

D. K-theory

Kasparov [346] developed relevant index theorems for the K-theory for C^*-algebra, building on earlier work of Atiyah and Brown-Douglas-Fillmore. A little thought provides a commutative diagram

$$\begin{array}{ccc} H_n(B\pi;\, \mathbb{L}_\bullet) & \longrightarrow & L_n(\mathbb{Z}[\pi]) \\ \downarrow & & \downarrow \\ K_n(B\pi) & \longrightarrow & K_n(C^*\pi) \end{array}$$

Because of the close connection between $\mathbb{L}_\bullet$ and KO away from 2, one can even obtain information on odd torsion from the injectivity of the bottom arrow. There is such a commutative diagram, but it does not commute with the periodicities of the two theories because periodicity emerges from the Dirac operator in K-theory, but is associated to the signature operator in surgery. As in L-theory, one predicts that the bottom arrow is an isomorphism if π is torsion-free.

In K-theory, one can interpret $K_n(B\pi)$ as the π-equivariant K-homology of $E\pi$. The bottom assembly map can be defined even for spaces where the π-action is not free; it suffices to have a proper action. If π is finite, for example, one can examine the π-equivariant K-homology of a point. In this case, the range of the Baum-Connes map $K_n(B\pi) \to K_n(C^*\pi)$ becomes a study of representation theory.

Baum and Connes conjectured that the "correct" assembly map $K_n(\underline{E\pi})^\pi \to K_n(C^*\pi)$ should be an isomorphism, where $\underline{E\pi}$ is the universal space for proper π-actions, analogous to $E\pi$ as the universal space of free π-actions. This conjecture has been proved in many cases, notably by Higson-Kasparov [304], for groups that act freely and isometrically on a Hilbert space, called *a-T-menable groups*, and by Lafforgue [383] and Mineyev-Yu [462] for hyperbolic groups. Lafforgue introduced an important "strong property (T)" that is a natural boundary to all known methods [384].

In algebraic K-theory and L-theory, the analogue of the Baum-Connes conjecture is harder to explain. In the next chapter, we will explain how Nil in the Bass-Heller-Swan formula and examples of Cappell in L-theory appear in this case. The Farrell-Jones conjecture is the right analogue. However, it says, for example, that the correct conjecture would assert that

$$H_n(E\pi/\pi; \mathbb{L}_\bullet(\mathbb{Q}[\pi_p])) \to L_n(\mathbb{Q}[\pi])$$

is an isomorphism if one used $\mathbb{Q}[\pi]$ instead of $\mathbb{Z}[\pi]$ in L-theory. Here π_p denotes the isotropy group of a point p in E, and the left-hand side should be interpreted with the help of sheaf theory.

E. Metric spaces

Kasparov's theorem was based on the nonpositive curvature of G/K, i.e. of the universal cover of an aspherical manifold, at least in the important special case when G is semisimple. The proof by Higson-Kasparov [304] of the Baum-Connes conjecture for a-T-menable groups showed that infinite-dimensional non-positive curvature is fairly common (see Guentner-Higson [276]). Skandalis, Tu, and Yu saw that, for the map to be injective, it is only necessary for π to have a uniform embedding in Hilbert space, not necessarily an action. This idea gives many injectivity results that are, as yet, inaccessible to just topological methods. For example, Guentner-Higson-Weinberger [277] proves the C^*-algebra K-theoretic Novikov conjecture for all linear groups, while the best current result only allows linear groups defined over the algebraic numbers. See Guentner-Tessera-Yu [278].

The property of having a uniform embedding is one about the group π as a metric space. Gromov hyperbolicity is also such a metric condition, aimed at imitating negative curvature. Unlike the algebraic tools that have been arising until this point in our discussions of L-groups, metric conditions on the groups are now understood to be critical.

In the last chapter we will introduce the bounded category of a metric space, and advocate the rethinking of topological problems in that setting. The same is true on the analytic side. The analogues of the Novikov conjecture, the Borel conjecture, and the Baum-Connes conjecture are all interesting and useful in the course of understanding even the ordinary conjecture. On the left-hand side, it would require a metric space $E\pi$ without knowing the group structure on π, but knowing only its metric structure. The methods for tackling these kinds of problems have appeared in Roe [547, 548] and Block-Weinberger [58]. In fact, they are now used in topological data analysis (see e.g. Carlsson [136] and Boissonnat-Chazal-Yvinec [62]). The analogues of the right-hand side were developed by Higson and Roe.

Gromov introduced the notion of finite asymptotic dimension, a large-scale analogue of Lebesgue's covering dimension of a compact metric space. Yu first proved the coarse Baum-Connes conjecture for proper metric spaces of finite asymptotic dimension by analytic methods, and therefore proved the Novikov conjecture for classifying spaces $B\pi$ when π has finite asymptotic dimension as a metric space. Afterwards, Bartels [39] and Chang-Ferry-Yu [151] gave topological versions, based on the α-approximation theorem. Dranishnikov-Ferry-Weinberger [204] gave another proof that only handles those groups of finite asymptotic dimension with finite Eilenberg-Maclane spaces $K(\pi, 1)$ using just the localized L-spectrum $\mathbb{L}_\bullet(e)_{(2)}$ rather than the whole L-spectrum. This proof has recently led to new ideas in quantitative topology (see e.g. Ferry-Weinberger [246] and Chambers-Dotterrer-Manin-Weinberger [148]).

5.1.2 Applications of functoriality

In this section, for the sake of simplicity, we will replace all instances of $\mathcal{S}^{Top}(M)$ with $\mathcal{S}^{Top}(M \times \mathbb{D}^4)$ when we deal with the periodic spectrum rather than the connective version, i.e. ignore the slight lack of periodicity that occurs the first time we cross a manifold by $\mathbb{D}^4$. When discussing the periodicity of $\mathcal{S}_n^{Top}(X)$ for a finite complex X, we use the model given by $\mathcal{S}^{Top}(M)$, the rel boundary structures of any oriented n-manifold that is simple homotopy equivalent to X. If there is no such model, we use $\mathcal{S}^{Top}(M^{n+4k})$, which will exist for k sufficiently large. To gain some more uniqueness, one might insist that M be parallelizable, although we know the isomorphism class of the group itself will be unchanged by different choices. If X is infinite, or even infinite-dimensional, then one just takes the limit over finite subcomplexes of X.

$$
\begin{array}{ccccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & L_{n+1}(\pi_1(X)) & \longrightarrow & \mathcal{S}_{n+1}^{Top}(X) & \longrightarrow & H_n(X;\mathbb{L}_\bullet^{\langle 1\rangle}) & \longrightarrow & L_n(\pi_1(X)) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & L_{n+1}(\pi_1(X),\pi_1(Y)) & \longrightarrow & \mathcal{S}_{n+1}^{Top}(X,Y) & \longrightarrow & H_n(X,Y;\mathbb{L}_\bullet^{\langle 1\rangle}) & \longrightarrow & L_n(\pi_1(X),\pi_1(Y)) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & L_n(\pi_1(Y)) & \longrightarrow & \mathcal{S}_n^{Top}(Y) & \longrightarrow & H_{n-1}(Y;\mathbb{L}_\bullet^{\langle 1\rangle}) & \longrightarrow & L_{n-1}(\pi_1(Y)) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & L_n(\pi_1(X)) & \longrightarrow & \mathcal{S}_n^{Top}(X) & \longrightarrow & H_{n-1}(X;\mathbb{L}_\bullet^{\langle 1\rangle}) & \longrightarrow & L_{n-1}(\pi_1(X)) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
$$

Remark 5.20. *Although we had not discussed it earlier, there is an obvious modification of this process that is possible for pairs (X, Y) in which one uses manifold pairs (M, N) with $N \subseteq \partial M$, a codimension 0 submanifold of the boundary of M. Then $\mathcal{S}_n^{Top}(X, Y)$ agrees with the structures on the pair when (X, Y) is a manifold pair of the right dimension and orientation character, and one considers structures on the pair that are trivialized on the part of ∂X not in Y, or alternatively by redoing the PL controlled construction to accommodate pairs. Refer to Definition 4.49 for the algebraic structure sets $\mathcal{S}_*^{Top}(X)$.*

Here we have suppressed notation concerning orientations, and we will continue to do so.

We recall that, if $B\pi$ is a finite complex, or even homotopy equivalent to one, then we can speculate that the algebraic structure sets $\mathcal{S}_n^{Top}(B\pi) = 0$ for all n. We will consider the Borel conjecture to be the following statement, i.e. following our previous section and ignoring any Whitehead group issues for now.

Conjecture 5.2. *If π is a torsion-free group, then $\mathcal{S}_n^{Top}(B\pi) = 0$ for all n.*

Remark 5.21. *In the next chapter, we will explain why $\mathcal{S}_{4k+3}^{Top}(B\pi) \otimes \mathbb{Q}$ is nonzero whenever π has torsion, at least in the residually finite case, where no analysis is necessary. A group is residually finite if the intersection of all its subgroups of finite index is trivial.*

We can prove that the Borel conjecture for aspherical manifolds implies the Borel conjecture for aspherical complexes. The proof will use a construction of Davis and a well-known fact about retracts.

Theorem 5.22. *(Davis [196]) Suppose that π is a group for which $B\pi$ is homotopy equivalent to a finite CW complex. Then there is a closed aspherical manifold M which retracts onto $B\pi$.*

Remark 5.23. *Recall that retracts do not preserve homotopy type. As a result, this version vastly extends the Borel conjecture beyond Poincaré duality groups. If Y is a retract of X, then $H_n(X; \mathbb{Z}) \cong H_n(Y; \mathbb{Z}) \oplus H_n(X, Y; \mathbb{Z})$ for all $n \geq 0$.*

Theorem 5.24. *Suppose that the Borel conjecture for manifolds holds. If π is a group for which $B\pi$ is a finite complex, then $\mathcal{S}_n^{Top}(B\pi) = 0$ for all n.*

5.1.3 Poincaré duality groups

Given a finitely presented group π, we would like to know whether there is a closed aspherical manifold with fundamental group π. An obvious condition is that $K(\pi, 1)$ must be a Poincaré complex; when this condition holds, then π is a *Poincaré duality group* . First we note that a Poincaré duality group must be torsion-free, since it is necessary for $K(\pi, 1)$ to be finite-dimensional.

Proposition 5.25. *The fundamental group of an aspherical finite-dimensional CW-complex X is torsion-free.*

Proof. Let $C \leq \pi_1(X)$ be a finite cyclic subgroup of $\pi_1(X)$. We have to show that C is trivial. Since X is aspherical, the quotient $\widetilde{X}/C$ is a finite-dimensional model for BC. Hence $H_k(BC) = 0$ for large k. Since $H_*(BC) \neq 0$ for all non-trivial cyclic groups, the result follows. □

Conjecture 5.26. *The following are posed by Wall.*

1. *If π is a group that satisfies Poincaré duality, then there is a closed aspherical manifold M whose fundamental group is π.*

2. *An aspherical manifold M is a product of two manifolds N_1 and N_2 iff the fundamental group π of M is a non-trivial product.*

Remark 5.27. *The Borel conjecture would imply that a homology ANR manifold is possible for (1), as will follow from our discussion in Chapter 8.*

Some view the following as positive evidence for Wall's conjecture. Others do not. In the years since Wall made his conjecture, there have been a number of new constructions of aspherical manifolds, but there remain few new constructions of $K(\pi, 1)$ Poincaré duality groups that are not built to be manifolds. The following is essentially the only example.

Theorem 5.28. *(Bartels-Lück-Weinberger [45]) Let π be a torsion-free hyperbolic group and let $n \geq 6$. Then π is the fundamental group of a closed aspherical manifold if the boundary of π is homeomorphic to an $(n-1)$-dimensional sphere. Indeed, in these dimensions, only the Gromov boundary of π determines whether π is such a fundamental group.*

Remark 5.29. *The paper of Ferry-Lück-Weinberger [242] proves this result for $n = 5$ and shows that such a manifold exists stably in lower dimensions, i.e. after crossing with some $\mathbb{Z}^k$.*

5.2 APPLICATIONS OF PERIODICITY AND FUNCTORIALITY

In the following, we give some immediate consequences of the periodicity of the structure set $\mathcal{S}_*^{Top}(M)$ for a Top manifold M. Recall that one reason for the introduction of the homology term $H_*(M; \mathbb{L}_\bullet^{\langle 1 \rangle})$ in the surgery exact sequence is to resolve the problems of functoriality. Indeed, if M is a Top n-manifold, then the function space $[M : F/Top]$ is contravariantly functorial and the L-groups $L_n(\mathbb{Z}[\pi])$ are covariantly functorial. As such the structure set $\mathcal{S}_n^{Cat}(M)$ is not functorial at all. When we replace $[X : F/Top]$ with $H_n(X; \mathbb{L}_\bullet^{\langle 1 \rangle})$, as is constructed in Section 4.4, it becomes conceivable that $\mathcal{S}_n^{Cat}(X)$ is now functorial. In the following we present a few results for a group π under the assumption that the Borel conjecture or the Novikov conjecture holds for π.

The next proposition shows why $\mathcal{S}^H$ is preferable $\mathcal{S}^{Top}$.

Proposition 5.30. *Let M^n be a closed Top n-manifold. If $\pi = \pi_1(M)$ satisfies the Borel conjecture, then*

$$\mathcal{S}_n^H(M^n) \cong H_{n+1}(B\pi, M; \mathbb{L}_\bullet),$$

where as usual $H_*(B\pi, M; \mathbb{L}_\bullet^{\langle 1 \rangle})$ is the homology of the mapping cylinder Cyl$(B\pi, M)$ *of the map $M \to B\pi$.*

Proof. The functoriality of $\mathcal{S}_*^H$ provides us with a sequence

$$\mathcal{S}_{n+1}^H(B\pi) \to \mathcal{S}_{n+1}^H(\mathrm{Cyl}(B\pi, M)) \to \mathcal{S}_n^H(M) \to \mathcal{S}_n^H(B\pi).$$

Since π satisfies the Borel conjecture, we have $\mathcal{S}_{n+1}^H(B\pi) = 0$ and $\mathcal{S}_n^H(B\pi) = 0$, so $\mathcal{S}_{n+1}^H(\mathrm{Cyl}(B\pi, M)) \cong \mathcal{S}_n^H(M)$, but by the π-π theorem we have

$$\mathcal{S}_{n+1}^H(\mathrm{Cyl}(B\pi, M)) \cong H_{n+1}(\mathrm{Cyl}(B\pi, M); \mathbb{L}_\bullet).$$

□

Remark 5.31. *One can compute $\mathcal{S}^{Top}(M)$ as the kernel of the map $\mathcal{S}^H(M) \to \mathcal{S}^H(\mathbb{S}^n) \cong \mathbb{Z}$ induced by the degree one map $M \to \mathbb{S}^n$.*

Definition 5.32. *We say that a CW complex X with fundamental group π is h-aspherical or haspherical if the map $X \to B\pi$ classifying its universal cover induces a $\mathbb{Z}$-homology isomorphism.*

Proposition 5.33. *If the Borel conjecture holds for an aspherical manifold M with $\pi_1(M) = \pi$, then there is a non-aspherical h-aspherical manifold with fundamental π which is topologically rigid.*

Proof. Let $M = K(\pi, 1)$ be n-dimensional. Execute surgery on M by removing $\mathbb{S}^1 \times \mathbb{D}^{n-1}$ from M and glueing in a knot complement J with $\pi_1(J) = \mathbb{Z}$ to form the space N. Here the knot is an embedded $(n-2)$-sphere in n-dimensional space, and one can always find a knot complement J with fundamental group $\mathbb{Z}$. Now

$$N = M \cup_{\mathbb{S}^1 \times \mathbb{S}^{n-2}} J$$

has fundamental group π. Note that $H_*(J; \mathbb{Z}) \cong H_*(\mathbb{S}^1; \mathbb{Z})$. By the Mayer-Vietoris theorem, it follows that $H_*(N; \mathbb{Z}) \cong H_*(M; \mathbb{Z}) \cong H_*(B\pi; \mathbb{Z})$. □

Let M be an h-aspherical manifold with fundamental group π, but which is not aspherical. We have a commutative diagram as below, in which the vertical isomorphisms $H_*(M; \mathbb{Z}) \to H_*(B\pi; \mathbb{Z})$ are given by h-asphericality, and the horizontal isomorphisms are given by the Borel conjecture.

$$\begin{array}{ccccccccc}
H_{n+1}(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) & \xrightarrow{\cong} & L_{n+1}(\mathbb{Z}[\pi]) & \xrightarrow{0} & \mathcal{S}^{Top}(M) & \longrightarrow & H_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) & \xrightarrow{\cong} & L_n(\mathbb{Z}[\pi]) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
H_{n+1}(B\pi; \mathbb{L}_\bullet^{\langle 1 \rangle}) & \xrightarrow[\cong]{} & L_{n+1}(\mathbb{Z}[\pi]) & \xrightarrow[0]{} & \mathcal{S}_{n+1}^{Top}(B\pi) & \longrightarrow & H_n(B\pi; \mathbb{L}_\bullet^{\langle 1 \rangle}) & \xrightarrow[\cong]{} & L_n(\mathbb{Z}[\pi])
\end{array}$$

From the diagram, it is clear that $\mathcal{S}^{Top}(M) = 0$.

Remark 5.34. *Other rigid non-aspherical manifolds can be found in Kreck-Lück [369].*

In the following we see how the Novikov conjecture for the group π gives a relationship between $\mathcal{S}^{Top}_{n+1}(M)$ and $\mathcal{S}^{Top}_{n+1}(B\pi)$ when M is a Top n-manifold with $n \geq 5$ and fundamental group π. We will use the general (and not entirely obvious) fact that, if the group homomorphism $C \to D$ is a rational injection, then there is a splitting $D \otimes \mathbb{Q} \to C \otimes \mathbb{Q}$, although this splitting is usually not canonical.

Theorem 5.35. *Let M be a Top n-manifold with $n \geq 5$ and fundamental group π. If the Novikov conjecture holds for π, then*

$$\mathcal{S}^{Top}_{n+1}(M) \cong H_{n+1}(B\pi, M; \mathbb{L}^{\langle 1\rangle}_\bullet) \oplus \mathcal{S}^{Top}_{n+1}(B\pi) \otimes \mathbb{Q}.$$

Proof. Consider the following diagram:

$$\begin{array}{ccccccccc}
H_{n+2}(B\pi, M; \mathbb{L}^{\langle 1\rangle}_\bullet) & \longrightarrow & 0 & \longrightarrow & \mathcal{S}^{Top}_{n+2}(B\pi, M) & \xrightarrow{\cong} & H_{n+1}(B\pi, M; \mathbb{L}^{\langle 1\rangle}_\bullet) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+1}(M; \mathbb{L}^{\langle 1\rangle}_\bullet) & \longrightarrow & L_{n+1}(\mathbb{Z}[\pi]) & \longrightarrow & \mathcal{S}^{Top}_{n+1}(M) & \longrightarrow & H_n(M; \mathbb{L}^{\langle 1\rangle}_\bullet) & \xrightarrow{A} & L_n(\mathbb{Z}[\pi]) \\
\downarrow & & \downarrow\cong & & \downarrow f_* & & \downarrow & & \downarrow\cong \\
H_{n+1}(B\pi; \mathbb{L}^{\langle 1\rangle}_\bullet) & \xrightarrow{(1-1)\otimes\mathbb{Q}} & L_{n+1}(\mathbb{Z}[\pi]) & \xrightarrow[\beta]{(onto)\otimes\mathbb{Q}} & \mathcal{S}^{Top}_{n+1}(B\pi) & \xrightarrow{0\otimes\mathbb{Q}} & H_n(B\pi; \mathbb{L}^{\langle 1\rangle}_\bullet) & \xrightarrow[A']{(1-1)\otimes\mathbb{Q}} & L_n(\mathbb{Z}[\pi])
\end{array}$$

where the isomorphism on the top row is given by the π-π theorem. First we choose a splitting $s\colon L_n(\mathbb{Z}[\pi]) \to H_n(B\pi; \mathbb{L}^{\langle 1\rangle}_\bullet)$. Since $\mathcal{S}^{Top}_{n+1}(M) \to H_n(M; \mathbb{L}^{\langle 1\rangle}_\bullet) \to H_n(B\pi; \mathbb{L}^{\langle 1\rangle}_\bullet)$ is the zero map, we know that there is a lift $f\colon \mathcal{S}^{Top}_{n+1}(M) \to H_{n+1}(B\pi, M; \mathbb{L}^{\langle 1\rangle}_\bullet)$. We then have a natural map

$$h = f \oplus g\colon \mathcal{S}^{Top}_{n+1}(M) \to \mathcal{S}^{Top}_{n+1}(B\pi) \oplus H_{n+1}(B\pi, M; \mathbb{L}^{\langle 1\rangle}_\bullet).$$

We would like to show that it is an isomorphism.

Suppose that $N \in \mathcal{S}^{Top}_{n+1}(M)$ lies in the kernel of h. Since $f(N) = 0$, then there is $N' \in \mathcal{S}^{Top}_{n+2}(B\pi, M)$ that maps to N. But $g(M) = 0$ in $H_{n+1}(B\pi, M; \mathbb{L}^{\langle 1\rangle}_\bullet)$, so $N' = 0$. Therefore $N = 0$. To show the surjectivity of h, consider first $(0, \alpha) \in \mathcal{S}^{Top}_{n+1}(B\pi) \oplus H_{n+1}(B\pi, M; \mathbb{L}^{\langle 1\rangle}_\bullet)$. Then there is a corresponding element $N_\alpha \in \mathcal{S}^{Top}_{n+2}(B\pi, M)$. Let N be the image of N_α in $\mathcal{S}^{Top}_{n+1}(M)$. Clearly $h(N) = (0, \alpha)$. Now consider $(N', 0) \in \mathcal{S}^{Top}_{n+1}(B\pi) \oplus H_{n+1}(B\pi, M; \mathbb{L}^{\langle 1\rangle}_\bullet)$. Let $N \in \ker(g)$. Choose $\gamma \in L_{n+1}(\mathbb{Z}[\pi])$ such that $\beta(\gamma) = N'$. Let γ act on N to give a manifold structure $N'' \in \mathcal{S}^{Top}_{n+1}(M)$. By construction we have $h(N'') = (N', 0)$. □

Remark 5.36. *We often call* $g : \mathcal{S}^{Top}_{n+1}(M) \to H_{n+1}(B\pi, M; \mathbb{L}^{\langle 1 \rangle}_{\bullet})$*, if it exists, a solution to the Novikov conjecture. Whenever the Novikov conjecture is true, there is a solution to the Novikov conjecture. Some may consider it to be a higher* ρ*-invariant or a refined* L*-class. Most proofs of the Novikov conjecture in special cases directly produce such a lift* g*.*

Remark 5.37. *The Novikov conjecture is known for one-dimensional cohomology classes by our previous discussions, and for two-dimensional classes by the work of Connes-Gromov-Moscovici (see Mathai [431] and Hanke-Schick [291]). As a consequence, if the cohomological dimension* $\mathrm{cd}_{\mathbb{Q}}(\pi_1(M)) \leq 2$*, we have*

$$\mathcal{S}^{Top}_{n+1}(M^n) \cong \bigoplus_i \ker\left(H_{n-4i}(M;\mathbb{Q}) \to H_{n-4i}(B\pi;\mathbb{Q})\right)$$

$$\oplus L_{n+1}(\mathbb{Z}[\pi])/\mathrm{im}\, H_k(B\pi) \otimes \mathbb{Q},$$

where $k = 0, 1,$ *or* 2*, whichever satisfies* $k \equiv (n+1) \bmod 4$*. In the next chapter, we will examine the interesting case of* π *finite.*

We close the section with another application of functoriality. The following is an example in which the Cappell UNil group appears in a structure set calculation. We discuss the UNil more fully in Section 6.3. The reader may wish to return to the remainder of this section after reading it.

In the following, if Σ is a hypersurface dividing a Top manifold M into two pieces M_1 and M_2, let $\mathcal{S}^{split}(M)$ be the collection of homotopy equivalences $f : N \to M$ such that f is also a homotopy equivalence when restricted to $f^{-1}(\Sigma)$ and $f^{-1}(M_i)$ for both $i = 1, 2$.

Proposition 5.38. *(Cappell [111] Theorem 3.1) Suppose that* M *is a Top* n*-manifold with* $n \geq 6$ *and fundamental group* π*. If* M *has a hypersurface* Σ *which divides* M *into two pieces, then the Cappell UNil group is a summand of the structure group* $\mathcal{S}^{Top}(M)$*. In fact, we have* $\mathcal{S}^{Top}(M) \cong \mathcal{S}^{split}(M) \oplus \mathrm{UNil}(\mathbb{Z}[\pi])$*, where* $\mathrm{UNil}(\mathbb{Z}[\pi])$ *will be shorthand for* $\mathrm{UNil}_{n+1}(\mathbb{Z}[\pi]; \mathbb{Z}[\pi_1(C)], \mathbb{Z}[\pi_1(D)])$*.*

Proof. Let M be divided by a hypersurface Σ into sides C and D. There is a group $\mathcal{S}^{split}(M)$ fitting into the sequence

$$\mathcal{S}^{Top}(\Sigma \oplus I) \to \mathcal{S}^{split}(M) \to \mathcal{S}^{Top}(C) \oplus \mathcal{S}^{Top}(D) \to \mathcal{S}^{Top}(\Sigma)$$

where maps are appropriate restrictions of manifold structures. There is a Mayer-Vietoris map on homology:

$$H_n(C; \mathbb{L}^{\langle 1 \rangle}_{\bullet}) \oplus H_n(D; \mathbb{L}^{\langle 1 \rangle}_{\bullet}) \to H_n(M; \mathbb{L}^{\langle 1 \rangle}_{\bullet}) \to H_{n-1}(\Sigma; \mathbb{L}^{\langle 1 \rangle}_{\bullet}).$$

Therefore Mayer-Vietoris is valid on the structure sets iff it is valid on L-groups. We then have a Mayer-Vietoris sequence

$$\to L_n(\mathbb{Z}[\pi_1(C)]) \oplus L_n(\mathbb{Z}[\pi_1(D)]) \to L_n^{split}(\mathbb{Z}[\pi_1(M)]) \to L_{n-1}(\mathbb{Z}[\pi_1(\Sigma)])$$
$$\to L_{n-1}(\mathbb{Z}[\pi_1(C)]) \oplus L_{n-1}(\mathbb{Z}[\pi_1(D)]).$$

Cappell proves that $L_{n+1}^{split}(\mathbb{Z}[\pi_1(M)]) \oplus \mathrm{UNil}(\mathbb{Z}[\pi_1(M)]) \cong L_{n+1}(\mathbb{Z}[\pi_1(M)])$. We consider the diagram

$$\begin{array}{ccccccccc}
 & & \mathrm{UNil}(\mathbb{Z}[\pi]) & \nwarrow & & & & & \\
 & & \downarrow {\scriptstyle 1-1} & & & & & & \\
H_{n+1}(M;\mathbb{L}_\bullet^{\langle 1\rangle}) & \longrightarrow & L_{n+1}(\mathbb{Z}[\pi_1(M)]) & \longrightarrow & \mathcal{S}^{Top}(M) & \longrightarrow & H_n(M;\mathbb{L}_\bullet^{\langle 1\rangle}) & \longrightarrow & L_n(\mathbb{Z}[\pi]) \\
\downarrow & & {\scriptstyle onto}\downarrow & & {\scriptstyle \beta}\uparrow\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle 1-1} \\
H_{n+1}^{split}(M;\mathbb{L}_\bullet^{\langle 1\rangle}) & \longrightarrow & L_{n+1}^{split}(\mathbb{Z}[\pi_1(M)]) & \longrightarrow & \mathcal{S}^{split}(M) & \longrightarrow & H_n^{split}(M;\mathbb{L}_\bullet^{\langle 1\rangle}) & \longrightarrow & L_n^{split}(\mathbb{Z}[\pi])
\end{array}$$

The isomorphism of homology terms comes from the fact that every homotopy is normally cobordant to a split homotopy equivalence. Also, the image of $H_{n+1}(M;\mathbb{L}_\bullet^{\langle 1\rangle})$ is $L_{n+1}^{split}(\mathbb{Z}[\pi_1(M)])$ and the map $\mathcal{S}^{Top}(M) \to \mathcal{S}^{split}(M)$ splits. Let $N \in \mathcal{S}^{Top}(M)$ and consider $N' = \beta(\alpha(N))$. The difference of these two elements is sent to zero in $H_n(M;\mathbb{L}_\bullet^{\langle 1\rangle})$ so there is $t_N \in L_{n+1}(\mathbb{Z}[\pi_1(M)])$ that is sent to $N - \beta(\alpha(N))$ in $\mathcal{S}^{Top}(M)$. By Cappell's result, there is a projection map $L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathrm{UNil}(\mathbb{Z}[\pi])$. Consider the image t'_N in $\mathrm{UNil}(\mathbb{Z}[\pi])$ of t under this map. Therefore we have an assignment $g\colon \mathcal{S}^{Top}(M) \to \mathrm{UNil}(\mathbb{Z}[\pi])$ given by $N \mapsto t'_N$. A diagram chase shows that g is a well-defined homomorphism. We can now define a map $(\alpha, g)\colon \mathcal{S}^{Top}(M) \to \mathcal{S}^{split}(M) \oplus \mathrm{UNil}(\mathbb{Z}[\pi])$. A diagram chase shows that this map is an isomorphism. □

5.3 AUTOMATIC VARIABILITY OF CHARACTERISTIC CLASSES

5.3.1 Acyclic groups

We say that a space X is *acyclic* if it is homologically equivalent to a point. In addition, we say that a group π is *acyclic* if its classifying space $B\pi$ is acyclic. One of the earliest examples of an acyclic group is Higman's four-generator, four-relator group [303] given by

$$\pi = \langle x_0, x_1, x_2, x_3 : x_{i+1} x_i x_{i+1}^{-1} = x_i^2 \rangle.$$

In this subsection we study the surgery exact sequence in the context of acyclic groups. Acyclic groups are perhaps an oddity; most groups have some homology, but acyclic groups do appear in a variety of situations and can be a useful tool. For example, Baumslag-Dyer-Heller [50] prove that every finitely generated group embeds in some finitely presented acyclic group. As another example, the Alexander horned sphere has two components in its complement; one is simply connected and the other is an acyclic aspherical manifold. We will see that some simple formulas hold for acyclic groups. In particular, if π is acyclic and M is a manifold with fundamental group π, we can compute formulas for $\mathcal{S}_n^{Top}(B\pi)$ and $\mathcal{S}^{Top}(M)$. We start with the following observation, true for any finitely presented group.

Proposition 5.39. *If π is acyclic, then the Integral Novikov conjecture holds for π.*

Proposition 5.40. *Suppose that M is an oriented Top n-manifold with $n \geq 5$ whose fundamental group π is acyclic. Then the normal invariant map gives a well-defined surjection $\mathcal{S}^{Top}(M) \to \widetilde{H}_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle})$.*

Proof. From the previous proposition, we know that the Integral Novikov conjecture holds for π. Consider the diagram

$$\begin{array}{ccccc} \mathcal{S}^{Top}(M) & \xrightarrow{\alpha} & H_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) & \xrightarrow{A} & L_n(\mathbb{Z}[\pi]) \\ \downarrow & & \downarrow \beta & & \downarrow \\ \mathcal{S}_n^{Top}(B\pi) & \longrightarrow & H_n(B\pi; \mathbb{L}_\bullet^{\langle 1 \rangle}) & \xrightarrow[1-1]{} & L_n(\mathbb{Z}[\pi]) \end{array}$$

Therefore the map $\mathcal{S}_n^{Top}(B\pi) \to H_n(B\pi; \mathbb{L}_\bullet^{\langle 1 \rangle})$ is the zero map, forcing $\beta \circ \alpha$ to be the zero map. Therefore $\operatorname{im} \alpha \subseteq \ker \beta$. But Lemma 5.48 gives us the reverse inclusion. Therefore $\operatorname{im} \alpha = \ker \beta$ and it is easily checked that the required surjection can be constructed. □

Remark 5.41. *For all groups π, we have the exact sequence, defining reduced L-groups*

$$\widetilde{L}_{n+1}(\mathbb{Z}[\pi]) \to L_n(\mathbb{Z}[\pi]) \xrightarrow{pr} L_n(\mathbb{Z}[e])$$

of oriented L-groups, where the last map splits. Therefore $L_n(\mathbb{Z}[\pi]) = L_n(\mathbb{Z}[e]) \oplus \widetilde{L}_{n+1}(\mathbb{Z}[\pi])$.

Proposition 5.42. *Let π be an acyclic group. Then $\mathcal{S}_n^{Top}(B\pi) = \widetilde{L}_{n+1}(\mathbb{Z}[\pi])$.*

Proof. Since the Integral Novikov conjecture holds for π, the surgery exact sequence gives

$$H_{n+1}(B\pi; \mathbb{L}_\bullet^{\langle 1 \rangle}) \xrightarrow{1-1} L_{n+1}(\mathbb{Z}[\pi]) \xrightarrow{onto} \mathcal{S}_n^{Top}(B\pi) \xrightarrow{0} H_n(B\pi; \mathbb{L}_\bullet^{\langle 1 \rangle}) \xrightarrow{1-1} L_n(\mathbb{Z}[\pi]).$$

If we reduce the homology and L-theory terms, then we note that $\widetilde{H}_{n+1}(B\pi; \mathbb{L}_\bullet^{\langle 1 \rangle}) = 0$ because π is acyclic. Therefore by exactness, we have $\mathcal{S}_n^{Top}(B\pi) = \widetilde{L}_{n+1}(\mathbb{Z}[\pi])$. □

Given the results from Propositions 5.40 and 5.42, we may be led to believe that, if M has an acyclic fundamental group, then $\mathcal{S}^{Top}(M)$ can be decomposed into a homology term and an L-theory term. The next proposition describes this decomposition.

Proposition 5.43. *Let M be an oriented Top n-manifold with $n \geq 5$ whose fundamental group π is acyclic. Then $\mathcal{S}^{Top}(M) = \widetilde{H}_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) \oplus \widetilde{L}_{n+1}(\mathbb{Z}[\pi])$.*

Proof. Recall first that the Integral Novikov conjecture holds for π. Consider the commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & L_{n+1}(\mathbb{Z}[\pi]) & \longrightarrow & \mathcal{S}^{Top}(M) & \longrightarrow & H_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) & \longrightarrow & L_n(\mathbb{Z}[\pi]) \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle onto} & & \downarrow & & \downarrow{\scriptstyle \cong} \\
\cdots & \xrightarrow[1-1]{} & L_{n+1}(\mathbb{Z}[\pi]) & \xrightarrow[onto]{} & \mathcal{S}_n^{Top}(B\pi) & \xrightarrow[0]{} & H_n(B\pi; \mathbb{L}_\bullet^{\langle 1 \rangle}) & \xrightarrow[1-1]{} & L_n(\mathbb{Z}[\pi])
\end{array}$$

From Proposition 5.40, we know that the $\mathcal{S}^{Top}(M)$ maps surjectively onto $\widetilde{H}_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle})$.

$$\begin{array}{ccc}
 & & \widetilde{H}_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) \\
 & \nearrow{\scriptstyle onto} & \downarrow \\
\mathcal{S}^{Top}(M) & \longrightarrow & H_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) \\
\downarrow{\scriptstyle onto} & & \downarrow \\
\mathcal{S}_n^{Top}(B\pi) & \xrightarrow[0]{} & H_n(B\pi; \mathbb{L}_\bullet^{\langle 1 \rangle})
\end{array}$$

A diagram chase shows that $\mathcal{S}^{Top}(M) = \widetilde{H}_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) \oplus \widetilde{L}_{n+1}(\mathbb{Z}[\pi])$. Here characteristic class variation occurs in the homology term. □

Corollary 5.44. *In the above situation, if the Borel conjecture holds for the acyclic group π, then $\widetilde{L}_{n+1}(\mathbb{Z}[\pi]) = 0$ and $\mathcal{S}^{Top}(M) \cong \widetilde{H}_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle})$.*

The following trivial fact exploiting $\mathbb{Q}$-acyclicity has an interesting consequence.

Corollary 5.45. *If π is a finite group, then the rational assembly map*

$$A \otimes \mathbb{Q} : \ H_n(B\pi; \mathbb{L}_\bullet^{\langle 1 \rangle}) \otimes \mathbb{Q} \to L_n(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$$

is injective.

Proof. If π is finite, then the standard model for $B\pi$ has finite skeleta, so the homology is finitely generated. A standard transfer argument using the contractible universal cover shows that homology is annihilated by $|\pi|$, so the homology groups of $B\pi$ are finite, and therefore $B\pi$ is $\mathbb{Q}$-acyclic. Then

$$H_n(B\pi; \mathbb{L}_\bullet^{\langle 1 \rangle}) \otimes \mathbb{Q} = H_n(pt; \mathbb{L}_\bullet^{\langle 1 \rangle}) \otimes \mathbb{Q} = L_n(\mathbb{Z}[e]) \otimes \mathbb{Q},$$

so the rational assembly map becomes $A \otimes \mathbb{Q} : L_n(\mathbb{Z}[e]) \otimes \mathbb{Q} \to L_n(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$. We know by Remark 5.41 that $L_n(\mathbb{Z}[e])$ is a summand of $L_n(\mathbb{Z}[\pi])$, so $A \otimes \mathbb{Q}$ is certainly an injection. □

Corollary 5.46. *In the case when π is finite, we have $\mathcal{S}^{Top}(M) \otimes \mathbb{Q} = \widetilde{H}_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) \otimes \mathbb{Q} \oplus \widetilde{L}_{n+1}(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$. Here the characteristic classes lie in the homology term, and the so-called ρ-invariants lie in the L-theory term.*

Remark 5.47. *This result is particularly useful because $L_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$ is quite understandable, as we saw in Chapter 2.*

5.3.2 Varying Pontrjagin classes

Our next result generalizes a classical result of Kahn [342] that the only possible linear combination of L-classes (equivalently rational Pontrjagin classes) which can be a homotopy invariant of simply connected manifolds is the top L-class of a manifold whose dimension is divisible by 4. In this section, we denote by L_M the total Hirzebruch L-class of the space M.

A simple diagram chase gives the following.

Lemma 5.48. *Let M be an oriented Top n-manifold with $n \geq 5$ and fundamental group π. Then*

$$\ker\left(H_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) \to H_n(B\pi; \mathbb{L}_\bullet^{\langle 1 \rangle})\right) \subseteq \operatorname{im}\left(\mathcal{S}^{Top}(M) \to H_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle})\right).$$

Recall from Section 4.6 that we have a relationship between topological and smooth surgery for any compact smooth manifold M.

Proposition 5.49. *If M is a compact smooth manifold of dimension at least 5 (or at least 6 if it has boundary), then the image of $\mathcal{S}^{Diff}(M) \to \mathcal{S}^{Top}(M)$ contains a subgroup of finite index.*

Therefore we can use topological surgery and its functoriality properties to deduce smooth results. For example, the infinitude of a smooth structure set is independent of the category, and we can also prove the following converse to the Novikov conjecture.

Theorem 5.50. *Let M be a closed, oriented smooth manifold of dimension $n \geq 5$, together with a map $f : M \to B\pi$ to the classifying space of a discrete group, inducing* *an isomorphism on fundamental group. Suppose we have a cohomology class*

$$L = L_1 + L_2 + \cdots \in H^{4*}(M;\mathbb{Q})$$

with $L_i \in H^{4i}(M;\mathbb{Q})$ for all i satisfying $f_*(L \cap [M]) = 0$ in $H_{n-4*}(B\pi;\mathbb{Q})$. Then there is a nonzero integer R such that, for any multiple r of R, there is a homotopy equivalence $h\colon M \to N$ of closed smooth manifolds so that $h^*(L_N) - L_M = rL$, where L_M and L_N are the total Pontrjagin classes of M and N, respectively.

Proof. We use the commutative diagram

$$\begin{array}{ccccc} \mathcal{S}^{Top}(M) & \xrightarrow{\tau} & H_n(M;\mathbb{L}_\bullet^{\langle 1\rangle}) & \xrightarrow{A} & L_n(\mathbb{Z}[\pi]) \\ & & \downarrow & & \downarrow \\ & & H_n(B\pi;\mathbb{L}_\bullet^{\langle 1\rangle}) & \xrightarrow[A']{} & L_n(\mathbb{Z}[\pi]) \end{array}$$

Let $\alpha = L \cap [M] \in H_{n-4*}(M;\mathbb{L}_\bullet^{\langle 1\rangle})$. Since $\beta \equiv f_*(\alpha) = 0$ in $H_{n-4*}(B\pi;\mathbb{Q})$, we know that there is an integer R_1 such that $R_1\beta = 0$ in $H_n(B\pi;\mathbb{L}_\bullet^{\langle 1\rangle})$. By the commutativity of the square, we know that $A(R_1\alpha) = 0$ in $L_n(\mathbb{Z}[\pi])$. Let R_2 be the constant guaranteed by the lemma above. Choose $R = R_2R_1$ and let r be a multiple of R. Since $A(r\alpha) = 0$, it follows that there is a Top manifold structure $g_r\colon N_r \to M$ in $\mathcal{S}^{Top}(M)$ such that $\tau(N_r, g) = r\alpha$, so $\tau(N_r, g) = rL \cap [M]$. If h is a homotopy inverse of g_r, then the left side of the equation is $(h^*(L_N) - L_M) \cap [M]$. Therefore $h^*(L_N) - L_M = rL$ as desired. ☐

Remark 5.51. *For smooth manifolds M with trivial fundamental group, the theorem of Browder and Novikov states that the Diff structure set $\mathcal{S}^{Diff}(M)$ is "commensurable" with $\oplus H^{4i}(M)$ subject to the Hirzebruch signature theorem. We will prove it below. It now follows immediately from the PL version proved in Chapter 3.*

Remark 5.52. *Let $f\colon M \to B\pi$ be a map. The Novikov conjecture says that, if $\alpha \in \bigoplus H_{4i}(M;\mathbb{Q})$ is of the form $\alpha = L_i(M) \cap [M] - L_i(M') \cap [M']$, then $f_*(\alpha) = 0$ in $\bigoplus H_{4i}(B\pi;\mathbb{Q})$. Therefore the above theorem is a partial converse, because if α is in the kernel of $\bigoplus H_{4i}(M;\mathbb{Q}) \to \bigoplus H_{4i}(B\pi;\mathbb{Q})$, then there is an integer r along with a homotopy equivalence $h\colon M \to M'$ such that*

$$r\alpha = L_i(M) \cap [M] - L_i(M') \cap [M'];$$

i.e. $r\alpha$ comes from some $h\colon M \to M'$ in $\mathcal{S}^{Top}(M)$.

The theorem above shows that one can vary the Pontrjagin class rationally within the same homotopy equivalence for simply connected manifolds M for which the Hirzebruch signature formula does not pose an obstruction. For non-simply connected manifolds with fundamental group π, the same result is true when the Novikov conjecture holds for the group π, i.e. when the higher signatures are homotopy invariants for manifolds with fundamental group π. For example, if the fundamental group is $\mathbb{Z}$, there is now one additional restriction on the variation of L-classes within a homotopy type by the homotopy invariance of the *codimension one signature*, i.e. the signature of the transverse inverse image of a point for a smooth map of M to the circle inducing an isomorphism on fundamental group. Note the homotopy invariance of this signature is not at all obvious, and it is the only new restriction.

With respect to the surgery exact sequence, we can use functoriality. The surgery homological surgery map $H_n(M, \mathbb{L}_\bullet^{\langle 1\rangle}) \to L_n(\mathbb{Z}[\pi])$ factors through $H_n(B\pi; \mathbb{L}_\bullet^{\langle 1\rangle})$. Rationally the map $H_n(M; \mathbb{L}_\bullet^{\langle 1\rangle}) \to H_n(B\pi; \mathbb{L}_\bullet^{\langle 1\rangle})$ is the higher signature map.

We return to the following diagram

$$\begin{array}{ccc} H_*(M; \mathbb{L}_\bullet^{\langle 1\rangle}) & \xrightarrow{\quad A \quad} & L_*(\mathbb{Z}[\pi]) \\ & {\scriptstyle \rho} \searrow \qquad \nearrow {\scriptstyle \phi} & \\ & H_*(B\pi; \mathbb{L}_\bullet^{\langle 1\rangle}) & \end{array}$$

The Novikov conjecture implies that ϕ is rationally injective. Therefore ker $\rho =$ ker A rationally. By Poincaré duality, we know that

$$\ker \rho \otimes \mathbb{Q} = \operatorname{coker}\,(H^*(B\pi; \mathbb{L}_\bullet^{\langle 1\rangle}) \otimes \mathbb{Q} \to H^*(M; \mathbb{L}_\bullet^{\langle 1\rangle}) \otimes \mathbb{Q}).$$

Since $\mathbb{L}_\bullet \otimes \mathbb{Q} = \bigoplus_{i=1}^{\infty} K(\mathbb{Q}, 4i)$, we have

$$\ker \rho \otimes \mathbb{Q} = \operatorname{coker}\,(H^*(B\pi; \mathbb{L}_\bullet^{\langle 1\rangle}) \otimes \mathbb{Q} \to \bigoplus_i H^{4i}(M; \mathbb{Q})).$$

Remark 5.53. *One can rephrase the above criterion for variability of $\mathbb{Q}$-Pontrjagin classes. If M is a Top n-manifold with $n \geq 5$ and fundamental group π, and if the cokernel of $H^{4i}(B\pi; \mathbb{Q}) \to H^{4i}(M; \mathbb{Q})$ is non-trivial for some i with* $0 < 4i < \dim(M)$, *then $|S^{Top}(M)| = \infty$ and $|S^{Diff}(M)| = \infty$, if nonempty, for a reason detected by rational Pontrjagin classes. If the Novikov Conjecture is assumed, this condition is necessary as well.*

Chapter Six

Beyond characteristic classes

This chapter is devoted to the existence of interesting manifolds using surgery. We will see how torsion in the fundamental group of a manifold forces the existence of such manifolds. We will also discuss Cappell's manifolds that are homotopy equivalent to connected sums that are not themselves connected sums. As shown in Chapter 1 this phenomenon is impossible for simply connected manifolds. It is a deep theorem of Cappell that it is also impossible when the fundamental group has no 2-torsion. We will also discuss the space form problem, which identifies those groups which act freely on some sphere.

6.1 A SECONDARY SIGNATURE INVARIANT

In their analysis of free involutions on the sphere, Browder and Livesay [93] introduced an invariant which they used to study the existence and uniqueness question for desuspensions. In particular, they produced infinitely many manifolds homotopy equivalent, but not diffeomorphic or even homeomorphic, to $\mathbb{RP}^{4k-1}$ for $k \geq 2$, although they did not know the topological invariance of their invariant at that time. Hirzebruch [319] soon gave an alternative definition of this invariant as follows. For any manifold M of dimension $4k-1$ with $\pi_1(M) = \mathbb{Z}_2$, there is a $4k$-dimensional manifold W^{4k} such that $\partial W = rM$ induces an isomorphism $\pi_1(M) \to \pi_1(W)$. Here rM means the disjoint union $M \coprod \cdots \coprod M$ of r copies of M. If $\widetilde{W}$ denotes the universal cover of W, they define an invariant

$$\tau(M) = \frac{1}{r}\left(\frac{\operatorname{sig}(\widetilde{W})}{2} - \operatorname{sig}(W)\right).$$

The fact that $\tau(M)$ is independent of W can be seen by a bordism argument which we will explain below.

In this section, we will provide a generalization of the Browder-Livesay invariant to study manifolds whose fundamental group has torsion. We will limit our discussion to orientable manifolds with dimension $4k+3$ with $k \geq 1$, and a broad class of fundamental groups.

Definition 6.1. *We say that a group G is residually finite if, for each non-trivial element $g \in G$, there is a normal subgroup H of finite index such that $g \notin H$. Equivalently, a group is residually finite if the intersection of all its subgroups of finite index is trivial.*

Let G be a residually finite group with torsion, and let g be an element in G of finite order n. Choose H as in the definition and set $N = G/H$. We will now define an invariant that measures the difference between the signature (projection) map $\mathrm{sig} : L_0(\mathbb{Z}[N]) \to L_0(\mathbb{Z}[e]) \cong \mathbb{Z}$ and the transfer map $\mathrm{sig}_N : L_0(\mathbb{Z}[N]) \to L_0(\mathbb{Z}[e]) \cong \mathbb{Z}$.

Definition 6.2. *Let H be a normal subgroup of G of finite index and let $N = G/H$ be the quotient group. Denote by $\mathrm{sig}_N : L_0(\mathbb{Z}[N]) \to L_0(\mathbb{Z}[e])$ the associated transfer map. Define the τ-invariant for $N = G/H$ to be the map $\tau_N : L_0(\mathbb{Z}[N]) \to \mathbb{Q}$ given by*

$$\tau_N(V) = \frac{\mathrm{sig}_N(V)}{|N|} - \mathrm{sig}(V)$$

for all $V \in L_0(\mathbb{Z}[N])$.

Lemma 6.3. *Let g and H be as given above with $\Gamma = \langle g \rangle \cong \mathbb{Z}_n$. If $i : \mathbb{Z}_n \to N$ is the induced projection map, then the composition*

$$L_{4k}(\mathbb{Z}[\Gamma]) \xrightarrow{i_*} L_{4k}(\mathbb{Z}[N]) \xrightarrow{\tau_N} \mathbb{Q}$$

is equal to τ_Γ and is non-trivial.

Proof. Let $i_* : L_{4k}(\mathbb{Z}[\Gamma]) \to L_{4k}(\mathbb{Z}[N])$ be the map induced by i. By the induction property of the signature map, i.e.

$$\tau_N(i_*(W)) = \frac{\mathrm{sig}_N(i_*W)}{|N|} - \mathrm{sig}(i_*V) = \frac{1}{|N|} \cdot \frac{|N|}{|\Gamma|} \mathrm{sig}_\Gamma(W) - \mathrm{sig}(W) = \tau_\Gamma$$

for all $W \in L_{4k}(\mathbb{Z}[\Gamma])$, we have the commutative diagram

$$\begin{array}{ccc} L_{4k}(\mathbb{Z}[\Gamma]) & \xrightarrow{\tau_\Gamma} & \mathbb{Q} \\ {\scriptstyle i_*}\downarrow & \nearrow{\scriptstyle \tau_N} & \\ L_{4k}(\mathbb{Z}[N]) & & \end{array}$$

We now check that $\tau_\Gamma : L_{4k}(\mathbb{Z}[\Gamma]) \to \mathbb{R}$ is non-trivial. The argument is very similar to that in Section 2.3. We know that, localized away from 2, the L-groups for $\mathbb{Z}[\Gamma]$ and $\mathbb{Q}[\Gamma]$ are the same. Because of the Ranicki-Rothenberg sequence in Theorem 2.62, it also remains the same when we change the decoration (s, p, h). We therefore only need to produce a bilinear form on a projective module over $\mathbb{Q}[\Gamma]$. In particular, we can choose the 1×1 bilinear form [1] on the module $\mathbb{Q}$, which is projective over $\mathbb{Q}[\Gamma]$. By definition, our invariant is $\frac{1}{|\Gamma|} - 1$ for this element, which is always non-trivial for

$n \geq 2$. □

We now define the τ-invariant for manifolds.

Thom's work on cobordism implies that every compact, odd-dimensional, oriented manifold M with fundamental group π has a multiple rM which is the boundary of an oriented Top manifold W. The statement holds even if we insist that the fundamental group of the null-cobordism also be π. We can extend to residually finite groups since the theorem can be used on the finite quotients.

Remark 6.4. *Actually using the result of Baumslag-Dyer-Heller mentioned before, one can prove Hausmann's theorem [298] that any null-cobordant manifold M bounds a manifold W so that $\pi_1(M) \to \pi_1(W)$ is injective.*

One can actually also prove that any null-cobordant manifold with finite fundamental group also bounds a manifold with finite fundamental group, where the group is of much larger order.

Definition 6.5. *Let M^{4k-1} be a Cat manifold with $k \geq 2$ and let $G = \pi_1(M)$. Choose $r \geq 1$ and W^{4k} so that $\partial W = rM$ and $\pi_1(M)$ injects into $\pi_1(W)$. Let H be a normal subgroup of G of finite index and define the τ-invariant to be the rational number*

$$\tau_\Gamma(M) = \frac{1}{r}\left(\frac{\mathrm{sig}(W_\Gamma)}{[G:\Gamma]} - \mathrm{sig}(W)\right)$$

where W_Γ is the Γ-cover of W. Note that $\pi_1(W_\Gamma) = \Gamma$.

Remark 6.6. *If W is a closed manifold such that Γ is a subgroup of $G = \pi_1(W)$, then* $\mathrm{sig}(W_\Gamma) = [G:\Gamma]\,\mathrm{sig}(W)$ *by the multiplicative property of the signature. In this case, the right-side of the above equation is zero. Therefore, for manifolds W with boundary, this difference in the displayed equation measures a sort of signature defect.*

Proposition 6.7. *Let M be as given in the definition. The τ-invariant $\tau_\Gamma(M^{4k-1})$ is independent of the choice of W.*

Proof. Let Γ be a finite-index subgroup of G. Let W^{4k} and Y^{4k} be manifolds such that $\partial W = rM$ and $\partial Y = sM$, with the additional property that G injects into both $\pi_1(W)$ and $\pi_1(Y)$. Let $j = [G:\Gamma]$. Take s copies of W and r copies of Y. Attach them by their boundaries to obtain a space X without boundary. Since signature behaves multiplicatively with respect to covers, we have $\mathrm{sig}(X_\Gamma) = j\,\mathrm{sig}(X)$. Novikov additivity implies that the signatures behave additively with respect to splitting, so

$$s\,\mathrm{sig}(W_\Gamma) - r\,\mathrm{sig}(Y_\Gamma) = j(s\,\mathrm{sig}(W) - r\,\mathrm{sig}(Y)).$$

Rearranging terms, we have

$$\frac{1}{r}\left(\frac{\text{sig}(W_\Gamma)}{[G:\Gamma]} - \text{sig}(W)\right) = \frac{1}{s}\left(\frac{\text{sig}(Y_\Gamma)}{[G:\Gamma]} - \text{sig}(Y)\right). \qquad \square$$

Theorem 6.8. *Let $k \geq 2$. Consider any compact oriented (smooth) manifold M^{4k-1} such that $G = \pi_1(M)$ has non-trivial torsion and is residually finite. Then the simple structure set $\mathcal{S}_s^{Cat}(M)$ is infinite for all three choices of Cat.*

Proof. Suppose that $G = \pi_1(M)$ is a residually finite group with torsion. Choose $g \in G$ with prime order p. By residual finiteness, there is a finite-index normal subgroup $H \lhd G$ that misses g. Define $\Gamma = \langle g \rangle \cong \mathbb{Z}_p$ and $N = G/H$. Let $i : \mathbb{Z}_p \to N$ be the induced map. By Lemma 6.3, since τ_Γ is non-trivial, then τ_N is also non-trivial. Therefore τ_N has an infinite image in $\mathbb{Q}$. Choose $V_{\eta_1}, V_{\eta_2}, \ldots$ in $L_0(\mathbb{Z}[N])$ such that, for all distinct pairs i and j, we have $\tau_N(V_{\eta_i}) \neq \tau_N(V_{\eta_j})$. For each i, let (M_i, f_i) be the Top manifold structure in $\mathcal{S}^{Top}(M)$ produced when V_{η_i} acts on the trivial manifold structure $id : M \to M$. Then the M_i are all homotopy equivalent to each other; we will show that they are all homeomorphically distinct.

Each V_{η_i} can be regarded geometrically as a cobordism between M and M_i. If $\tau_N(M) = \frac{1}{r}\left(\frac{\text{sig}_G(W_G)}{[G:H]} - \text{sig}(W)\right)$, then one can modify W by glueing a copy of the cobordism V_{η_i} along M to each boundary component of W, forming a space W' such that $\partial W' = rM_i$. By Novikov additivity, the difference $\tau_N(M_i) - \tau_N(M)$ can be identified with the quantity $\tau_N(V_{\eta_i})$. Therefore, for any distinct i and j, we have

$$\tau_N(M_i) - \tau_N(M_j) = \tau_N(M_i) - \tau_N(M) + \tau_N(M) - \tau_N(M_j) = \tau_N(V_{\eta_i}) - \tau_N(V_{\eta_j}) \neq 0.$$

Therefore infinitely many homeomorphically distinct manifolds are detected by τ_N. $\square$

Remark 6.9.

1. *The analogue for M^{4k+1} is false. For instance, the real projective space $\mathbb{RP}^{4k+1}$ has a finite structure set. See López de Medrano [403] as well as Section 6.2. Also, we cannot dispense with the orientability hypothesis as $\mathbb{RP}^2 \times \mathbb{S}^5$ has finite structure set.*
2. *If M is smooth, all the M_i can be taken smooth as well and are not Top homeomorphic. The extension of the τ-invariant from smooth to topological manifolds can be done by bordism arguments. See Chang-Weinberger [152] or the discussion in Wall Chapter 13. Note that this result is stronger than asserting that they are distinct elements of a structure set, since the structure set may contain a manifold structure where the map is exotic but the manifold is not.*
3. *Using L^2-cohomological ideas, Cheeger-Gromov [163] extend the τ-invariants to groups Γ that are not residually finite. The theorem above asserts that, whenever $\pi_1(M)$ has torsion, then τ_Γ is not a homotopy invariant. Chang-Weinberger [152] give a proof of this result that does not assume residual finiteness. Mathai*

[430] conjectured that the converse is true for τ_Γ *and proved this converse for torsion-free crystallographic fundamental groups. Chang [150] then proved that Mathai's conjecture is actually a consequence of the Borel conjecture for* $\pi_1(M)$ *and is therefore true in a great many cases.*

Remark 6.10. *Hirzebruch's invariant was only shown a bit later to be equal (up to a sign) to the Browder-Livesay invariant by Hirzebruch and López de Medrano [403].*

6.2 MANIFOLDS WITH FUNDAMENTAL GROUP $\mathbb{Z}_2$

The study of $\mathbb{RP}^n$ and its PL manifold structures is equivalent to the classification of fixed-point free involutions of homotopy spheres. Any such involution T on a homotopy sphere Σ^n gives rise to a homotopy equivalence $\Sigma^n/T \to \mathbb{RP}^n$. Furthermore, up to homotopy, this homotopy equivalence is unique for all even n and also for odd n if we additionally demand that the map be orientation-preserving. Once the L-groups for $\mathbb{Z}$ are known, it immediately follows that $\mathcal{S}^{PL}(\mathbb{RP}^n)$ is finite except when $n \equiv 3 \bmod 4$. In this case, an additional summand of $\mathbb{Z}$ is given by the *Browder-Livesay invariant*. The torsion parts in all cases are detected by the normal invariant set $\mathcal{N}^{PL}(\mathbb{RP}^n)$. At the end we will also give some computations for $\mathcal{S}^{Top}(\mathbb{RP}^n)$ that are somewhat simpler.

Browder and Livesay gave an approach to this problem using ideas from simply connected surgery. They consider pairs (M^n, N^{n-1}) where $f : M \to \mathbb{RP}^n$ is a homotopy equivalence and N is the transverse inverse image of $\mathbb{RP}^{n-1}$. If they provide a homotopy for f such that N is homotopy equivalent to $\mathbb{RP}^{n-1}$, then M would be the Thom space of the (unique) non-trivial real line bundle over N, and one would have reduced the classification problem in dimension n to that in dimension $n-1$. Therefore, Browder and Livesay studied the series of *desuspension obstructions* and their relative versions, i.e. the extent to which the desuspension of a given free involution is unique. We will largely follow López de Medrano's discussion interjected with the Wall and Browder-Livesay perspectives.

The PL surgery exact sequence for this problem is given by

$$\cdots \to L_{n+1}(\mathbb{Z}[\mathbb{Z}_2^{\pm}]) \to \mathcal{S}^{PL}(\mathbb{RP}^n) \to [\mathbb{RP}^n : F/PL] \to L_n(\mathbb{Z}[\mathbb{Z}_2^{\pm}]).$$

The first goal of this section is to compute $[\mathbb{RP}^n : F/PL]$ for all $n \geq 5$. We will see that the answer depends only on $n \bmod 4$. We will also give attention to the orientation of each projective space in order to identify the L-groups $L_n(\mathbb{Z}[\mathbb{Z}_2^{\pm}])$ correctly.

Since KO has no odd torsion, there is no contribution of $F/PL[1/2]$ to the computation of the normal invariant set. Therefore all the information is encoded in $F/PL_{(2)}$. Let Y be the two-stage Postnikov system of $F/PL_{(2)}$ with $\pi_2(Y) = \mathbb{Z}_2$ and $\pi_4(Y) = \mathbb{Z}$. The computation of

$$[\mathbb{RP}^n : F/PL_{(2)}] = [\mathbb{RP}^n : Y] \times \prod_{k=1}^{\infty} H^{4k-2}(\mathbb{RP}^n; \mathbb{Z}_2) \times \prod_{k=1}^{\infty} H^{4k}(\mathbb{RP}^n; \mathbb{Z}_{(2)})$$

reduces to the calculation $[\mathbb{RP}^n : Y]$ since we know that the cohomology terms are all $\mathbb{Z}_2$.

Lemma 6.11. *Consider the map $\phi : \mathbb{RP}^{2n+1} \to \mathbb{CP}^n$ induced from the natural fibration $\mathbb{S}^{2n+1} \to \mathbb{CP}^n$ by identifying antipodal points in each fiber. If $\pi_j(X) = 0$ for all odd $j \le 2n+1$, then the induced map $[\mathbb{CP}^n : X] \to [\mathbb{RP}^{2n+1} : X]$ is a surjection.*

Lemma 6.12. *Consider the cofibration given by $\mathbb{S}^n \xrightarrow{\pi_n} \mathbb{RP}^n \xrightarrow{i_n} \mathbb{RP}^{n+1}$. If X is an H-space, there is an induced Puppe exact sequence*

$$\cdots \to [\Sigma\mathbb{RP}^{n+1} : X] \xrightarrow{\Sigma i_n^*} [\Sigma\mathbb{RP}^n : X] \xrightarrow{\Sigma\pi_n^*} \pi_{n+1}(X) \xrightarrow{j_{n+1}^*} [\mathbb{RP}^{n+1} : X]$$
$$\xrightarrow{i_n^*} [\mathbb{RP}^n : X] \xrightarrow{\pi_n^*} \pi_n(X).$$

Lemma 6.13. *The maps in the Puppe sequence above satisfy the following.*

1. *If n is odd, then* $\operatorname{im} \pi_n^* = 2\pi_n(F/PL)$ *and* $\ker j_{n+1}^* = 2\pi_{n+1}(F/PL)$.
2. *If n is even, then* $\operatorname{im} \pi_n^* = 0$ *and* $\ker j_{n+1}^* = 0$.

Remark 6.14. *These kernels and images are calculated from the structure of F/PL. The entire calculation is cohomology, except for the k-invariant.*

We also note that, since $F/PL_{(2)}$ has no odd homotopy groups, we can conclude that $[\mathbb{RP}^{4k} : F/PL] \cong [\mathbb{RP}^{4k+1} : F/PL]$ and $[\mathbb{RP}^{4k+2} : F/PL] \cong [\mathbb{RP}^{4k+3} : F/PL]$. We are finally able to compute $[\mathbb{RP}^n : F/PL]$.

Theorem 6.15. *For all $n \ge 4$, let $r(n) = [\frac{n-4}{2}]$. Then $[\mathbb{RP}^n : F/PL] \cong \mathbb{Z}_4 \times \mathbb{Z}_2^{r(n)}$.*

Proof. It suffices to prove that $[\mathbb{RP}^n : Y] = \mathbb{Z}_4$. Since $[\mathbb{RP}^n : Y] = [\mathbb{RP}^4 : Y]$, we use the Puppe exact sequence

$$[\Sigma\mathbb{RP}^3 : Y] \xrightarrow{\Sigma\pi_3^*} \pi_4(Y) \xrightarrow{j_4^*} [\mathbb{RP}^4 : Y] \xrightarrow{i_3^*} [\mathbb{RP}^3 : Y] \xrightarrow{\pi_3^*} \pi_3(Y),$$

which by Lemma 6.13 reduces to

$$0 \to \mathbb{Z}_2 \to [\mathbb{RP}^4 : Y] \to [\mathbb{RP}^3 : Y] \to 0.$$

We know $[\mathbb{RP}^3 : Y] \cong [\mathbb{RP}^2 : Y] \cong \mathbb{Z}_2$ and $[\mathbb{RP}^5 : Y] \cong [\mathbb{RP}^4 : Y]$. From Lemma 3.46 we know that $[\mathbb{CP}^2 : F/PL] \cong \mathbb{Z}_2$. Therefore $[\mathbb{RP}^4 : Y]$ has order 4. By Lemma

6.11 there is therefore a sequence of surjections

$$\mathbb{Z} \cong [\mathbb{CP}^2 : F/PL] \to [\mathbb{RP}^5 : F/PL] \cong [\mathbb{RP}^4 : F/PL] \to [\mathbb{RP}^4 : Y].$$

Hence $[\mathbb{RP}^4 : Y]$ is cyclic and must be isomorphic to $\mathbb{Z}_4$. □

Remark 6.16. *Note that we can also write*

$$[\mathbb{RP}^n : F/PL] = \mathbb{Z}_4 \times \sum_{i=6}^{n} \pi_i(F/PL) \otimes \mathbb{Z}_2.$$

In brief, if $k \geq 1$, we have

$$[\mathbb{RP}^{4k+r} : F/PL] = \begin{cases} \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-2} & \text{i} \quad r = 0, \\ \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-2} & \text{if} \quad r = 1, \\ \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-1} & \text{if} \quad r = 2, \\ \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-1} & \text{if} \quad r = 3. \end{cases}$$

When $\pi = \mathbb{Z}_2$ there are two possibilities for the orientation map $w : \pi \to \mathbb{Z}_2$, which we shall denote by $\mathbb{Z}_2^+$ or $\mathbb{Z}_2^-$. We remind the readers of the relevant L-groups. These groups were computed by Wall [665, 672] and were discussed in Section 2.6. The nonoriented copies of $\mathbb{Z}_2$ are just Arf invariants from $L_2(\mathbb{F}_2)$. The L-groups in the nonoriented case are 2-periodic when there is a central orientation-reversing element of order 2.

$n \bmod 4$	0	1	2	3
$L_n(\mathbb{Z}[\mathbb{Z}_2^+])$	$\mathbb{Z}^2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$L_n(\mathbb{Z}[\mathbb{Z}_2^-])$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0

Definition 6.17. *Let $n \geq 5$ and let $f : M^n \to \mathbb{RP}^n$ be a homotopy equivalence, i.e. $(M, f) \in \mathcal{S}^{PL}(\mathbb{RP}^n)$. The mapping cone of the double covering map $\mathbb{S}^n \to M^n$ defines a map $\Sigma : \mathcal{S}^{PL}(\mathbb{RP}^n) \to \mathcal{S}^{PL}(\mathbb{RP}^{n+1})$ for all $n \geq 5$. We call Σ the structure set suspension map.*

Both of the following lemmas can be proved by adding on a Kervaire manifold to the surgery problem when required.

Lemma 6.18. *The map $\mathbb{Z}_2 \cong L_{4k+2}(\mathbb{Z}[\mathbb{Z}_2]) \to \mathcal{S}^{PL}(\mathbb{RP}^{4k+1})$ is zero.*

Lemma 6.19. *The map $\theta : [\mathbb{RP}^{4k+2} : F/PL] \to \mathbb{Z}_2$ is surjective.*

Theorem 6.20. *Let $k \geq 1$. The PL structure sets of real projective spaces are as given:*

$$\mathcal{S}^{PL}(\mathbb{RP}^{4k+r}) = \begin{cases} \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-3} & \text{if } \quad r = 0, \\ \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-2} & \text{if } \quad r = 1, \\ \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-2} & \text{if } \quad r = 2, \\ \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-2} \times \mathbb{Z} & \text{if } \quad r = 3. \end{cases}$$

Proof. We have the diagram of structure sequences given by the following. Notice that the L-groups respect orientability.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{S}^{PL}(\mathbb{RP}^{4k}) & \xrightarrow{\text{(mono)}} & [\mathbb{RP}^{4k} : F/PL] & \xrightarrow{\theta'} & \mathbb{Z}_2 \\
 & & \Sigma \big\downarrow \text{(mono)} & & \cong \big\uparrow i^* & & \\
\mathbb{Z}_2 & \xrightarrow{0} & \mathcal{S}^{PL}(\mathbb{RP}^{4k+1}) & \xrightarrow{\cong} & [\mathbb{RP}^{4k+1} : F/PL] & \longrightarrow & 0 \\
 & & \Sigma \big\downarrow \cong & & \text{(onto)} \big\uparrow i^* & & \\
0 & \longrightarrow & \mathcal{S}^{PL}(\mathbb{RP}^{4k+2}) & \xrightarrow{\text{(mono)}} & [\mathbb{RP}^{4k+2} : F/PL] & \xrightarrow{\text{(onto)}} & \mathbb{Z}_2 \\
 & & \Sigma \big\downarrow \text{(mono)} & & \cong \big\uparrow i^* & & \\
\mathbb{Z}^2 & \longrightarrow & \mathcal{S}^{PL}(\mathbb{RP}^{4k+3}) & \xrightarrow{\alpha} & [\mathbb{RP}^{4k+3} : F/PL] & \xrightarrow{\theta} & \mathbb{Z}_2 \\
 & & \Sigma \big\downarrow & & \big\uparrow i^* & & \\
0 & \longrightarrow & \mathcal{S}^{PL}(\mathbb{RP}^{4k+4}) & \xrightarrow{\text{(mono)}} & [\mathbb{RP}^{4k+4} : F/PL] & \xrightarrow{\theta'} & \mathbb{Z}_2
\end{array}$$

Lemma 6.18 shows that $\mathbb{Z}_2 \to \mathcal{S}^{PL}(\mathbb{RP}^{4k+1})$ is the zero map. Lemma 6.19 shows that $[\mathbb{RP}^{4k+2} : F/PL] \to \mathbb{Z}_2$ is surjective. We wish to discuss a few other maps, namely

1. $\mathbb{Z}^2 \to \mathcal{S}^{PL}(\mathbb{RP}^{4k+3})$
2. $[\mathbb{RP}^{4k+3} : F/PL] \to \mathbb{Z}_2$
3. $[\mathbb{RP}^{4k} : F/PL] \to \mathbb{Z}_2$

(1) Forming the connected sum with a Milnor manifold, we have a map

$$0 \to L_0(\mathbb{Z}[\mathbb{Z}_2])/L_0(\mathbb{Z}[e]) = \mathbb{Z} \to \mathcal{S}^{Top}(\mathbb{RP}^{4k+3}).$$

The previous term lies in the simply connected summand because of the assembly map considerations of the previous section and the torsionness of $H^*(\mathbb{RP}^\infty)$. The Browder-Livesay definition immediately gives directly a desuspension invariant defined on $\mathcal{S}^{Top}(\mathbb{RP}^{4k+3})$, i.e. the signature of the quadratic form (u, Tv) on the codimension one invariant submanifold in the 2-fold cover, which splits this exact sequence.

(2) The map $[\mathbb{RP}^{4k+3} : F/PL] \to L_3(\mathbb{Z}[\mathbb{Z}_2^+])$ represents the codimension one Arf invariant. The nonzero element of order 2 in $L_3(\mathbb{Z}[\mathbb{Z}_2^+])$ lies in the image of $L_3(\mathbb{Z}[e]) \times L_2(\mathbb{Z}[e]) \cong L_3(\mathbb{Z}[\mathbb{Z}]) \to L_3(\mathbb{Z}[\mathbb{Z}_2])$. Therefore, if $f : M \to \mathbb{RP}^{4k+3}$ is a normal invariant, we can form the connect sum $M\#(\mathbb{S}^1 \times K^{4k+2})$ along a circle generating the fundamental group. The surgery obstruction of the new normal invariant is now zero. Therefore the surgery map is surjective.

(3) Because the map $L_{4k}(\mathbb{Z}[e]) \to L_{4k}(\mathbb{Z}[\mathbb{Z}_2^-])$ is zero, it will not help us to add a Milnor manifold. Instead, we puncture $\mathbb{RP}^{4k}$ to arrive at an exact sequence

$$\mathcal{S}^{Top}(\mathbb{RP}^{4k}\backslash\mathbb{D}^{4k}, \mathbb{S}^{4k-1}) \to [\mathbb{RP}^{4k}\backslash\mathbb{D}^{4k} : F/Top] \to L_{4k}(\mathbb{Z}[\mathbb{Z}_2^-], \mathbb{Z}[e]).$$

The middle term is the same as $[\mathbb{RP}^{4k-1} : F/Top]$. There is a commutative triangle

$$\begin{array}{ccc} & & L_3(\mathbb{Z}[\mathbb{Z}_2^+]) \\ & \nearrow & \Big\downarrow {\scriptstyle tr} \\ [\mathbb{RP}^{4k-1} : F/Top] & \longrightarrow & L_0(\mathbb{Z}[\mathbb{Z}_2^-],\ \mathbb{Z}[e]) \end{array}$$

From (2) we know that the diagonal arrow given by the codimension one Arf invariant is surjective. Now $L_3(\mathbb{Z}[\mathbb{Z}_2^+]) \cong \mathbb{Z}_2$ where the nonzero element is the codimension one Arf invariant. We will now show that $[\mathbb{RP}^{4k} : F/Top] \to L_{4k}(\mathbb{Z}[\mathbb{Z}_2^-])$ is a surjection. Again we use the trick of puncturing and using the Poincaré conjecture (cf. Proposition 1.71). Given the identification $\mathcal{S}^{Top}(\mathbb{RP}^{4k}) = \mathcal{S}^{Top}(\mathbb{RP}^{4k}\backslash\mathbb{D}^{4k}, \mathbb{S}^{4k-1})$, we have the following sequence

$$\mathcal{S}^{Top}(\mathbb{RP}^{4k}\backslash\mathbb{D}^{4k}, \mathbb{S}^{4k-1}) \to [\mathbb{RP}^{4k}\backslash\mathbb{D}^{4k} : F/Top] \to L_{4k}(\mathbb{Z}[\mathbb{Z}_2^-], \mathbb{Z}[e]).$$

Note obviously that $[\mathbb{RP}^{4k}\backslash\mathbb{D}^{4k} : F/Top] \cong [\mathbb{RP}^{4k-1} : F/Top]$, and its image in $L_{4k}(\mathbb{Z}[\mathbb{Z}_2^-], \mathbb{Z}[e])$ is the same as the transfer of the normal invariant $[\mathbb{RP}^{4k-1} : F/Top]$ in $L_{4k-1}(\mathbb{Z}[\mathbb{Z}_2^-])$, i.e. the codimension one Arf invariant under the induced line bundle map. One can find a suitable cobordism of it to the product of the Arf invariant and the skew-symmetric form on $\mathbb{RP}^2$. It is nonzero for the same algebraic reason that the product of any non-trivial surgery obstruction and the bilinear form on $\mathbb{CP}^2$ is nonzero.

The desired computations for $\mathcal{S}^{PL}(\mathbb{RP}^n)$ can be performed with all this information. □

Here we repeat the previous diagram with all calculations in place. The first line corresponds to $\mathbb{RP}^{4k}$.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-3} & \xrightarrow{\text{(mono)}} & \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-2} & \xrightarrow{\theta'} & \mathbb{Z}_2 \\
 & & \Sigma \Big\downarrow \text{(mono)} & & \cong \Big\uparrow i^* & & \\
\mathbb{Z}_2 & \xrightarrow{0} & \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-2} & \xrightarrow{\cong} & \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-2} & \longrightarrow & 0 \\
 & & \Sigma \Big\downarrow \cong & & \text{(onto)} \Big\uparrow i^* & & \\
0 & \longrightarrow & \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-2} & \xrightarrow{\text{(mono)}} & \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-1} & \xrightarrow{\text{(onto)}} & \mathbb{Z}_2 \\
 & & \Sigma \Big\downarrow \text{(mono)} & & \cong \Big\uparrow i^* & & \\
\mathbb{Z}^2 & \longrightarrow & \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-2} \times \mathbb{Z} & \xrightarrow{\alpha} & \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-2} & \xrightarrow{\theta} & \mathbb{Z}_2 \\
 & & \Sigma \Big\downarrow & & \Big\uparrow i^* & & \\
0 & \longrightarrow & \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-1} & \xrightarrow{\text{(mono)}} & \mathbb{Z}_4 \times \mathbb{Z}_2^{2k} & \xrightarrow{\theta'} & \mathbb{Z}_2
\end{array}
$$

Remark 6.21. *In the case of $\mathcal{S}^{PL}(\mathbb{RP}^{4k+3}) \cong \mathbb{Z}_4 \times \mathbb{Z}_2^{2k-2} \times \mathbb{Z}$, the finite part is detected by the normal invariant, and the infinite cyclic part by the Browder-Livesay invariant, which provides a precise obstruction to finding an invariant $(4k+2)$-sphere of a free differentiable involution T of a homotopy $(4k+3)$-sphere. López de Medrano shows how to construct free involutions with arbitrary Browder-Livesay invariant on some homotopy $(4k+3)$-spheres and hence that there exist infinitely many distinct involutions. The set $\mathcal{S}^{PL}(\mathbb{RP}^n)$ can be interpreted as the PL equivalence class of PL involutions on $\mathbb{S}^n$.*

We now discuss the Top category briefly. From Theorem 3.97 we have

$$F/Top_{(2)} \simeq \prod_{k=1}^{\infty} K(\mathbb{Z}_2, 4k-2) \times \prod_{k=1}^{\infty} K(\mathbb{Z}_{(2)}, 4k),$$

which coincides with $F/PL_{(2)}$ except for the initial two-term Postnikov system which we had denoted earlier by Y. Parallel to Remark 6.16, we have the following.

Proposition 6.22. *Let $n \geq 5$. The normal invariant set for $\mathbb{RP}^n$ is*

$$[\mathbb{RP}^n : F/Top] = \sum_{i=2}^{n} \pi_i(F/PL) \otimes \mathbb{Z}_2 = \mathbb{Z}_2^{[n/2]}.$$

The remaining arguments for the PL case can be reused for the Top case. Note that we merely replace the $\mathbb{Z}_4$ in all the structure sets $\mathcal{S}^{PL}(\mathbb{RP}^n)$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Theorem 6.23. *Let $k \geq 1$. The Top structure sets of real projective spaces are as given:*

$$\mathcal{S}^{Top}(\mathbb{RP}^{4k+r}) = \begin{cases} \mathbb{Z}_2^{2k-1} & \text{if} \quad r = 0, \\ \mathbb{Z}_2^{2k} & \text{if} \quad r = 1, \\ \mathbb{Z}_2^{2k} & \text{if} \quad r = 2, \\ \mathbb{Z}_2^{2k} \times \mathbb{Z} & \text{if} \quad r = 3. \end{cases}$$

Remark 6.24. *The difference between $\mathcal{S}^{PL}(\mathbb{RP}^n)$ and $\mathcal{S}^{Top}(\mathbb{RP}^n)$ gives us a counterexample to the Hauptvermutung in all dimensions at least 5.*

Remark 6.25. *With Top surgery, the map $[\mathbb{RP}^n : F/PL] \to [\mathbb{RP}^n : F/Top]$ is described as the direct sum of an isomorphism with the map $\mathbb{Z}_4 = [\mathbb{RP}^4 : F/PL] \to [\mathbb{RP}^4 : F/Top] = \mathbb{Z}_2 \times \mathbb{Z}_2$, sending $\mathbb{Z}_4$ to $\mathbb{Z}_2 = \pi_2(F/Top)$.*

Remark 6.26. *When $n = 4k + r$ is even, the structure set $\mathcal{S}^{Top}(\mathbb{RP}^n)$ is detected by the normal invariants $[\mathbb{RP}^n : F/Top] \cong \mathbb{Z}^{[n/2]}$, and can be interpreted as the splitting invariants for splitting along copies of $\mathbb{RP}^{2k}$ inside, similarly to the $\mathbb{CP}^n$ situation which we discussed at the end of Section 3.4. In dimension 2 mod 4 there are no additional pieces, but in dimension 0 mod 4, there is an extra copy of $\mathbb{Z}$ that comes from the pairing of the codimension two piece with $\pi_2(F/Top)$.*

Remark 6.27. *Freedman's work on four-dimensional surgery shows that everything works in dimension 4, so $\mathcal{S}^{Top}(\mathbb{RP}^4) = \mathbb{Z}_2$; i.e. there is one topological homotopy $\mathbb{RP}^4$. It cannot be given a smooth or PL structure because its normal invariant does not lie in the image of $[\mathbb{RP}^4 : F/PL] \to [\mathbb{RP}^4 : F/Top]$. On the other hand, the set $\mathcal{S}^{PL}(\mathbb{RP}^4)$ computed formally would suggest another smooth homotopy $\mathbb{RP}^4$ that is homeomorphic to $\mathbb{RP}^4$. Of course, nowadays there is no good existence theory for smooth 4-manifolds. However, Cappell-Shaneson [124] constructed such a manifold. Fintushel-Stern [248] also constructed such a manifold whose two-fold cover is the standard 4-sphere. Freedman's work proves that these manifolds are Top homeomorphic to $\mathbb{RP}^4$.*

6.3 GENERAL SPLITTABILITY

This section returns to the problem of codimension one splitting, which is stated as follows. Let $f : W' \to W$ be a homotopy equivalence and let M be a distinguished connected submanifold of W. The problem is to decide whether there is a homotopy from f to another homotopy equivalence $g : W' \to W$ such that g is transverse to M and the restriction $g|_{g^{-1}(M)} : g^{-1}(M) \to M$ is also a homotopy equivalence.

When we studied applications of the π-π theorem in Section 1.3, we saw that, if $\pi_1(M) \to \pi_1(W)$ an isomorphism and if M is a locally separating submanifold, i.e. one with trivial normal bundle, then splitting is always possible. As a result, for example, being a connected sum is a homotopy invariant for simply connected PL or Top manifolds in high dimensions. If W_+ is a component of $W \backslash M$, then this splitting is also possible when $\pi_1(M) \to \pi_1(W_+)$ is an isomorphism.

However, when the normal bundle is non-trivial, there are Browder-Livesay obstructions to a solution for this problem, as we discussed in the previous section. For the countably infinite number of homotopy projective spaces $\mathbb{RP}^{4k+3}$, one cannot split the homotopy equivalence to $\mathbb{RP}^{4k+3}$ along $\mathbb{RP}^{4k+2}$ despite the fact that the map $\mathbb{RP}^{4k+2} \to \mathbb{RP}^{4k+3}$ induces an isomorphism on their fundamental groups.

In this section, we will study some theorems of Cappell [111] about this problem. He discovered, for example, that the L-group $L_2(\mathbb{Z}[D_\infty])$ is infinitely generated, and that there are infinitely many manifolds homotopy equivalent to $\mathbb{RP}^{4k+1} \# \mathbb{RP}^{4k+1}$ that are not themselves connected sums. On the other hand, when the fundamental group has no 2-torsion, then being a connected sum is a homotopy invariant, as in the simply connected case.

The obstruction to splitting is the L-group $L_*(\Phi)$ of a quadrad Φ, whose vanishing would provide a Mayer-Vietoris sequence for L-groups of an amalgamated free product. Cappell gives conditions for which the group vanishes. Instead of the L-group of some quadrad, he uses bimodules to give an alternative algebraic description of the L-group as a UNil group. Here UNil is a Hermitian analogue of the Nil groups that enter in calculations of Whitehead and other K-groups of Laurent polynomials of rings, or amalgamated free products and HNN extensions of groups. See Bass, Bass-Murthy [48], and Waldhausen [659]. This group is always 2-primary, indeed of exponent 4, and the UNil group always splits off of the L-group of the amalgamated free product or HNN extension.

Virtually cyclic groups are always amalgamated free products $A *_B C$ or HNN extensions $A *_B$ of finite groups, where the subgroup B is of index two in both A and C. The Farrell-Jones conjecture implies that one needs to understand these groups in order to understand all L-groups.

In this section, we will first review the surgery theory for manifold triads and tetrads, and explain how UNil appears in the computation for the surgery obstruction. Following this treatment we will quickly define the UNil groups and state some of its most important properties. The example of Cappell mentioned above shows that the splitting problem is highly non-trivial. Lastly we will discuss some recent results in UNil computation and describe some of its immediate consequences.

6.3.1 Quadrads

Suppose that W is a closed manifold and M a codimension one submanifold of W with a trivial normal bundle. In this case, van Kampen's theorem states that $\pi_1(W)$ is either an amalgamated free product

$$\pi_1(W) \cong \pi_1(W^+) *_{\pi_1(M)} \pi_1(W^-),$$

where W^+ and W^- are the two components of the complement if the complement is disconnected, or an HNN extension $\pi_1(W) \cong \pi_1(W^+) *_{\pi_1(M)}$ if the complement is connected. For simplicity of notation we will focus on the disconnected case.

A fairly technical issue that we will avoid is that of decorations. Recall that, because of product formulas, a simple homotopy equivalence $W' \to M \times \mathbb{S}^1$ can at most split to produce a homotopy equivalence. If one has a split homotopy equivalence $W' \to M \times \mathbb{S}^1$, then its torsion is automatically a norm by the Milnor duality theorem and addition theorems for Whitehead torsion. This issue arose when we discussed the Farrell-Hsiang splitting theorem 2.102 and is discussed in all generality in Cappell's papers [111]. One can safely ignore this problem by using the $L^{-\infty}$ decoration. Here $L_n^{-\infty}$ is the limit of the maps

$$L_n^p(\mathbb{Z}[\pi]) \to L_n^{-1}(\mathbb{Z}[\pi]) \to L_n^{-2}(\mathbb{Z}[\pi]) \to \cdots$$

where $L_n^{-1}(\mathbb{Z}[\pi]) = \ker (L_{n+1}^p(\mathbb{Z}[\mathbb{Z} \times \pi]) \to L_{n+1}^p(\mathbb{Z}[\mathbb{Z} \times \pi]))$ and $L_n^{-k}(\mathbb{Z}[\pi]) = \ker (L_{n+1}^{-k+1}(\mathbb{Z}[\mathbb{Z} \times \pi]) \to L_{n+1}^{-k+1}(\mathbb{Z}[\pi]))$ for all k.

In the situation (W, W^+, W^-) above, let Φ denote the diagram of groups

$$\begin{array}{ccc} \pi_1(M) & \xrightarrow{f^+} & \pi_1(W^+) \\ {\scriptstyle f^-}\downarrow & & \downarrow \\ \pi_1(W^-) & \longrightarrow & \pi_1(M) \end{array}$$

Let $i^+ : \pi_1(W^+) \to \pi_1(W)$ and $i^- : \pi_1(W^-) \to \pi_1(W)$ be induced by the inclusion maps. We can then define L-groups as an analogue of Wall's Chapter 9 but with surgery on *manifolds with corners* instead of merely manifolds with boundary. The construction immediately gives the exact sequences[1]

$$\cdots \to L_{n+1}(\Phi) \to L_n(\pi_1(M) \to \pi_1(W^-)) \to L_n(\pi_1(W^+) \to \pi_1(W)) \to L_n(\Phi) \to \cdots$$

and

$$\cdots \to L_{n+1}(\Phi) \to L_n(\pi_1(M) \to \pi_1(W^+)) \to L_n(\pi_1(W^-) \to \pi_1(W)) \to L_n(\Phi) \to \cdots$$

Here the term $L_n(\pi_1(M) \to \pi_1(W^{\pm}))$ is an alternative notation for $L_n(\pi_1(W^{\pm}), \pi_1(M))$. If $L_n(\Phi) = 0$ for all n, the usual argument that excision is equivalent to Mayer-Vietoris gives an exact sequence

$$\cdots \to L_n(\pi_1(M)) \to L_n(\pi_1(W^+)) \times L_n(\pi_1(W^-))$$

$$\to L_n(\pi_1(W)) \to L_{n-1}(\pi_1(M)) \to \cdots$$

By considering a pair of manifolds with boundary whose boundaries are identified, we

[1] Here $L_*(\Phi)$ is shorthand for $L_*(\mathbb{Z}[\Phi])$.

can use the same techniques to define L-groups $L_n^{split}(\pi_1(M) \to \pi_1(W^+), \pi_1(W^-))$ which would fit into an exact sequence

$$\cdots \to L_n(\pi_1(M)) \to L_n(\pi_1(W^+)) \times L_n(\pi_1(W^-))$$
$$\to L_n^{split}(\pi_1(M) \to \pi_1(W^+), \pi_1(W^-)) \to L_{n-1}(\pi_1(M)) \to \cdots$$

Then there is a forgetful map $L_n^{split}(\pi_1(M) \to \pi_1(W^+), \pi_1(W^-)) \to L_n(\pi_1(W))$ which fits in the expected exact sequence

$$\cdots \to L_{n+1}(\Phi) \to L_n^{split}(\pi_1(M) \to \pi_1(W^+), \pi_1(W^-)) \to L_n(\pi_1(W)) \to \cdots$$

Therefore, the Mayer-Vietoris sequence is equivalent to the statement that the forgetful map $L_n^{split}(\pi_1(M) \to \pi_1(W^+), \pi_1(W^-)) \to L_n(\pi_1(W))$ is an isomorphism; i.e. the obstruction to split surgery is identical to the obstruction to surgery. These ideas are all generalized in Section 8.7 in the discussion of Browder-Quinn's stratified L-groups.

The proof of the π-π splitting case of codimension one splitting shows the following.

Theorem 6.28. *The obstruction to codimension one splitting lies in $L_{n+1}(\Phi)$, and $L_*(\Phi) = 0$ in the π-π situation.*

To be explicit, if I is the unit interval, consider the map $W' \times I \to (W, M) \times I$, and decompose $W' \times \{0\}$ by the transverse inverse image of M to obtain a split surgery problem with a normal cobordism to an unsplit homotopy equivalence, i.e. an element of $L_{n+1}(\Phi)$.

Theorem 6.29. *(Cappell [111]) Let R be a ring containing $1/2$. If one considers $R[\pi]$ instead of $\mathbb{Z}[\pi]$, then $L_n(R[\Phi]) = 0$ for all n. For any ring R, we have $L_n(R[\Phi]) \otimes \mathbb{Z}[1/2] = 0$. In all cases, the L-group $L_*(\Phi)$ splits off $L_*(\pi_1(W))$ as a summand.*

Remark 6.30. *In fact, Ranicki's localization theorem [535] proves that the second statement follows from the first. Then $L_n(R[\Phi])$ has exponent dividing 8.*

There is another 2-torsion aspect to the splitting obstruction.

Definition 6.31. *A subgroup $A \subseteq B$ is square root closed if it has the following property: for all $a \in A$, if $a = b^2$ for some $b \in B$, then $b \in A$.*

Theorem 6.32. *In the context above, if $\pi_1(M) \subseteq \pi_1(W)$ is square root closed, then $L_n(\Phi) = 0$.*

Remark 6.33. *This theorem implies the positive results about homotopy invariance of connected sum in the absence of 2-torsion in the fundamental group.*

The final aspect of the theorem, that $L_*(\Phi)$ splits off of $L_*(\pi_1(W))$, is a geometric statement of "the nilpotent normal cobordism theorem" that one can find a unitary nilpotent normal cobordism of a homotopy equivalence between split manifolds to a split one.

See Cappell [111].

6.3.2 A non-splittable homotopy equivalence

Cappell's first example of a non-splittable homotopy equivalence

$$M \to \mathbb{RP}^{4k+1} \# \mathbb{RP}^{4k+1}$$

is obtained by the Wall realization of an explicit element γ in $L_{4k+2}(\mathbb{Z}[D_\infty])$. This γ is called the *Cappell element*. We will show that γ acts on the identity on $\mathbb{RP}^{4k+1}\#\mathbb{RP}^{4k+1}$ to produce a non-trivial manifold structure in $\mathcal{S}^{Top}(\mathbb{RP}^{4k+1}\#\mathbb{RP}^{4k+1})$ which is not a connected sum. By the exactness of the surgery exact sequence, it is enough to show that γ does not lie in the image of $H_{4k+2}(BD_\infty; \mathbb{L}_\bullet)$. Further, it suffices to consider the 2-localization so it reduces to ordinary homology.

Our strategy is to use an elementary property of group homology. Note that the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ has subgroups Γ_n of index n for any n. These Γ_n are abstractly isomorphic to D_∞. When n is odd, the transfer $H_*(BD_\infty) \to H_*(B\Gamma_n)$ associated to the inclusion map $\Gamma_n \to D_\infty$ is an isomorphism. Our strategy is to examine the behavior of the Cappell element under transfer to large covers.

Recall that $L_{2n}(\mathbb{Z}[\pi])$ is constructed from $(-1)^n$-symmetric quadratic forms over $\mathbb{Z}[\pi]$. Consider the dihedral group $D_\infty \cong \mathbb{Z}_2 * \mathbb{Z}_2$ generated by involutions $g: x \mapsto -x$ and $h: x \mapsto 1-x$. Then $t = gh$ is an affine translation of $\mathbb{Z}$ in the usual view of D_∞ as the set of affine isomorphisms of $\mathbb{Z}$. It is useful actually to construct a whole family of Cappell elements to demonstrate the size of $L_{4k+2}(\mathbb{Z}[D_\infty])$.

Definition 6.34. *For each $n \geq 1$, the Cappell element γ_n in $L_{4k+2}(\mathbb{Z}[D_\infty])$ is defined on a two-dimensional quadratic form, given by (e, f) with $\lambda(e,e) = \lambda(f,f) = 0$ and $\lambda(e,f) = 1$, and with $\mu(e) = g$ and $\mu(f) = t^n g t^{-n}$, where n is odd. Alternatively, we can define $\gamma_n = i_*(\gamma_1)$, where the map $i_*: L_{4k+2}(\mathbb{Z}[\Gamma_n]) \to L_{4k+2}(\mathbb{Z}[D_\infty])$ is induced by the inclusion $\Gamma_n \to D_\infty$.*

The argument is quite short. The augmentation map $L_{4k+2}(\mathbb{Z}[D_\infty]) \to L_{4k+2}(\mathbb{Z}[e])$ sending g and h to the trivial element takes γ_n to the standard Arf invariant 1 element of $L_{4k+2}(\mathbb{Z}[e])$. Let tr denote the transfer maps in both homology and L-theory.

For each n, we choose a larger odd m for which $\mu(f) = t^n g t^{-n}$ is no longer contained in Γ_m, and so $\mathrm{Arf}(tr(\gamma_n)) = 0$. See the diagram below. If γ_n had come from an element β in the homology group, then $\mathrm{tr}(\gamma_n)$ would also come from β, a contradiction. Therefore none of elements is zero.

$$\begin{array}{ccccc}
H_{4k+2}(BD_\infty; \mathbb{L}_\bullet)_{(2)} & \longrightarrow & L_{4k+2}(\mathbb{Z}[D_\infty]) & \xrightarrow{Arf} & L_{4k+2}(\mathbb{Z}[e]) \\
{\scriptstyle tr}\downarrow{\scriptstyle \cong} & & {\scriptstyle tr}\downarrow & & \downarrow \\
H_{4k+2}(B\Gamma_m; \mathbb{L}_\bullet)_{(2)} & \longrightarrow & L_{4k+2}(\mathbb{Z}[\Gamma_m]) & \xrightarrow{Arf} & L_{4k+2}(\mathbb{Z}[e])
\end{array}$$

One can check that different values of n give different elements because they vanish on different transfers, so we have an infinite number of copies of $\mathbb{Z}_2$ in $L_2(\mathbb{Z}[D_\infty])$. Note that splittable maps to $\mathbb{RP}^{4k+1}\#\mathbb{RP}^{4k+1}$ must lie in the image of

$$\mathcal{S}^{Top}(\mathbb{RP}^{4k+1}) \times \mathcal{S}^{Top}(\mathbb{RP}^{4k+1}) \to \mathcal{S}^{Top}(\mathbb{RP}^{4k+1}\#\mathbb{RP}^{4k+1}),$$

and the domain is finite from our calculations in the previous section. Therefore infinitely many of our elements γ_n must lie outside the image, each representing a nonsplittable homotopy equivalence.

Remark 6.35. *Cappell achieves more in his paper. He constructs explicit manifolds that are not connected sums at all, while the theorem above merely constructs nonsplittable maps to* $\mathbb{RP}^{4k+1}\#\mathbb{RP}^{4k+1}$.

6.3.3 Definition of UNil and its basic properties

In the computation of the L-group $L_n^h(\Phi)$, Cappell shows that there is a canonically defined group $\mathrm{UNil}_n^h(R[H]; R[\widehat{G}_1], R[\widehat{G}_2])$ that splits as a direct summand of $L_n^h(\Phi)$. In some cases, these two groups can simply be identified. We include the definition here for completeness, as well as its most important consequences. However, for our purposes only its properties are of importance. More information is found in Cappell's announcements in [112] and [115]. It is unfortunate that he never published a detailed account.

Let R be a (regular) ring with unit and involution, and let E be an R-bimodule with involution. Then E is equipped with a homomorphism $x \mapsto \overline{x}$ such that $\overline{\overline{x}} = x$ and $\overline{axb} = \overline{b}\,\overline{x}\,\overline{a}$ for all $a, b \in R$ and $x \in E$. If D is an R-bimodule, let $\overline{D} = \{\overline{x} : x \in D\}$. We say that E is *hyperbolic* if there is an R-bimodule D for which $E = D \times \overline{D}$. If $\varepsilon \in \{1, -1\}$, we say that an ε-Hermitian form over E is a triple (P, λ, μ), where P is a finitely generated free right R-module and $\lambda : P \times P \to E$ and $\mu : P \to E/\{x - \varepsilon\,\overline{x} : x \in E\}$ are functions that satisfy the usual axioms of L-groups.

If E_1 and E_2 are R-bimodules with involution which are free left R-modules, then an *ε-UNil form over the pair* (E_1, E_2) is a pair of ε-Hermitian forms $(P_1, \lambda_1, \mu_1; P_2, \lambda_2, \mu_2)$ such that $P_1 = P_2^*$ and for which there are filtrations

$$P_1 = P_1^0 \supset P_1^1 \supset P_1^2 \supset \cdots \supset P_1^n = 0,$$

$$P_2 = P_2^0 \supset P_2^1 \supset P_2^2 \supset \cdots \supset P_2^n = 0.$$

Letting $\lambda_1^* : P_1 \to P_1^* \otimes_R E_1 = P_2 \otimes_R E_1$ and $\lambda_2^* : P_2 \to P_2^* \otimes_R E_2 = P_1 \otimes_R E_2$ be the obvious induced maps, we have

$$\lambda_1^*(P_1^k) \subseteq P_2^{k+1} \otimes_R E_1, \qquad \lambda_2^*(P_2^k) \subseteq P_1^{k+1} \otimes_R E_2$$

for all k. If C is such an ε-UNil form, we define its negative $-C$ to be the same pair except that the functions are negated. We say that C is a *kernel* if, for $i = 1, 2$, there are free summands V_i of P_i such that

1. $V_2 \subseteq P_2 = P_1^*$ is the annihilator of $V_1 \subseteq P_1$;
2. $\lambda_i|_{V_i \times V_i}$ and $\mu_i|_{V_i}$ are the zero maps.

Among the ε-UNil forms, define the equivalence relation $A \sim B$ iff $A \times (-B)$ is a kernel. If $\varepsilon = (-1)^k$, form the abelian group $\mathrm{UNil}^h_{2k}(R; E_1, E_2)$ from the equivalence classes of ε-UNil forms under direct sum. By considering simple ε-UNil forms and simple kernels, one can similarly define $\mathrm{UNil}^s_{2k}(R; E_1, E_2)$. Let $R[t, t^{-1}]$ be endowed with the involution given by $\overline{xt^i} \equiv \overline{x}t^{-i}$ and let $M_i' = E_i \otimes_R R[t, t^{-1}]$ be endowed with the induced involution. Define

$$\mathrm{UNil}^h_{2k-1}(R; E_1, E_2) = \mathrm{UNil}^s_{2k}(R[t, t^{-1}]; E_1', E_2')/\mathrm{UNil}^s_{2k}(R; E_1, E_2).$$

There is a periodicity $\mathrm{UNil}^h_n(R; E_1, E_2) \cong \mathrm{UNil}^h_{n+2}(R; E_1^-, E_2^-)$, where E_i^- is the same as E_i but equipped with the involution $x \mapsto -\overline{x}_i$. Similarly there is an isomorphism $\mathrm{UNil}^s_n(R; E_1, E_2) \cong \mathrm{UNil}^s_{n+2}(R; E_1^-, E_2^-)$. If R is a regular ring, then

$$\mathrm{UNil}^h_{2k-1}(R; E_1, E_2) = \mathrm{UNil}^s_{2k-1}(R; E_1, E_2).$$

Let $R \subseteq \Lambda_1$ and $R \subseteq \Lambda_2$ be inclusions of unital rings with involution. Suppose that each Λ_i has an R-bimodule decomposition $\Lambda_i = R \times \widehat{\Lambda}_i$, where $\widehat{\Lambda}_i$ is a free left R-module. Then an ε-UNil form $(P_1, \lambda_1, \mu_1, P_2, \lambda_2, \mu_2)$ over $(\widehat{\Lambda}_1, \widehat{\Lambda}_2)$ determines an ε-Hermitian form (P, λ, μ) over $\Lambda_1 *_R \Lambda_2$ with $P = (P_1 \times P_2) \otimes_R (\Lambda_1 *_R \Lambda_2)$ with the following properties:

1. for all $x \in P_2$ and $y \in P_1$ we have $\lambda(x, y) = \langle x, y \rangle$;
2. for all $i = 1, 2$ and $x, y \in P_i$, we have $\lambda(x, y) = \lambda_i(x, y)$;
3. for all $i = 1, 2$ and $x \in P_i$, we have $\mu(x) = \mu_i(x)$.

The computations for other values of λ and μ follow from the axioms for a Hermitian form. This construction gives a homomorphism $\mathrm{UNil}^h_n(R; \widehat{\Lambda}_2, \widehat{\Lambda}_2) \to L^h_n(\Lambda_1 *_R \Lambda_2)$.

Theorem 6.36. *(Cappell [115])*

1. *The image of the map* $\mathrm{UNil}^h_n(R; \widehat{\Lambda}_2, \widehat{\Lambda}_2) \to L^h_n(\Lambda_1 *_R \Lambda_2)$ *is 2-primary; i.e. the group is torsion and the order of each element is a power of 2.*
2. *The image is zero if* $\widehat{\Lambda}_1$ *and* $\widehat{\Lambda}_2$ *are hyperbolic, or if 2 is invertible in* R.

Let R be a ring with $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$, and let H, G_1, and G_2 be finitely presented groups with $H \subseteq G_1 \cap G_2$. Endow G_1 and G_2 with homomorphisms $\omega_i : G_i \to \mathbb{Z}_2 = \{\pm 1\}$ with $\omega_1|_H = \omega_2|_H$. The group ring $R[G_i]$ on G_i can be given an involution with

$\overline{g} = \omega(g)g^{-1}$ for all $g \in G_i$. Let $R[\widehat{G}_i] \subseteq R[G_i]$ be the $R[H]$-subbimodule additively generated by elements in $G_i \backslash H$.

The following are the statements made earlier.

Theorem 6.37. *The homomorphism*

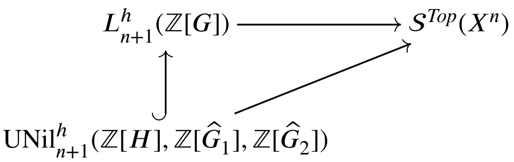
$$\mathrm{UNil}_n^h(R[H]; R[\widehat{G}_1], R[\widehat{G}_2]) \to L_n^h(R[G_1 *_H G_2])$$

is a split monomorphism.

Corollary 6.38. *The group* $\mathrm{UNil}_n^h(R[H]; R[\widehat{G}_1], R[\widehat{G}_2])$ *is 2-primary. If* $1/2 \in R$*, then the group is zero.*

Corollary 6.39. *The group* $\mathrm{UNil}_n^h(R[H]; R[\widehat{G}_1], R[\widehat{G}_2])$ *is trivial if* H *is square root closed in* G_1 *and* G_2*.*

While there is a notion of smooth splittability, we will concentrate mostly on the topological category. Again, let $n \geq 5$ and let X be a Top n-manifold with a codimension 1 separating hypersurface S. Suppose that S divides X into two subsets X_1 and X_2. Let $H = \pi_1(S)$ and let $G_i = \pi_1(X_i)$ for $i = 1, 2$. Then the group $G = \pi_1(X)$ will be the product $G_1 *_H G_2$. The rings $\mathbb{Z}[\widehat{G}_1]$ and $\mathbb{Z}[\widehat{G}_2]$ are $\mathbb{Z}[H]$-bimodules with involution. One can form the group $\mathrm{UNil}_{n+1}^h(\mathbb{Z}[H]; \mathbb{Z}[\widehat{G}_1], \mathbb{Z}[\widehat{G}_2])]$ that injects into $L_{n+1}^h(\mathbb{Z}[G])$. In fact, there is a diagram



where the action of the UNil group on $\mathcal{S}^{Top}(X^n)$ is free on the orbit of the action of $L_{n+1}^h(\mathbb{Z}[G])$. In the general situation, the map $\mathrm{UNil}_{n+1}^h(\mathbb{Z}[H], \mathbb{Z}[\widehat{G}_1], \mathbb{Z}[\widehat{G}_2]) \to \mathcal{S}^{Top}(X^n)$ is a split injection, and we denote by

$$s: \mathcal{S}^{Top}(X^n) \to \mathrm{UNil}_{n+1}^h(\mathbb{Z}[H], \mathbb{Z}[\widehat{G}_1], \mathbb{Z}[\widehat{G}_2])$$

the splitting.

Some important properties about splittability are specific to manifolds X_i with fundamental group $\mathbb{Z}_2$. Note that, if H is trivial and $G_1 = G_2 = \mathbb{Z}_2$, then the rings $\mathbb{Z}[\widehat{G}_i]$ are both rank 1 over $\mathbb{Z}$, i.e. $\mathbb{Z}[\widehat{G}_i] \cong \mathbb{Z}$. Since the Whitehead group of $\mathbb{Z}_2$ vanishes, there is no difference between homotopy equivalence and simple homotopy equivalence. Hence the UNil group can be written as $\mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$. More specifically there are decorations $\pm$ that are required when dealing with manifolds that may not necessarily be orientable, so in principle we have $\mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^{\pm}, \mathbb{Z}^{\pm})$. Some of the results that emerge from the UNil theory of Cappell [112, 113] include the following:

Theorem 6.40. *(Cappell [113]) Let $n \geq 4$ and suppose that M^n is a Top n-manifold. Suppose that X_1 and X_2 are n-manifolds with fundamental group $\mathbb{Z}_2$ and let $f : M^n \to X_1 \# X_2 \equiv X$ be a homotopy equivalence.*

1. *The map f is splittable iff the splitting obstruction $s(f)$ vanishes in the group $\mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^{\pm}, \mathbb{Z}^{\pm})$.*
2. *Given $\gamma \in \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^{\pm}, \mathbb{Z}^{\pm})$, we can find a topological manifold M^n when $n \geq 4$ and a homotopy equivalence $f : M^n \to X_1 \# X_2$ such that $s(f) = \gamma$.*
3. *The group $\mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^{\pm}, \mathbb{Z}^{\pm})$ acts on $\mathcal{S}^{Top}(Y)$ freely on the orbits of the action of $L^h_{n+1}(\mathbb{Z}[G])$. If $x \in \mathcal{S}^{Top}(Y)$ and $\alpha \in \mathrm{UNil}(\mathbb{Z}; \mathbb{Z}^{\pm}, \mathbb{Z}^{\pm})$, then x and $\alpha.x$ are normally cobordant and $s(\alpha.x) = s(x) + \alpha$.*
4. *Let $X_1 = X_2 = \mathbb{RP}^4$ and let $s \in \mathrm{UNil}_5(\mathbb{Z}; \mathbb{Z}^-, \mathbb{Z}^-)$ be nonzero. The above realization identifies a topological manifold M^4 with a non-splittable homotopy equivalence $f : M^4 \to \mathbb{RP}^4 \# \mathbb{RP}^4$.*

There are some results about the computation of these UNil groups, in its simplest non-trivial use.

1. $\mathrm{UNil}_0(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) = 0$ (Cappell [115] and Connolly-Koźniewski [179]);
2. $\mathrm{UNil}_1(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) = 0$ (Connolly-Ranicki [180]);
3. $\mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ is infinitely generated (Cappell [114] and Connolly-Koźniewski [179]);
4. $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ is infinitely generated (Connolly-Ranicki [180]) and has an element of exponent 4 (Ranicki).

The non-splittable map $f : W^{4k+1} \to \mathbb{RP}^{4k+1} \# \mathbb{RP}^{4k+1}$ constructed in the previous section is detected by the nonzero Cappell splitting invariant $s(f)$ in $\mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$. The computation $\mathrm{UNil}_0(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) = 0$ shows that, for all $k \geq 1$, a homotopy connected sum $\mathbb{RP}^{4k+3} \# \mathbb{RP}^{4k+3}$ is a connected sum.

6.3.4 UNil calculations and the Verschiebung algebra

In this section we describe without proof some UNil calculations by Connolly and Davis in [178]. In it they apply their results to compute the L-theory groups of D_∞, the infinite dihedral group. Banagl-Ranicki gave an independent calculation of $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ using generalized Arf invariants in [35].

Definition 6.41. *Let $R[t^+]$ denote the polynomial ring $R[t]$ with the involution $\sum r_i t^i \mapsto \sum \overline{r_i} t^i$ and similarly let $R[t^-]$ be endowed with the involution $\sum r_i t^i \mapsto \sum (-1)^i \overline{r_i} t^i$. Denote by $\varepsilon_0 : R[t] \to R$ the evaluation map $\varepsilon_0(f) = f|_0$ at zero. It is a split surjection with the obvious splitting map $s_0 : R \to R[t]$. Define*

$$N^{\pm} L_n(R) = \ker(\varepsilon_{0*} : L_n(R[t^{\pm}]) \to L_n(R)).$$

By the splitting we have $L_n(R[t^{\pm}]) \cong L_n(R) \times N^{\pm}L_n(R)$.

Theorem 6.42. *(Connolly-Ranicki [180]) There is an isomorphism*

$$r\colon \mathrm{UNil}_n(R; R^{\pm}, R) \to N^{\pm}L_n(R),$$

which is natural in R.

Now we consider the ring endomorphism $V_i\colon R[t] \to R[t]$ with $V_i(p) = p|_{t^i}$, where $p|_f$ denotes the polynomial by replacing every instance of t in $p \in R[t]$ with the element $f \in R[t]$. Notice that $V_iV_j = V_{ij}$ for all $i, j \geq 1$. Let $\mathcal{M} = \{V_1, V_2, \dots\}$ be the monoid of endomorphisms with multiplication given as above.

Definition 6.43. *Let* $\mathcal{V} = \mathbb{Z}[\mathcal{M}]$ *be called the Verschiebung algebra. Notice that* $\mathcal{V}$ *can be expressed as the polynomial ring* $\langle V_2, V_3, V_5, \dots\rangle_{\mathbb{Z}}$ *generated by just the prime terms. The subalgebra* $\langle V_3, V_5, \dots\rangle_{\mathbb{Z}}$ *generated by the odd prime terms is denoted by* $\mathcal{V}_{od}$.

Connolly and Davis compute the UNil groups in terms of the Verschiebung algebra:

1. $\mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \cong \mathcal{V}/\langle 2, V_2 - 1\rangle$;
2. $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \cong \mathcal{V}/\langle 4, 2V_2 - 2\rangle \times \bigoplus_{i=0}^{\infty} \mathcal{V}/\langle 2, V_2\rangle$;
3. $\mathrm{UNil}_0(\mathbb{Z}; \mathbb{Z}^-, \mathbb{Z}) = \mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}^-, \mathbb{Z}) \cong \mathcal{V}_{od}/\langle 2\rangle$;
4. $\mathrm{UNil}_1(\mathbb{Z}; \mathbb{Z}^-, \mathbb{Z}) = \mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}^-, \mathbb{Z}) \cong \bigoplus_{i=1}^{\infty} \mathcal{V}_{od}/\langle 2\rangle$.

If R is a ring with involution $r \mapsto \bar{r}$ and A_1 and A_2 are R-bimodules with involution, then one can define $\mathrm{UNil}_n^h(R; A_1, A_2)$. If A_i^- denotes the bimodule with involution given by $a \mapsto -\bar{a}$, then there are isomorphisms

$$\mathrm{UNil}_n(R; A_2, A_1) \cong \mathrm{UNil}_n^h(R; A_1, A_2) \cong \mathrm{UNil}_{n+2}^h(R; A_1^-, A_2^-).$$

Let $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ be the dihedral group generated by elements a_1 and a_2 of order 2. Let $w\colon D_\infty \to \{\pm 1\}$ be a homomorphism with $w_i = w|_{\langle a_i\rangle}$. Let $\varepsilon_i = w(a_i) = \pm 1$. Cappell's Mayer-Vietoris sequence is given by

$$\cdots \to L_n(\mathbb{Z}[e]) \to L_n(\mathbb{Z}[\mathbb{Z}_2], w_1) \times L_n(\mathbb{Z}[\mathbb{Z}_2], w_2)$$
$$\to L_n(\mathbb{Z}[D_\infty], w)/\mathrm{UNil}_n^h(\mathbb{Z}; \mathbb{Z}^{\varepsilon_1}, \mathbb{Z}^{\varepsilon_2}) \to L_{n-1}(\mathbb{Z}[e]) \to \cdots$$

For any group G, let $\widetilde{L}_n(\mathbb{Z}[G], w) = \mathrm{coker}(L_n(\mathbb{Z}) \to L_n(\mathbb{Z}[G], w))$. Then we have the following decompositions.

Theorem 6.44.

1. *If* $n \equiv 1 \bmod 4$ *and the* w_i *are both non-trivial, then we have*

$$L_n(\mathbb{Z}[D_\infty], w) = \mathrm{UNil}^h_n(\mathbb{Z}; \mathbb{Z}^-, \mathbb{Z}^-) \times L_{n-1}(\mathbb{Z}[e]).$$

   Note that these groups are all 4-periodic, so we can replace n with 1 in the above without losing any information.

2. *Otherwise, we have the following decomposition, with the assumption that* $\varepsilon_1 \leq \varepsilon_2$*:*

$$\widetilde{L}_n(\mathbb{Z}[D_\infty], w) = \widetilde{L}_n(\mathbb{Z}[\mathbb{Z}_2], w_1) \times L_n(\mathbb{Z}[\mathbb{Z}_2], w_2) \times \mathrm{UNil}^h_n(\mathbb{Z}; \mathbb{Z}^{\varepsilon_1}, \mathbb{Z}^{\varepsilon_2}).$$

The groups $L_n(\mathbb{Z}[\mathbb{Z}_2], w)$ are computed in Section 2.6. The relation of the L-theory of the dihedral group to the algebra $\mathbb{Z}[t]$ used in this section is a descendant of the Browder-Livesay approach discussed in Section 6.2.

6.4 CONNECTED SUMS OF PROJECTIVE SPACES

The computation of $\mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\pm, \mathbb{Z}^\pm)$ makes it possible to classify all closed manifolds homotopy equivalent to $\mathbb{RP}^n \# \mathbb{RP}^n$ up to Top homeomorphism. Brookman-Davis-Khan [77] did computations and accomplished somewhat more. Recall that the structure set does not classify up to homeomorphism type; there may be exotic self-homotopy equivalences that are not homotopic to a homeomorphism appearing in the calculation. The computations in [77] give us an understanding of $\mathcal{S}^{Top}(\mathbb{RP}^n \# \mathbb{RP}^n)$, which we will outline in this section. Note that the fundamental group of such a connected sum is the free product $\mathbb{Z}_2 * \mathbb{Z}_2$. By Stallings [611], its Whitehead torsion satisfies $\mathrm{Wh}(\mathbb{Z}_2 * \mathbb{Z}_2) = \mathrm{Wh}(\mathbb{Z}_2) \oplus \mathrm{Wh}(\mathbb{Z}_2) = 0$, so there is no difference between homotopy equivalence and simple homotopy equivalence in this context. In this section all manifolds have dimension at least 5.

Definition 6.45. *Suppose that* $n \geq 5$*. Let* I_n *and* J_n *denote the set of Top homeomorphism classes of closed manifolds homotopy equivalent to* $\mathbb{RP}^n$ *and* $\mathbb{RP}^n \# \mathbb{RP}^n$*.*

If $h\mathrm{Aut}(M)$ denotes the group of homotopy classes of (simple) self-homotopy equivalences $M \to M$, then $h\mathrm{Aut}(M)$ acts on $\mathcal{S}^{Top}(M)$ by post-composition, and the homeomorphism classes of closed manifolds (simple) homotopy equivalent to M are given by $\mathcal{S}^{Top}(M)/h\mathrm{Aut}(M)$. Therefore we have the identifications $I_n = \mathcal{S}^{Top}(\mathbb{RP}^n)/h\mathrm{Aut}(\mathbb{RP}^n)$ and

$$J_n = \mathcal{S}^{Top}(\mathbb{RP}^n \# \mathbb{RP}^n)/h\mathrm{Aut}(\mathbb{RP}^n \# \mathbb{RP}^n).$$

One of the goals of this section is to understand the sets I_n and J_n, which in general are not the same as the structure sets of $\mathbb{RP}^n$ and $\mathbb{RP}^n \# \mathbb{RP}^n$. We start by giving some known information about the manifold structures of $\mathbb{RP}^n$.

We recall the result of Theorem 6.2.

Theorem 6.46. *Let $n \geq 4$ and write $n = 4m + k$, where $m \geq 0$ and $k \in \{1, 2, 3, 4\}$. The topological structure sets $\mathcal{S}^{Top}(\mathbb{RP}^n)$ for $n \geq 4$ satisfy the following bijective relationships:*

$$g : \mathcal{S}^{Top}(\mathbb{RP}^{4m+k}) = \begin{cases} \mathbb{Z}_2^{2m} & \text{if } k = 1, \\ \mathbb{Z}_2^{2m} & \text{if } k = 2, \\ \mathbb{Z}_2^{2m} \oplus \mathbb{Z} & \text{if } k = 3, \\ \mathbb{Z}_2^{2m+1} & \text{if } k = 4. \end{cases}$$

So in the case when $k \neq 3$, the topological structure set of $\mathbb{RP}^{4m+k}$ is finite and has precisely $2^{2m+[k/4]}$ elements. In addition, the map $\mathbb{Z} \to \mathcal{S}^{Top}(\mathbb{RP}^{4m+3})$ is given by Wall realization $\mathbb{Z} \cong \widetilde{L}_{4k+4}(\mathbb{Z}[\mathbb{Z}_2]) \to \mathcal{S}^{Top}(\mathbb{RP}^{4k+3})$ and the map $\mathcal{S}^{Top}(\mathbb{RP}^{4k+3}) \to \mathbb{Z}$ is a Browder-Livesay desuspension invariant of a free $\mathbb{Z}_2$-action on $\mathbb{S}^{4k+3}$ to an action on some embedded $\mathbb{S}^{4k+2}$.

Theorem 6.47. *Let g be the correspondence above. Then there is a bijection $I_n \to \mathcal{S}^{Top}(\mathbb{RP}^n)/\sim$, where $h_1 \sim h_2$ in $\mathcal{S}^{Top}(\mathbb{RP}^n)$ iff $g(h_1) = \pm g(h_2)$, Note that $\sim$ only matters in the case when $k = 3$.*

Proof. For n even the group $h\mathrm{Aut}(\mathbb{RP}^n)$ is trivial, and for n odd it is $\mathbb{Z}_2$. When $k = 1$, the action of $\mathbb{Z}_2$ is trivial. In the case $k = 3$, the non-trivial element ϕ of $h\mathrm{Aut}(\mathbb{RP}^n)$ is represented by an orientation-reversing self-homeomorphism of $\mathbb{RP}^{4k+3}$, e.g. a map that negates one coordinate in $\mathbb{RP}^{4k+3}$. Therefore the induced map $\phi_* : \mathcal{S}^{Top}(\mathbb{RP}^{4k+3}) \to \mathcal{S}^{Top}(\mathbb{RP}^{4k+3})$ multiplies by -1 in the $\mathbb{Z}$-coordinate because a change in orientation reverses the sign of the fundamental class and therefore of any signature invariant. □

This previous theorem completes the computation for $\mathbb{RP}^n$ in all dimensions $n = 4m+k$.

	$n \equiv 1 \mod 4$	$n \equiv 2 \mod 4$	$n \equiv 3 \mod 4$	$n \equiv 4 \mod 4$
$\mathcal{S}^{Top}(\mathbb{RP}^n)$	$\mathbb{Z}_2^{2m}$	$\mathbb{Z}_2^{2m}$	$\mathbb{Z}_2^{2m} \oplus \mathbb{Z}$	$\mathbb{Z}_2^{2m+1}$
I_n	$\mathbb{Z}_2^{2m}$	$\mathbb{Z}_2^{2m}$	$\mathbb{Z}_2^{2m} \oplus \mathbb{Z}_{\geq 0}$	$\mathbb{Z}_2^{2m+1}$

We now proceed to a discussion about the connected sums $\mathbb{RP}^n \# \mathbb{RP}^n$. The computation for the structure set of $\mathbb{RP}^n \# \mathbb{RP}^n$ quickly emerges from Cappell's calculation of $L_*(\Phi)$.

Theorem 6.48. *Let $\mathcal{S}_{sp}^{Top}(\mathbb{RP}^n \# \mathbb{RP}^n)$ be the subset of $\mathcal{S}^{Top}(\mathbb{RP}^n \# \mathbb{RP}^n)$ consisting of homotopy equivalences $h : W \to \mathbb{RP}^n \# \mathbb{RP}^n$ which can be non-trivially split, i.e. which are homeomorphic to $h_1 \# h_2 : W_1 \# W_2 \to \mathbb{RP}^n \# \mathbb{RP}^n$. Let $n \geq 5$. (In fact, the following result holds for $n = 4$ from results of Jahren and Kwasik [334].) Define $\varepsilon = (-1)^{n+1}$.*

1. *Let $s(h) \in \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon)$ be the Cappell splitting invariant for a Top manifold structure h in $\mathcal{S}^{Top}(\mathbb{RP}^n \# \mathbb{RP}^n)$. This invariant was defined in Section 6.3.*

There is a bijection

$$\rho : \mathcal{S}^{Top}(\mathbb{RP}^n \# \mathbb{RP}^n) \to \mathcal{S}^{Top}_{sp}(\mathbb{RP}^n \# \mathbb{RP}^n) \times \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon)$$

such that $\rho(h) = (-s(h) \cdot h, s(h))$. *Here the dot* $\cdot$ *indicates the action of an element of the* L*-group on the structure set.*

2. *Connected sum gives a bijection* $\# : \mathcal{S}^{Top}(\mathbb{RP}^n) \times \mathcal{S}^{Top}(\mathbb{RP}^n) \to \mathcal{S}^{Top}_{sp}(\mathbb{RP}^n \# \mathbb{RP}^n)$.

Proof. Note that $\mathcal{S}^{Top}_{sp}(\mathbb{RP}^n \# \mathbb{RP}^n)$ is actually a subgroup of $\mathcal{S}^{Top}(\mathbb{RP}^n \# \mathbb{RP}^n)$ because it is the kernel of the Cappell map. Recall also that, if $\varepsilon = -1$, then $\mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon) \cong \mathrm{UNil}_{n-1}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$. Note that, in both parts, we require the fact that the Whitehead group $\mathrm{Wh}(\mathbb{Z}_2)$ and subsequently $\mathrm{Wh}(\mathbb{Z}_2 * \mathbb{Z}_2)$ are both zero by Cappell's nilpotent normal cobordism construction. See Cappell [111] for more information.

For (1) the map ρ is given by Cappell's nilpotent normal cobordism construction [117], which gives a normal cobordism between a given splitting problem and a split homotopy equivalence. By part (c) of Theorem 6.40, we know that $-s(h) \cdot h$ is split using the formula $s(-s(h) \cdot h) = -s(h) + s(h) = 0$. In other words, it belongs to $\mathcal{S}^{Top}_{sp}(\mathbb{RP}^n \# \mathbb{RP}^n)$. See Cappell [113].

We now prove (2). For $n \geq 5$ surjectivity of the connected sum operation follows from the validity of the Poincaré conjecture for homotopy spheres. For injectivity in all dimensions $n \geq 4$, let h_1, h_2, h_1', and h_2' represent elements of $\mathcal{S}^{Top}(\mathbb{RP}^n)$ and suppose that $h_1 \# h_2 : M_1 \# M_2 \to \mathbb{RP}^n \# \mathbb{RP}^n$ and $h_1' \# h_2' : M_1' \# M_2' \to \mathbb{RP}^n \# \mathbb{RP}^n$ both represent the same element in $\mathcal{S}^{Top}_{sp}(\mathbb{RP}^n \# \mathbb{RP}^n)$; i.e. they are s-bordant by some map $H : W \to (\mathbb{RP}^n \# \mathbb{RP}^n) \times (I, \{0\}, \{1\})$ such that the restrictions of H to the two boundary components of W give $h_1 \# h_2$ and $h_1' \# h_2'$.

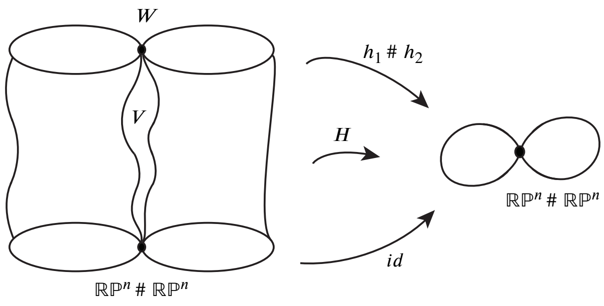


Figure 6.1: A map to the connected sum of projective spaces

If $\mathbb{S}^{n-1}$ is the connecting sphere in $\mathbb{RP}^n \# \mathbb{RP}^n$, then $V = H^{-1}(\mathbb{S}^{n-1} \times I)$ is a submanifold of W with two boundary components, both of which are connecting $(n-1)$-spheres. Let $s_V(H \text{ rel } \partial)$ be the splitting invariant of H relative to V. The relative form of the nilpotent normal cobordism construction [117] provides a normal bordism $(J, H, H') : (L, W, W') \to (\mathbb{RP}^n \# \mathbb{RP}^n) \times I$ relative to ∂H such that

1. the relative surgery obstruction of J is $s_V(H \text{ rel } \partial)$;
2. the splitting of $\partial H' = \partial H$ along $\mathbb{S}^{n-1} \times \partial I$ extends to a splitting of H' along $\mathbb{S}^{n-1} \times I$.

The inverse image $V' = (H')^{-1}(\mathbb{S}^{n-1} \times I)$ is a simply connected h-cobordism, so V' is homeomorphic to $\mathbb{S}^{n-1} \times I$, and H' restricts to a homotopy equivalence on the complement of V'. Therefore H is the boundary connected sum of two homotopy equivalences

$$(H_i', h_i, h_i') : (W_i', M_i, M_i') \to \mathbb{RP}^n \times (I, \{0\}, \{1\})$$

for $i = 1, 2$. Hence $[h_1] = [h_1']$ and $[h_2] = [h_2']$ in $\mathcal{S}^{Top}(\mathbb{RP}^n)$. □

The formula $\mathcal{S}^{Top}(\mathbb{RP}^n \# \mathbb{RP}^n) \cong \mathcal{S}^{Top}(\mathbb{RP}^n) \oplus \mathcal{S}^{Top}(\mathbb{RP}^n) \oplus \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^{\varepsilon}, \mathbb{Z}^{\varepsilon})$ gives the complete characterization. Here we use the 4-periodicity of UNil and the formula $\mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^-, \mathbb{Z}^-) \cong \mathrm{UNil}_{n-1}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$. We can compute the structure sets by using the isomorphisms $\mathrm{UNil}_0(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) = \mathrm{UNil}_1(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) = 0$ and $\mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) = \mathbb{Z}_2^{\infty}$ and $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) = \mathbb{Z}_2^{\infty} \oplus \mathbb{Z}_4^{\infty}$:

$$\mathcal{S}^{Top}(\mathbb{RP}^{4m+k} \# \mathbb{RP}^{4m+k}) \cong \begin{cases} \mathbb{Z}_2^{\infty} \oplus \mathbb{Z}_4^{\infty} & \text{if } k \equiv 0 \mod 4, \\ \mathbb{Z}_2^{\infty} & \text{if } k \equiv 1 \mod 4, \\ \mathbb{Z}_2^{4m} & \text{if } k \equiv 2 \mod 4, \\ \mathbb{Z}_2^{4m} \oplus \mathbb{Z}^2 & \text{if } k \equiv 3 \mod 4. \end{cases}$$

We now examine the sets J_n. Denote by i the injection $i : \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^{\varepsilon}, \mathbb{Z}^{\varepsilon}) \hookrightarrow L_{n+1}(\mathbb{Z}[\mathbb{Z}^{\varepsilon} * \mathbb{Z}^{\varepsilon}])$ and let $\sigma : \mathcal{N}^{Top}((\mathbb{RP}^n \# \mathbb{RP}^n) \times I) \to L_{n+1}(\mathbb{Z}[\mathbb{Z}^{\varepsilon} * \mathbb{Z}^{\varepsilon}])$ be the surgery map. If we have $h_1 : X_1 \to \mathbb{RP}^n$ and $h_2 : X_2 \to \mathbb{RP}^n$ and an element $\vartheta \in \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^{\varepsilon}, \mathbb{Z}^{\varepsilon})$, then Wall realization gives a normal bordism $g : W \to (\mathbb{RP}^n \# \mathbb{RP}^n) \times I$ that

1. restricts to $h_1 \# h_2 : X_1 \# X_2 \to \mathbb{RP}^n \# \mathbb{RP}^n$ and some $h : X \to \mathbb{RP}^n \# \mathbb{RP}^n$ on the boundary components of W;
2. satisfies $\sigma(g) = i(\vartheta)$ in $L_{n+1}(\mathbb{Z}[\mathbb{Z}^{\varepsilon} * \mathbb{Z}^{\varepsilon}])$.

Cappell [114] shows that Aut $(\mathbb{RP}^n \# \mathbb{RP}^n)$ is generated by three self-homeomorphisms, each of which can be defined by its action on the cover $\mathbb{S}^{n-1} \times \mathbb{S}^1$:

1. $\gamma_1[w, z] = [w, -z]$ which interchanges the two summands of $\mathbb{RP}^n \# \mathbb{RP}^n$;
2. $\gamma_2[(w_1, \ldots, w_n), z] = [(w_1, \ldots, w_{n-1}, -w_n), z]$ which reflects through the last coordinate in $\mathbb{RP}^{n-1} \# \mathbb{RP}^{n-1}$;
3. $\gamma_3[w, z] = [\tau(z^2)(w), z]$ if $\mathrm{Im}(z) \geq 0$ and $\gamma_3[w, z] = [\tau(\overline{z}^2)(w), z]$ if $\mathrm{Im}(z) \leq 0$, which Dehn twists along the connecting cylinder with the isotopy $\tau : \mathbb{S}^1 \to$

$SO_n(\mathbb{R})$ that generates $\pi_1(SO_n(\mathbb{R})) \cong \mathbb{Z}_2$.

When n is even, the map γ_2 is isotopic to the identity. Note that all these elements are splittable. The map γ_1 will be of special interest to us. The induced involution on $\mathbb{Z}_2 * \mathbb{Z}_2$, on $L_n(\mathbb{Z}[\mathbb{Z}_2^\varepsilon * \mathbb{Z}_2^\varepsilon])$, and on $\mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon)$ will all be denoted by γ_{1*}. See also Jahren-Kwasik [334].

Theorem 6.49. *(Brookman-Davis-Khan [77]) Let $n \geq 5$ and $\varepsilon = (-1)^{n+1}$. Denote by $P(I_n)$ be the set $(I_n \times I_n)/\sim$ of unordered pairs of elements in I_n.*

1. *There is a bijection of sets $\psi : P(I_n) \times \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon)/\gamma_{1*} \to J_n$ given by*

$$\psi : ((h_1, h_2), \vartheta) \mapsto [h]$$

   where the map $h : X \to \mathbb{RP}^n \# \mathbb{RP}^n$ is produced from h_1, h_2, and ϑ as above.

2. *If $[\vartheta] \neq 0$ in $\mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon)/\gamma_{1*}$, then X cannot be represented as a connected sum of two manifolds each with fundamental group $\mathbb{Z}_2$.*

Proof. Recall that g is the bijective map on the structure set of $\mathbb{RP}^n$ given in Theorem 6.46. We will prove (1). The second statement follows easily from the material in this section.

We use the bijection $J_n \to \mathcal{S}^{Top}(\mathbb{RP}^n \# \mathbb{RP}^n)/\mathrm{Aut}\,(\mathbb{RP}^n \# \mathbb{RP}^n)$ from Theorem 6.47 and determine how the automorphisms act on the structure set. Identify $\mathcal{S}^{Top}(\mathbb{RP}^n \# \mathbb{RP}^n)$ with $\mathcal{S}^{Top}(\mathbb{RP}^n) \times \mathcal{S}^{Top}(\mathbb{RP}^n) \times \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon)$. This identification is equivariant with respect to the action of Aut and the product. For an element (h_1, h_2, x) in this decomposition, we have

1. $\gamma_1 \cdot (h_1, h_2, x) = (h_2, h_1, \gamma_{1*}(x))$,
2. $\gamma_2 \cdot (h_1, h_2, x) = (g^{-1}(-g(h_1)), g^{-1}(-g(h_2)), x)$,
3. $\gamma_3 \cdot (h_1, h_2, x) = (h_1, h_2, x)$.

The quotient of the product by the automorphism group identifies $g^{-1}(z)$ with $g^{-1}(-z)$ in the first two terms by γ_3, and makes them unordered by γ_1. On the UNil piece, the automorphism γ_1 identifies x with $\gamma_{1*}(x)$. Therefore J_n is in bijective correspondence with $P(I_n) \times \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon)/(x \sim \gamma_{1*}(x))$, as required. □

The computation above of course leaves us to ascertain the exact nature of γ_{1*} in the cases in which $\mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon)$ is nonzero, i.e. when $n \equiv 0$ or $1 \bmod 4$, which correspond to $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ and $\mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$, respectively. The paper of Brookman-Davis-Khan resolves this problem. The proof is technical, so we will merely state the result.

Lemma 6.50. *(Connolly-Davis [178]) There is a $\mathbb{Z}$-module isomorphism given by*

$$t\mathbb{Z}_4[t]/A \times t\mathbb{Z}_2[t] \cong \mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$$

where $A = \{2p|_{t^2} - 2p : \ p \in t\mathbb{Z}_4[t]\}$.

Theorem 6.51. *Let γ_1 be the homeomorphism of $\mathbb{RP}^n \# \mathbb{RP}^n$ given by switching the two summands.*

1. *The map* $\gamma_{1*} : \mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \to \mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ *is the identity.*
2. *Let* $p : t\mathbb{Z}_4[t]/A \to t\mathbb{Z}_2[t]$ *be the obvious projection map. With respect to the direct sum decomposition of* $\mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$*, the map* $\gamma_{1*} : \mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \to \mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ *is represented by the matrix* $\gamma_{1*} = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$*, i.e.* $\gamma_{1*}(a, b) = (a, p(a) + b)$.

Note that, if $a \in t\mathbb{Z}_4[t]/A$ is a multiple of 2, then $p(a)$ vanishes, so for such a and any $b \in t\mathbb{Z}_2[t]$, the switch map γ_{1*} acts as the identity on (a, b).

6.5 LENS SPACES

In this section, we will discuss the structure set of lens spaces. Lens spaces have been the subject of study for over a century. The three-dimensional versions were introduced by Tietze [647] in 1908. They were the first known examples of 3-manifolds which were not determined by their homology and fundamental group alone. In addition, they are the simplest examples of closed manifolds whose homotopy type does not determine their homeomorphism type. In 1919, J. W. Alexander showed that lens spaces which have isomorphic fundamental groups and the same homology may actually have different homotopy types. De Rham proved the deep theorem that lens spaces corresponding to nonconjugate rotations on the sphere are not diffeomorphic. Other lens spaces even have the same homotopy type, but not the same homeomorphism type.

Homotopy lens spaces of higher odd dimension were intensively studied by Wall [672], who showed that the Top structure set of lens spaces are all non-trivial. Although he did not have the group structure on the structure set $\mathcal{S}^s(L_k^{2n+1})$, he essentially discovered that these groups are torsion-free. The surgery exact sequence has non-trivial extensions; i.e. the sequence does not split. The remaining determination of the structure set for even-order fundamental groups was done by Macko-Wegner [417] and Balko-Macko-Niepel-Rusin [34]. The structure set contains torsion, but the surgery exact sequence is non-split. The details of all these computations are extensive, and we will refer the reader to the original works for the entire treatment.

6.5.1 Homotopy type of lens spaces

Let m be an integer, and let $a_1, \dots, a_d$ be d integers that are coprime to m. Then $\mathbb{Z}_m$ acts freely on $\mathbb{S}^{2d-1}$ by considering the action of a generator $g \in \mathbb{Z}_m$ on the unit sphere $\mathbb{S}^{2d-1}$ of $\mathbb{C}^d$ given by

$$g : (z_1, \dots, z_d) \mapsto (e^{2\pi i a_1/m} z_1, \dots, e^{2\pi i a_d/m} z_d).$$

They are all of the free (linear) $\mathbb{Z}_m$-actions. By elementary representation theory, we can permute or change the sign of the a_i without altering the diffeomorphism type of the quotient. We define

$$L_m^{2d-1}(a_1, \dots, a_d) = \mathbb{S}^{2d-1}/\mathbb{Z}_m$$

acting as above. This quotient is a *linear lens space*. The a_i are called the *rotation numbers* determining the action.

Remark 6.52. *We are interested in the conjugacy of these actions in all three manifold categories. Unless otherwise stated, we will focus on orientation-preserving conjugacies. To consider arbitrary isomorphisms of the quotient manifold, one can ask whether there is an "orbit equivalence" between these actions, which just allows changing the generators by multiplying all the a_i by the same number k. We might be able to define invariants in $F(\mathbb{Z}_m)$ for various functors of $\mathbb{Z}_m$, and then we can act by* $\mathrm{Aut}(\mathbb{Z}_m)$ *to shift the invariants around. If an orientation is used in defining an invariant, then the orientation change is likely to introduce a sign.*

By covering space theory, these lens spaces have the same homotopy groups. However, they do not all belong to the same homotopy type. Much like $K(\mathbb{Z}_2, 1) \simeq \mathbb{RP}^\infty$, the classifying space $K(\mathbb{Z}_m, 1)$ has a model given by an infinite lens space denoted by L_m^∞.

Proposition 6.53. *Every $L_m^{2d-1}(a_1, \dots, a_d)$ has a map to $L_m^{2d-1}(1, , \dots, 1)$ which is the isomorphism identifying the fundamental group with the group that is acting on the sphere. The homotopy class of this map is determined by its degree, and has degree $(a_1 \cdots a_d)^{-1}$ mod m.*

Proof. We know that $L_m^{2d-1}(1, , \dots, 1)$ is the $2d-1$ skeleton of L_m^∞, so there is a map $L_m^{2d-1}(a_1, \dots, a_d) \to L_m^\infty$ whose image can be assumed to lie in this skeleton. Since this class is nonzero in $H_{2d-1}(L_m^\infty; \mathbb{Z})$, the map must have degree coprime to m.

Covering space theory implies that any two maps will only differ up to homotopy by a map on the top cell, which lifts to $\mathbb{S}^{2d-1}$. Consequently, the degree determines the homotopy class, and the degree can only change by a multiple of m, since the map $\mathbb{S}^{2d-1} \to L_m^{2d-1}(1, , \dots, 1)$ has degree m.

To complete the proof, we need to build an equivariant map between $\mathbb{S}^{2d-1}(a_1, \dots, a_d)$ and $\mathbb{S}^{2d-1}(1, , \dots, 1)$ (with the obvious notation). We can explicitly construct the map $\psi : (z_1, \dots, z_d) \mapsto (z_1^{b_1}, \dots, z_d^{b_d})$, where each b_i is the multiplicative inverse of a_i. The map obviously has degree $b_1 \cdots b_d$. □

Remark 6.54. *Of course, this $L_m^{2d-1}(1,,\dots,1)$ can be replaced by any other lens space, and then the degree will be related to the ratio of the products of the rotation numbers.*

Remark 6.55. *The proposition is regarded in terms of obstruction theory as a calculation of the first k-invariant of the lens space $L_m^{2d-1}(a_1,\dots,a_d)$ that lies in $H^{2d}(L_m^\infty;\mathbb{Z}) \cong \mathbb{Z}_m$. Of course, the action of $\mathbb{Z}_m^*$ on this $\mathbb{Z}_m$ is by multiplication by the $(-d)$-th power of the automorphism of $\mathbb{Z}_m$.*

Theorem 6.56. *Two lens spaces $L_m^{2d-1}(a_1,\dots,a_d)$ and $L_m^{2d-1}(b_1,\dots,b_d)$ are homotopy equivalent iff there is an integer r for which $a_1 \cdots a_d \equiv \pm r^2 b_1 \cdots b_d \bmod m$.*

Remark 6.57. *When $m = 2,3,5,6$, any two lens spaces of the above form are homotopy equivalent, since the only classes of numbers coprime to m are those of 1 and -1. The $m = 5$ case is the smallest for which there are counterexamples.*

Remark 6.58. *Without requiring a calculation for the k-invariant, the theorem and its proof imply a similar statement for (1) quotients of $\mathbb{Z}_m$-actions which are not necessarily free, and (2) free actions of any finite group G. By constructing the $2d$-skeleton of the Eilenberg-MacLane space $K(G,1)$, one can show that, if G acts freely on $\mathbb{S}^{2d-1}$, then $H^{2d}(K(G,1);\mathbb{Z}) \cong \mathbb{Z}_{|G|}$.*

Corollary 6.59. *Two linear lens spaces are homotopy equivalent, preserving orientation and identification of π_1 iff the products of their rotation numbers are the same.*

Example 6.60. *The lens spaces $L_5^3(1,1)$ and $L_5^3(1,2)$ have the same homotopy and homology groups, but are not homotopy equivalent. The lens spaces $L_7^3(1,1)$ and $L_7^3(1,2)$ are homotopy equivalent, but not by a homotopy equivalence that preserves the fundamental group.*

6.5.2 Homeomorphism type of lens spaces

The question of unrestricted homotopy equivalences is a more difficult question because there are more equivalence classes. For instance, if p is a prime congruent to 3 mod 4, then any two lens spaces will then become homotopy equivalent, while from the "polarized, oriented" perspective there will be $p-1$ types, or more generally $\phi(m)$ types if m is not prime. It is an exercise to show that any two lens spaces of the same dimension with the same fundamental group have the same localizations at every prime, or even at any finite set of primes.

The classification of lens spaces into their PL homeomorphism types requires more machinery. The classic method uses *Reidemeister torsion*. Reidemeister torsion is a subtle invariant that is only defined for a finite simplicial complex X with fundamental group G that admits a ring morphism $\mathbb{Z}[G] \to R$ to a commutative ring R (or more generally to a matrix ring $\rho\colon \mathbb{Z}[G] \to M_n(R)$) such that $H_*(X;R) = 0$ (respectively

$H_*(X;\rho) = 0$). In this case, the Reidemeister torsion element $\Delta(X)$ is a well-defined element in the units $R^\times$ of R, modulo $\pm\det(\rho(g))$ for all $g \in G$.

First we define the general notion of torsion.

Definition 6.61. *A based chain complex is a chain complex C^* of finitely generated free R-modules, such that each C_i is given a basis.*

We give a description of the torsion using the determinant. Note that a based chain complex C_* of R-modules of length 2, i.e. with A_i nonzero for only two consecutive indices, is an automorphism in $\mathrm{GL}(R)$. It has a well-defined determinant in $K_1(R)$ or, even more primitively, in the units R^* of R, which we define to be the *torsion*. For reasons related to Grothendieck groups, it is convenient to insert a sign $(-1)^{i-1}$ if the chain complex is of the form

$$\cdots \to 0 \to C_{i+1} \to C_i \to 0 \to \cdots .$$

For a three-term exact sequence

$$\cdots \to 0 \to C_{i+1} \to C_i \to C_{i-1} \to 0 \to \cdots$$

there is an isomorphism $C_i \to C_{i+1} \oplus C_{i-1}$ between based free abelian groups. Since these C_i are equipped with a particular basis, this isomorphism is not well-defined. However, if we represent an isomorphism by a square matrix, then the indeterminancy can be represented by an upper triangular unipotent matrix whose determinant is 1. Therefore a determinant of the isomorphism is well-defined, and we can define the torsion once again. The same idea works for arbitrary chain complexes of finite length. See Cohen [168] and Milnor [456].

Remark 6.62. *In the Grothendieck group of free modules with automorphisms, the upper triangular matrices are those with a filtration whose associated grading is the identity. In other words, they would be forced to be trivial.*

Definition 6.63. *If C^* is a finite-length acyclic based chain complex, then $\tau(C)$ is the torsion of the based equivalence $\oplus C_{2i+1} \to \oplus C_{2i}$. It is well-defined up to upper triangular matrices.*

Remark 6.64. *We always define torsions modulo $\pm$ since our chain complexes are cellular chain complexes and their basis is well-defined up to a choice of signs, i.e. orientations for each cell. Indeed, we usually will at least tacitly use the equivariant chain complex of the universal cover, and the basis as $\mathbb{Z}[G]$-modules will have an indeterminacy from the choice of a lift of each oriented cell downstairs, which introduces multiplication by the 1×1 matrix (g) for a group element g of G.*

Remark 6.65. *The torsion can also be expressed additively. A based R-module is a free, finite-dimensional R-module with a specified basis. An isomorphism $\alpha : A' \to A$ of based R-modules determines a class $[\alpha]$ in $\widetilde{K}_1(R)$. Suppose now that $f_* : C' \to C$*

is a chain isomorphism between based chain complexes. We define the torsion of f_* *by*

$$\tau(f_*) = \sum_n (-1)^n [f_n : C'_n \to C_n] \in \widetilde{K}_1(R),$$

where $\widetilde{K}_1(R) = \mathrm{coker}\,(K_1(\mathbb{Z}) \to K_1(R))$.

Now we define the Reidemeister torsion of a CW complex in the following way. Let (C, ∂) be a finite, based chain complex over a ring S. Let $\eta : S \to R$ be a homomorphism of rings. Then $(C \otimes_S R, \partial \times id_R)$ is a finite, based chain complex over R. In addition, if (C, ∂) is acyclic with chain contraction β, then $(C \otimes_S R, \partial \otimes id_R)$ is acyclic with chain contraction $\beta \otimes id_R$. The torsions of the two are related by $\tau(C \otimes_S R) = f_*\tau(C)$, where $f_* : \widetilde{K}_1(S) \to \widetilde{K}_1(R)$ is the induced homomorphism on K-theory. If R is a commutative ring, the determinant defines a homomorphism $\det : \widetilde{K}_1(R) \to R^\times/\langle \pm 1 \rangle$. If R is a field, then the determinant map is an isomorphism.

Definition 6.66. *Let* C_* *be a based chain complex over a ring* S*, and let* $\eta : S \to R$ *be a ring homomorphism to a commutative ring* R *or to* $\mathrm{GL}_n(R)$*. Suppose that* $C_* \otimes_S R$ *is acyclic. Then we define the Reidemeister torsion of* C_* *with respect to* $\eta : S \to R$ *to be the quantity* $\Delta_R(C_*) = \det(\tau(C_* \otimes_S R))$ *in* $R^\times/(\eta(g))$.

Definition 6.67. *Let* X *be a CW complex with fundamental group* G *and let* R *be a commutative ring. Let* $\rho : \mathbb{Z}[G] \to \mathrm{GL}_n(R)$ *be a ring homomorphism. When a choice of lifts and orientation of cells is given, the chain complex* $C'_* = C_*(\widetilde{X}) \otimes_{\mathbb{Z}[G]} R$*, if it is acyclic, has a torsion element* $\tau(C'_*) \in \widetilde{K}_1(R)$*. Therefore the Reidemeister torsion* $\Delta_R(X) = \Delta_R(C_*(X))$ *of* X *is defined to be the determinant* $\Delta_R(X) = \det \tau(C'_*)$ *in* $R^\times/(\pm g)$.

We give the relationship between the torsion of a homotopy equivalence and the Reidemeister torsions of the domain and range.

Proposition 6.68. *Suppose that* $f : X \to Y$ *is a homotopy equivalence between finite connected CW complexes, and consider the induced chain isomorphism* $f_* : C_*(X) \to C_*(Y)$*. Let* G *be their fundamental group and suppose that* $\eta : \mathbb{Z}[G] \to R$ *is a ring homomorphism to a commutative ring* R *such that* $C_*(\widetilde{X}) \otimes_{\mathbb{Z}[G]} R$ *and* $C_*(\widetilde{Y}) \otimes_{\mathbb{Z}[G]} R$ *are both acyclic. If* $I \subseteq R^\times$ *is the subgroup generated by* -1 *and* $\det(\sigma(g))$ *as* g *ranges through* G*, then*

1. *the Reidemeister torsions* $\Delta_R(X)$ *and* $\Delta_R(Y)$ *are well-defined when taken in* $R^\times/I$*;*
2. $\det \tau(f_*) = \Delta_R(Y) - \Delta_R(X)$ *in* $R^\times/I$*, or alternatively* $\Delta_R(Y)/\Delta_R(X)$ *if one is thinking multiplicatively.*

Remark 6.69. *The proposition implies that the image of the Whitehead torsion of* f

under the map

$$K_1(\mathbb{Z}[G]) \xrightarrow{p} K_1(\mathrm{GL}_n(R)) \xrightarrow{\det} R^\times$$

is given by $\Delta_R(Y)/\Delta_R(X)$.

For lens spaces, since $\mathbb{Z}_m$ acts trivially on the rational homology, we can use $R = \mathbb{Q}[\mathbb{Z}_m]/\Sigma$, where Σ denotes the ideal generated by the sum of the group elements. This ring is isomorphic to $\bigoplus\mathbb{Q}[\zeta_d]$, where d ranges over the divisors of m that are not 1. As a result, for each divisor of m, we obtain a unit in a cyclotomic field. To compute the Reidemeister torsion of a lens space, we can decompose L_m^{2k+1} as the join of L_m^{2k-1} and a circle; i.e. we can write $L_m^{2k+1} = L^{2k-1} \cup (\mathbb{S}^1 \times \mathbb{D}^{2k})$. Using the product and sum formulas for torsions from Appendix A.5 we can proceed inductively and show that $\Delta(L_m^{2d-1}(a_1, \dots, a_d)) = \prod \Delta(L_m^1(a_i))$. The chain complex for the base case is that of a circle $\mathbb{S}^1$, and $\Lambda(L_m^1(1)) = \zeta - 1$, where $\zeta = e^{2\pi i/m}$.

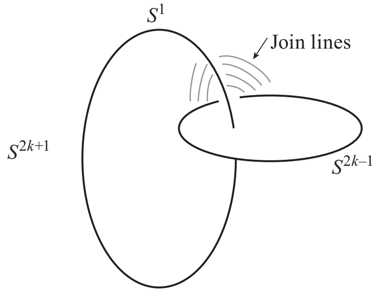


Figure 6.2: Constructing a lens space by join

Therefore $\Delta(L_m^{2d-1}(a_1, \dots, a_d)) = \prod(\zeta^{a_i} - 1)$. Since Cat isomorphic manifolds must have the same Reidemeister torsion in all three manifold categories, this invariant becomes a nice tool to distinguish between homotopy equivalent spaces. In the case of lens spaces, it may not be immediately obvious whether, given $a_1, \dots, a_d$ and $b_1, \dots, b_d$, we actually have an equality of Reidemeister torsions. Fortunately, the Franz independence lemma, a part of 19th century number theory, can be used to produce the following result.

Theorem 6.70. *(de Rham [199], see Cohen [168], Milnor [456]) Two lens spaces* $L_m^{2d-1}(a_1, \dots, a_d)$ *and* $L_m^{2d-1}(b_1, \dots, b_d)$ *are PL homeomorphic iff there is an integer* r *along with numbers* $\varepsilon_i = \pm 1$ *such that* $(a_1, \dots, a_k)$ *is a permutation of* $(\varepsilon_1 r a_1, \dots, \varepsilon_d r a_d)$.

Remark 6.71. *In fact, the homeomorphism classification is the same as the classification by simple homotopy equivalence and diffeomorphism. The theorem holds in Top as well. One can use the result of Chapman [158] and Kirby-Siebenmann [361] that homeomorphisms between compact manifolds are simple. See Appendix A.5.*

Example 6.72. *The classic example is that* $L_7^3(1)$ *and* $L_7^3(2)$ *are homotopy equivalent* *but not homeomorphic to each other.*

6.5.3 The ρ-invariant

Powerful as it is, the Reidemeister torsion is not an h-cobordism invariant. If an h-cobordism (W, M, N) has Whitehead torsion τ, then the ratio $\Delta(M)/\Delta(N)$ of the Reidemeister torsions of the boundary components is the norm $N(\tau)$. For cyclic groups, the Whitehead group is torsion-free, and $N(\tau) = \tau^2$ or 1, depending on the parity of the dimension. To some extent it can help identify pairs of linear lens spaces that are not h-cobordant, but not always.

However, Atiyah-Bott and Milnor showed in fact that linear lens spaces are *never* h-cobordant. They used ideas that would later develop into the G-signature theorem and the Atiyah-Patodi-Singer index theorem for manifolds with boundary. The purely topological center of this method was isolated by Wall and is called the *ρ-invariant* $\rho(M)$. It is defined for odd-dimensional oriented manifolds with finite fundamental group G, and in fact was tacitly defined in Chapter 2.

Recall that, for closed even-dimensional manifolds W with fundamental group G, we defined in Section 2.3 a signature invariant $\text{sign}_G(W) \in RO(G)^{(-1)^n}$, where the superscript indicates its relevant symmetry. This result also applies to Poincaré complexes. When n is even, the representation has a real character, and for n odd it has purely imaginary characters.

Remark 6.73. *Note that this invariant can be defined more generally for G-actions on manifolds or Poincaré complexes X, whether or not the action is free. If G is the fundamental group of X, then we can regard X as a G-space by mentally identifying it with the deck group action on its universal cover. Like the ordinary signature, this G-signature is a cobordism invariant and a homotopy invariant.*

Critical to the program of modifying this G-signature to obtain a non-homotopy invariant of free odd-dimensional G-manifolds is the following proposition, called *Novikov additivity* in the literature. In fact, the following proposition was tacitly used in our discussion of the Browder-Livesay invariant of homotopy $\mathbb{RP}^{4k-1}$.

Proposition 6.74. *(Novikov additivity) If W has boundary, one can define* $\text{sign}_G(W) \in RO(G)^{(-1)^n}$ *by taking the quadratic intersection form on $H_{2k}(M)$ and modding out by the torsion. Then* $\text{sign}_G(W_1 \cup W_2) = \text{sign}_G(W_1) + \text{sign}_G(W_2)$.

Remark 6.75. *Of course, just as in Section 2.3, we can take characters of this representation and obtain a collection of invariants* $\text{sign}(g, M)$ *for all $g \in G$. For some purposes, it is very useful to keep track of the rings in which these invariants take their values, despite the fact that they are torsion-free and do not lose any information when tensored with $\mathbb{C}$. However, the elegance of character theory makes it irresistible sometimes, and is frequently necessary for doing actual calculations.*

In the following, we use the notation $\widetilde{RO}$ to indicate that we must mod out by the regular representation to handle issues of well-definedness.

Definition 6.76. *Let M^{2n-1} be a closed oriented manifold with fundamental group G. Suppose that W is any oriented manifold with boundary $\partial W = rM$ with an isomorphism $\pi_1(M) \to \pi_1(W)$, where rM means the disjoint union of r copies of M. Then define the ρ-invariant of M to be $\rho(M) = \frac{1}{r}\operatorname{sign}_G(W)$, which lies in $RO(G)^- \otimes \mathbb{Q}$ when n is odd, and in $\widetilde{RO}(G)^+ \otimes \mathbb{Q}$ when n is even.*

Remark 6.77. *The definition requires that a multiple of M be the boundary of a free G-manifold. However, for the purposes of computation, one can use G-manifolds which are not free, and then use characteristic classes to correct the G-signature. It is more difficult if the action is really non-smooth. However, when there is some amount of "normal" smoothness available, then the correction is the term that enters the G-signature formula. See Atiyah-Patodi-Singer [26].*

The ρ-invariant for the lens space is not a simple calculation. We record here the formula as a character

$$\rho(L_m^{2d-1}(a_1, \dots, a_d)) = \frac{1+\zeta^{b_1}}{1-\zeta^{b_1}} \cdots \frac{1+\zeta^{b_d}}{1-\zeta^{b_d}},$$

where each b_i is the inverse of a_i mod m. Note that ρ is always a unit in the representation ring for linear lens spaces. See Atiyah-Bott [24] and Wall [672]. We also observe that the alternating product reflects the multiplicative nature of the determinant in (2) of Proposition 6.68.

Remark 6.78. *Because $\rho(g, N)$ is an h-cobordism invariant, we have the following. If N^{2k-1} is a manifold with $\pi_1(N) \cong G$, there is a function $\widetilde{\rho}_g : S^h(N) \to \mathbb{C}$ given by*

$$\widetilde{\rho}_g : [h : M \to N] \mapsto \rho(g, M) - \rho(g, N).$$

Theorem 6.79. *If M is an n-dimensional closed oriented manifold with finite fundamental group G, then $S^{Top}(M)$ is a finitely generated abelian group and*

$$S^{Top}(M) \otimes \mathbb{Q} \cong \widetilde{H}_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) \otimes \mathbb{Q} \text{ if } n \text{ is even}$$

and

$$S^{Top}(M) \otimes \mathbb{Q} \cong \widetilde{H}_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) \otimes \mathbb{Q} \times RO(G)^- \otimes \mathbb{Q}$$

if $n = 2k-1$ with k odd. Finally, if $n = 2k-1$ with k even, then

$$S^{Top}(M) \otimes \mathbb{Q} \cong \widetilde{H}_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) \otimes \mathbb{Q} \times \widetilde{RO}(G)^+ \otimes \mathbb{Q},$$

where the extra piece is given by the ρ-invariant and defines a lattice, i.e. a copy of $\mathbb{Z}^n$, in the target.

The proof follows from the bordism definition of the ρ-invariant and the calculation of $L_{2k}(\mathbb{Z}[G]) \otimes \mathbb{Q}$ in terms of the multisignature. It makes the abstract isomorphism $\mathcal{S}^{Top}(M) \cong \widetilde{H}_n(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) \otimes \mathbb{Q} \times \widetilde{L}_{n+1}(\mathbb{Z}[G]) \otimes \mathbb{Q}$ in Corollary 5.46 more concrete.

Wall showed that, in the case of lens spaces, the normal invariant and the ρ-invariant are closely integrally connected.

Remark 6.80. *It is not hard to use the material on assembly maps to define a structure group $\mathcal{S}^{hs}(M)$ of manifolds that are homotopy equivalent to M up to s-cobordism. The group does not fit directly in a Rothenberg sequence, but can be analyzed using the ideas of such a sequence. It is an exercise to compute the structure group and to extend the previous theorem from $\mathcal{S}^s$ or $\mathcal{S}^h$ to $\mathcal{S}^{hs}$. The latter can be achieved by using the ρ-invariant and the Reidemeister torsions for a manifold M whose fundamental group G acts trivially on the rational homology $H_*(\widetilde{M}; \mathbb{Q})$ of its universal cover, and the normal invariant, as well as by determining the group extensions.*

We end the section with some additional examples calculating with the ρ-invariant.

Proposition 6.81. *Let M be a $(4k-1)$-dimensional manifold with fundamental group G and let N be a 4ℓ-dimensional manifold. If $\widetilde{M}$ is the universal cover of G, then $\widetilde{M} \times N$ has a free G-action. Let K be the manifold given by the quotient $(\widetilde{M} \times N)/G$. Then $\rho(K) = \operatorname{sig}_G(N) \otimes \rho(M)$ in $R_{\mathbb{Q}}^{+}(G)/I_G$.*

It is an exercise to show that two homotopy equivalent three-dimensional lens spaces become diffeormorphic after crossing with a sphere $\mathbb{S}^{2k+1}$ with $k \geq 2$, but not after crossing with $\mathbb{S}^{2k}$ or $\mathbb{S}^1$. One can ask whether the remaining products are h-cobordant, and whether there is an extension to higher dimensions.

Example 6.82. *Consider a cyclic p-subgroup $\mathbb{Z}_p \subseteq SU(n)$ generated by some diagonal matrix $\operatorname{Diag}(\lambda_1, \ldots, \lambda_n)$ where $\prod_{i=1}^n \lambda_i = 1$ and λ_i are all p-th roots of unity. Let $M = SU(n)/\mathbb{Z}_p$ be the quotient of the action. Then $\rho(M)$ is zero.*

Example 6.83. *Since the signature depends on the orientation, the sign of the ρ-invariant also depends on the orientation. Let $n \equiv 3 \bmod 4$ and let $h\colon \mathbb{RP}^n \to \mathbb{RP}^n$ be the map given by $[x_0 : \cdots : x_n] \mapsto [-x_0 : \cdots : x_n]$. This map has degree -1 and induces an isomorphism on the fundamental group. Let $\mathbb{RP}^n_{op}$ denote the space with the opposite orientation. Then*

$$\rho(\mathbb{RP}^n) \overset{h_*}{=} \rho(\mathbb{RP}^n_{op}) = -\rho(\mathbb{RP}^n).$$

Since $\rho(\mathbb{RP}^n) = 0$ lies in a torsion-free group, it follows that it is zero.

6.5.4 Free cyclic group actions on the sphere

One of the most impressive achievements in Wall's book is the classification of homotopy lens spaces with odd-order fundamental group. Wall's theorem is somewhat more precise than the following.

Theorem 6.84. *Homotopy lens spaces with odd-order fundamental group are determined by their Reidemeister torsions and their ρ-invariants. There is an isomorphism given by the suspension map that joins an action on $\mathbb{S}^{2k-1}$ with the rotation by $2\pi/n$ on $\mathbb{S}^1$ to obtain a rotation on $\mathbb{S}^{2k+1}$.*

The last sentence is in strong contrast with the case of $\mathbb{Z}_2$, where the situation is already seen to be remarkable. The normal invariant $[\mathbb{S}^{2k+1}/\mathbb{Z}_m : F/Top]$ is clearly odd torsion and therefore can be computed by $KO_*(L_m^{2k+1})$. Additionally it has order around $m^{k/2}$ by the Atiyah-Hirzebruch spectral sequence. The size of the normal invariants grows in k, but amazingly it is irrelevant to the isomorphism of the structure sets. The normal invariant of a homotopy equivalence is determined by the ρ-invariant. Moreover, the suspension map takes normally cobordant homotopy lens spaces to homotopy lens spaces that are not necessarily normally cobordant.

Remark 6.85. *Incidentally, one can extend this theorem to dimension 3 if one classifies $\mathbb{Z}_n$-actions on integral homology 3-spheres up to s-concordance. See Freedman-Quinn [254]. The suspension map will still produce a continuous action on $\mathbb{S}^5$ because of the double suspension theorem (see Appendix A.5). In dimension 3, the calculation of the surgery exact sequence is obvious enough, since the normal invariant set for lens spaces with odd-order fundamental group is trivial in dimension 3.*

The following is an approach which can be used to provide a classification up to s-cobordism. Because $\mathrm{Wh}(\mathbb{Z}_n)$ is torsion-free, the Reidemeister torsion classifies all the possible simple homotopy types of homotopy lens spaces.

Using the join construction, we can decompose $L_m^{2k+1} = E \cup (\mathbb{S}^1 \times \mathbb{D}^{2k})$, where E is a $\mathbb{D}^2$-bundle over L_m^{2k-1}. We can split along $\mathbb{S}^1 \times \mathbb{S}^{2k-1}$ using the π-π case of splitting. There is an isomorphism $\mathcal{S}^{Top}(L_m^{2k+1}) \to \mathcal{S}^{Top}(E, \partial E)$ which requires us to show that the map induced by taking the disk bundle gives an isomorphism $\mathcal{S}^{Top}(L_m^{2k-1}) \to \mathcal{S}^{Top}(E, \partial E)$ with the Chern class $1 \in H^2(L; \mathbb{Z}) \cong \mathbb{Z}_n$. [2]

Consider the composition

$$L_i(\mathbb{Z}[e]) \to L_i(\mathbb{Z}[\mathbb{Z}_m]) \to L_{i+2}(\mathbb{Z}[\mathbb{Z}_m], \mathbb{Z}[\mathbb{Z}]) \to L_{i+1}(\mathbb{Z}[\mathbb{Z}]) \to L_i(\mathbb{Z}[e]),$$

where the first map is by inclusion, the second induced by the bundle E, and the next by the boundary, and the last is obtained by codimension one splitting, i.e. the Shaneson formula for $\mathbb{Z} \times \{e\}$. This composition is clearly the identity, and therefore we should consider the reduced L-groups. This calculation is at least reasonable by the methods described in Chapter 2. In Section 2.3 we discussed the calculation of the L-groups for all cyclic groups.

The last step to check is that this map induces an isomorphism, which, because of the connection between the L-groups and multisignature, requires that ρ be a unit. This

[2] Recall that the notation $\mathcal{S}^{Top}(E, \partial E)$ means not rel boundary.

fact, however, is explicit in Wall, and implicit in the work of Atiyah and Bott [24].

Remark 6.86. *Another completely different argument, not available to Wall at the time, uses PL equivariant signature operators applied to the cone of the action on the sphere. See Teleman [641]. This approach would give an element in* $KO^{\mathbb{Z}_m}_{2k+2}(\mathbb{D}, \mathbb{S})$ *that combines the ρ-invariant and the L-group; the suspension theorem is basically equivariant Bott periodicity. This argument still requires some special pleading at the prime 2 since the signature operator does not quite generally detect normal invariants at 2. But it does here by our observations about normal invariants.*

Wall's method is more direct. Using the classification of homotopy complex projective spaces $\mathbb{CP}^k$, he had many homotopy lens spaces at his disposal; i.e. the total space of circle bundles with Euler class $m \in H^2(\mathbb{CP}^k; \mathbb{Z}) \cong \mathbb{Z}$. He deduces their ρ-invariants from the G-signature formula, initially only known in the smooth case. Their actions, however, are *normally smooth*, i.e. there is a neighborhood of the fixed set that is equivariantly homeomorphic to the action on a G-vector bundle. These examples exhaust all of the normal invariants of lens spaces and show that the ρ-invariant, modulo the images of multisignatures of $L_{2k+2}(\mathbb{Z}[\mathbb{Z}_m])$, is strong enough to determine the normal invariant.

The case of general cyclic groups was only completed relatively recently. The proof combines phenomena that arise for $\mathbb{Z}_2$ and for $\mathbb{Z}_n$ when n is odd. The ρ-invariant detects part of the normal invariant, i.e. the rank of the abelian group of normal invariants for the even-order case grows with dimension, so it could not possibly detect all of it. Here is a description of the results. For more information, consult Balko-Macko-Niepel-Rusin [34] and Macko-Wegner [417] for the important case of powers of 2.

The general result is a mixture of cases for 2^k and m odd. With the complete answer that $S^s(L_m)$ is free abelian for m odd in this case, i.e. so that the surgery exact sequence is completely non-split, there is a need for torsion because of the calculation of the normal invariants

$$[L^{2d-1}_{2^k} : F/Top] \cong \bigoplus H^{4i+2}(L^{2d-1}_{2^k}; \mathbb{Z}_2) \oplus \bigoplus H^{4i}(L^{2d-1}_{2^k}; \mathbb{Z}_{(2)}).$$

The number of summands increases linearly with dimension, and the size of the L-group is more or less unchanged. Each of the first terms gives a copy of $\mathbb{Z}_2$ in the relevant dimension, and the second terms gives a copy of $\mathbb{Z}_{2^k}$.

This calculation was done by Macko-Wegner [417].

Theorem 6.87. *Let* $\mathfrak{L}^{2d-1}_m$ *be a homotopy lens space with* $\pi_1(\mathfrak{L}^{2d-1}_m) \cong \mathbb{Z}_m$ *where* $m = 2^K$ *and* $d \geq 3$. *Then*

$$S^s(\mathfrak{L}^{2d-1}_m) \cong \mathbb{Z}^k \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_2 \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_{2^{\min(K,2i)}},$$

where $c = [\frac{d-1}{2}]$ *and* $k = \frac{m}{2} - 1$ *if d is odd and* $\frac{m}{2}$ *if d is even.*

The calculation is similar to that of Wall, i.e. based on the comparison of ρ-invariants for normal invariants that come from the restriction of circle actions to a copy of $\mathbb{Z}_{2^k}$ inside and also on a small difference that comes from $L_3(\mathbb{Z}[\mathbb{Z}_{2m}]) = \mathbb{Z}_2$ given by a codimension one Arf invariant. However, the straightforward variation on the trick of taking a connected sum with a Kervaire manifold shows that, for any manifold M with a surjection $\pi \equiv \pi_1(M) \to \mathbb{Z}_2$, the normal invariant map $\mathcal{N}(M) \to L_3(\mathbb{Z}[\pi]) \to L_3(\mathbb{Z}[\mathbb{Z}_2])$ is a surjection.

Remark 6.88. *Wall and Macko-Wegner [417] and Balko-Macko-Niepel-Rusin [34] accomplish more. They classify homotopy lens spaces as an explicit combination of τ, ρ, and, for the even-order case, the remaining part of the normal invariant that is not in the image of ρ. Functoriality can be used to simplify these calculations. In light of functoriality we can use these results about lens spaces to understand arbitrary closed manifolds with cyclic fundamental group.*

Proposition 6.89. *If M is an oriented topological manifold with cyclic fundamental group π, then the sequence*

$$0 \to \mathcal{S}^s(M) \to \mathcal{S}^s(B\pi) \times H_n(M; \mathbb{L}_\bullet) \to H_n(B\pi; \mathbb{L}_\bullet)$$

is exact at the middle two terms.

This proposition determines all the extension problems in structure sets in these cases. Furthermore, Dress induction immediately tells us that it will be true for groups that are products of cyclic groups and groups of odd order.

The proof is a useful exercise in the assembly map and L-theory calculation.

Remark 6.90. *Our discussion of the oozing problem in Section 6.7 will show that a similar result holds for the h-decoration whenever π has a finite abelian 2-Sylow subgroup, but it is not true when π is quaternionic. On the other hand, for the projective p-decoration, the same holds for all finite fundamental groups, at least in the oriented case, because here the worst obstructions that arise from closed manifolds are collections of codimension one Arf invariants; i.e. they are detected by mapping to cyclic groups. These calculations show that it is fundamentally important to understand $\mathcal{S}^{Top}(B\pi)$, even when $B\pi$ is not a manifold and in situations distant from the Borel conjecture.*

6.6 THE TOPOLOGICAL SPACE FORM PROBLEM

The topological space form problem, raised by Hopf in 1925, is the question of classifying those manifolds whose universal cover is the sphere, or at least of identifying the groups that can act freely on it. P. A. Smith gave the first condition, that $\mathbb{Z}_p \times \mathbb{Z}_p$ cannot act in this way. Cartan-Eilenberg gave a generalization to the condition that the group cohomology of such a group must be periodic. Between 1935 and 1956, Zassenhaus, Suzuki, and Cartan-Eilenberg classified the groups of periodic cohomology. Wolf completely characterized the groups that act linearly and freely on some sphere.

Swan [631] proved that, if m is the period of the group G, then there is a finite complex of finite type with a free G-action that is homotopy equivalent to $\mathbb{S}^{km-1}$ for some integer k. At that time, however, the problem remained unresolved, whether periodic cohomology sufficed for a free action or whether nonlinear examples existed.

Milnor then showed that, when p is prime, the dihedral group D_{2p} does not act on a manifold homotopy sphere, despite having periodic cohomology. Petrie [505] then gave the first examples of groups that act freely on topological spheres but not linearly so, i.e. not by an orthogonal action. An example of such a group is the semidirect product $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$. Lee [393] showed that the generalized quaternionic groups $Q(16a, b, c)$ of period 4 act freely and linearly on $\mathbb{S}^7$ but not even freely on any sphere of the form $\mathbb{S}^{8k+3}$, despite the periodicity of the cohomology groups. Davis's semicharacteristic (discussed in Section 2.4) can be used to prove these results. Given a sphere $\mathbb{S}^r$ of a particular odd dimension r, we still do not know, as of the writing of this text, how to classify the precise groups that act freely on it, although it is clear that the issue is number-theoretic.

The classification problem in both the linear and general cases are deeply related to the notion of the pq-condition on a group. If p and q are primes (not necessarily distinct), we say that a given group G *satisfies the pq-condition* if every subgroup of G of order pq is cyclic. A group has periodic cohomology iff it satisfies the p^2-condition for all primes p. Wolf proves that, if G acts freely and orthogonally on a sphere, then it satisfies all pq-conditions [712]. The converse holds if G is solvable. However, the pq-conditions are not all necessary for free topological action: the examples of Petrie mentioned above showed that for p and q odd the pq-condition is never necessary: if m is odd q is an odd prime with $q|\phi(m)$ and $(q, m) = 1$, then any extension of $\mathbb{Z}_m$ by $\mathbb{Z}_q$ acts freely on $\mathbb{S}^{2q-1}$.

The result of Milnor [447] that a group G acting freely on some topological sphere cannot contain dihedral groups is exactly the assertion that they satisfy the $2p$-condition for all primes p, in contrast with Swan's theorem that they act on finite complexes of the right homotopy type. These Swan Poincaré complexes are interesting complexes that are not homotopy equivalent to closed manifolds. Milnor's argument was direct and elementary, but not surgery-theoretic. J. Davis's semicharacteristic puts them in a surgery context.

Madsen, Thomas, and Wall [422] finally proved the result that the p^2- and $2p$-conditions on a group G are both necessary and sufficient for a free spherical G-action to exist. With the development of non-simply connected surgery, the topological space form problem was completed in a series of papers by these three authors in the late 1970s. This section will explain how Dress induction and various facts about periodic groups combine in the proof of this theorem. We refer to the book of Davis-Milgram [192] and the original papers for more information.

6.6.1 Periodic groups

In this chapter all groups G are finite. To catalogue all such finite groups that act freely on some sphere, we first show that such groups must be periodic. These so-called P-groups have interesting homological properties and can be exhaustively enumerated. To begin, we see that only spheres of odd dimension admit interesting fixed-point-free actions.

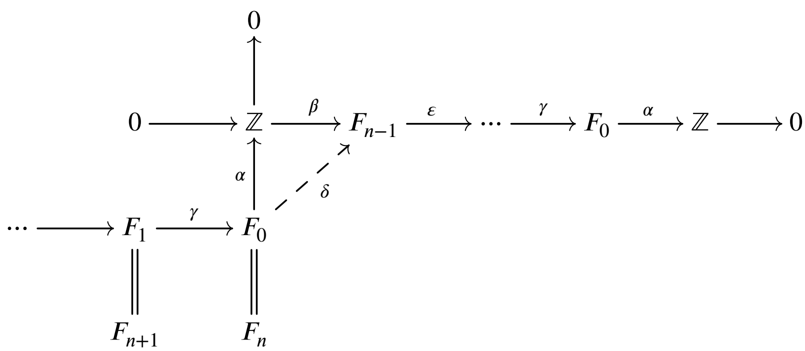
Proposition 6.91. *The only non-trivial group which acts freely on an even-dimensional sphere is* $\mathbb{Z}_2$.

Proof. By the Lefschetz fixed point theorem, any self-map of $\mathbb{S}^{2k}$ without fixed points must have degree -1. If G contains non-trivial elements g and h, then as fixed-point-free maps on the sphere both g and h have degree -1, and so gh has degree $(-1)(-1) = 1$. Therefore $gh = e$. But g and h are arbitrary, so the group G must be $\mathbb{Z}_2$. □

We first show that, if G acts freely on a sphere, then it has a free periodic resolution. This condition implies that all Sylow p-groups of G are cyclic or generalized quaternionic.

Definition 6.92. *Let G be a group and n a positive integer. By a free $\mathbb{Z}[G]$-resolution of period n of G we mean an exact sequence* $0 \to \mathbb{Z} \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0$ *of $\mathbb{Z}[G]$-modules, where the F_i are finitely generated and free, and G acts trivially on the two $\mathbb{Z}$ terms. In this case, we say that n is a period of G. The smallest such n is called the period of G. We also say that G is periodic.*

Despite the terminology, a free $\mathbb{Z}[G]$-resolution of period n of G is technically not a free $\mathbb{Z}[G]$-complex, since the leftmost $\mathbb{Z}$ is not a free $\mathbb{Z}[G]$-module. But the following diagram describes how, by concatenating the sequence $F_{n-1} \to \cdots \to F_1 \to F_0$ with itself repeatedly, we obtain infinite-length free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules.



Indeed, since α is surjective and β is injective, we have $\operatorname{im}\beta = \operatorname{im}(\beta\circ\alpha)$ and $\ker\delta = \ker\alpha$. So $\ker\varepsilon = \operatorname{im}\beta = \operatorname{im}(\beta\circ\alpha) = \operatorname{im}\delta$ and $\ker\delta = \ker\alpha = \operatorname{im}\gamma$. The new sequence is therefore exact at both F_{n-1} and $F_0 = F_n$.

One natural example of such a resolution is the chain complex of a *space form*, i.e. the quotient of a sphere by the free action of a group G.

Proposition 6.93. *If a finite group G acts freely on $\mathbb{S}^{n-1}$ and $n-1$ is odd, there is a free $\mathbb{Z}[G]$-resolution of period n.*

Proof. Consider the equivariant cellular chain complex of $\mathbb{S}^n$. □

Definition 6.94. *We say that a group G has periodic cohomology if there is $d \geq 2$ such that, for all $i \geq 1$, we have $H^i(G;\mathbb{Z}) = H^{i+d}(G;\mathbb{Z})$.*

Theorem 6.95. *(Artin-Tate [15], see Cartan-Eilenberg [141]) If G acts freely on $\mathbb{S}^{n-1}$ with $n-1$ odd, then*

1. *G has periodic cohomology;*
2. *$H^n(G;\mathbb{Z}) \cong \mathbb{Z}_{|G|}$;*
3. *all abelian subgroups of G are cyclic;*
4. *every Sylow p-group of G is cyclic or the generalized quaternionic group*

$$Q(2^m) = \langle x, y \mid x^{2^{m-1}} = 1,\ x^{2^{m-2}} = y^2,\ yxy^{-1} = x^{-1} \rangle$$

of 2^m elements for some m.

In fact these four conditions are equivalent.

Proof. Statement (1) is obvious using our iterated periodic resolution. We offer a proof of (2). Denote by $K(G,1)_{(n)}$ the n-skeleton of the Eilenberg-Maclane space $K(G,1)$. Let us write $(\mathbb{S}^{n-1}/G) \cup_\alpha e^n = K(G,1)_{(n)}$, where α is a map that sends elements of $\mathbb{S}^{n-1} = \partial e^n$ to a generator of $H_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$. If $\alpha : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}/G$ is the projection map, then α is the generating element of $\pi_{n-1}(\mathbb{S}^{n-1}/G) = \mathbb{Z}$ and $H_{n-1}(K(G,1)) \cong \mathbb{Z}/\operatorname{im}\alpha_* \cong \mathbb{Z}_{|G|}$. Since $H_n(G;\mathbb{Z})$ is finite, we know that $\operatorname{Hom}(H_n(G;\mathbb{Z}),\mathbb{Z}) = 0$, so $H^n(G;\mathbb{Z}) \cong \operatorname{Hom}(H_n(G;\mathbb{Z});\mathbb{Z}) \oplus \operatorname{Ext}(H_{n-1}(G;\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}_{|G|}$. Statement (3) is clearly necessary and (4) requires non-trivial calculations of group cohomology which we suppress. □

A group that satisfies any of these conditions is called a *P-group*. For example, the quaternionic group Q_8 is a P-group, but the dihedral group D_8, the automorphism group of the square, is not. We will continue to use the standard notation that, if p is prime, then a group H is a *p-group* if there is an integer k for which $|H| = p^k$.

Let us state some basic properties of P-groups and their periods. The period of $\mathbb{Z}_m$ is 2 and the period of $Q(2^n)$ is 4. If G_1 and G_2 are P-groups with periods n_1 and n_2, and if $(|G_1|, |G_{(2)}|) = 1$, then $G_1 \times G_2$ is a P-group with period $\operatorname{lcm}(n_1, n_2)$. We can also obtain P-groups by taking semidirect products. If G is a P-group all of whose Sylow

p-subgroups of G are cyclic, i.e. not generalized quaternionic, then G is a metacyclic group $\mathbb{Z}_a \rtimes_\phi \mathbb{Z}_b = A(a, b, \phi)$, where a, b are coprime and $\phi\colon \mathbb{Z}_b \to \operatorname{Aut} \mathbb{Z}_a$ is a homomorphism. For example, a P-group of odd order is metacyclic. Conversely, if a and b are coprime, then $A(a, b, \phi)$ is a P-group.

In the statement of Theorem 6.95 we defined the quaternionic group of 2^n elements. One can further define a quaternionic group of $4k$ elements for any integer $k \geq 2$:

$$Q(4k) = \langle x, y \mid x^{2k} = 1,\ x^k = y^2,\ yxy^{-1} = x^{-1} \rangle.$$

Yet another generalization, denoted by $Q(2^n a, b, c)$, where a, b, c are coprime and $n \geq 3$, arises from a sequence $0 \to \mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c \to Q(2^n a, b, c) \to Q(2^n) \to 0$. There the group $Q(2^n)$ is mapped into $\operatorname{Aut}(\mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c)$ so that x inverts elements in $\mathbb{Z}_a$ and $\mathbb{Z}_b$ while y inverts elements in $\mathbb{Z}_a$ and $\mathbb{Z}_c$.

Although we do not need all these groups for our purposes, it is worth realizing the extent to which P-groups can be completely listed.

At this point we begin our discussion on the complete classification of P-groups, primarily a result of work done by Zassenhaus in the solvable case and Suzuki in the non-solvable case. The classification separates these groups G into six classes, depending on the nature of the quotient group $G/O(G)$, where $O(G)$ is the maximal normal subgroup of odd order of G. Since the subgroups of P-groups are also P-groups, it is clear that $O(G)$ is metacyclic. In addition $G/O(G)$ has the same Sylow 2-subgroups as G and, if p is odd, the Sylow p-subgroups of $G/O(G)$ are cyclic.

Two of the six types for $G/O(G)$ are the cyclic group $\mathbb{Z}_{2^r}$ and the quaternionic group $Q(2^n)$. The other four are matrix groups or extensions of matrix groups.

Theorem 6.96. *(Suzuki [628]) Let* $\mathrm{SL}_2(\mathbb{F}_p)$ *be the collection of* 2×2 *matrices with determinant 1 with entries in the field* $\mathbb{F}_p$*. If* $p \geq 3$*, then* $\mathrm{SL}_2(\mathbb{F}_p)$ *is a* P*-group. The perfect* P*-groups are precisely the* $\mathrm{SL}_2(\mathbb{F}_p)$ *for* $p \geq 5$*.*

Note that both $\mathrm{SL}_2(\mathbb{F}_3)$ and $\mathrm{SL}_2(\mathbb{F}_5)$ are subgroups of $\mathbb{S}^3$ and $\mathbb{S}^3/\mathrm{SL}_2(\mathbb{F}_5)$ is the Poincaré sphere.

One of the key elements of the space form problem is to identify the p-hyperelementary subgroups of a given P-group, on which we can exercise Dress induction. We described this process in Section 2.5. The p-hyperelementary subgroups of P-groups fall into only two general classes, so that the analysis of them in the later subsections of the chapter will be simply a two-step process.

Definition 6.97. *Let p be a prime, and let n be a positive integer coprime to p. A p-hyperelementary group is a group G formed as a split extension* $0 \to \mathbb{Z}_n \to G \to G_p \to 0$*, where G_p is a p-group. It is expressed* $A(n, G_p, \phi)$*, where* $\phi\colon G_p \to \operatorname{Aut} \mathbb{Z}_n$ *is a homomorphism. In other words, a p-hyperelementary group is an extension of a p-group by a cyclic group.*

Theorem 6.98. *The p-hyperelementary P-groups are of the form* $A(a, p^r, \phi) = \mathbb{Z}_a \rtimes \mathbb{Z}_{p^r}$

and $\mathbb{Z}_d \rtimes Q(2^n a, b, c)$. In this notation we allow for the fact that the $\mathbb{Z}_a$ and $\mathbb{Z}_d$ do not appear, so that $\mathbb{Z}_{p^r}$ and $Q(2^n a, b, c)$ are also listed in the classification.

6.6.2 The *k*-invariant

The previous section showed that periodicity is a necessary condition for a finite group G to act freely on some sphere. To test the converse, we begin with a P-group G and determine whether, as a first step, it acts freely on some finite-dimensional CW complex that is homotopy equivalent to a sphere. Much of this discussion generalizes the material for homotopy lens spaces. The finiteness criterion is addressed by the Swan obstruction, which we will discuss in this section.

Definition 6.99. *Let G be a group and let $C_* = \{C_n\}$ be a chain complex of projective $\mathbb{Z}[G]$-modules. We say that C_* is finitely dominated if it is $\mathbb{Z}[G]$-chain homotopic to a finite projective complex $\{P_i\}_{i=0}^n$; i.e. the P_i are finitely generated projective $\mathbb{Z}[G]$-modules.*

Remark 6.100. *This terminology is motivated by Wall [664]. The Swan theory anticipated Wall's work in an important special case.*

Definition 6.101. *The space X is finitely dominated if there is a finite CW complex Y with maps $i : X \to Y$ and $r : Y \to X$ such that $r \circ i$ is homotopy equivalent to id_X.*

Theorem 6.102. *(Wall [681]) The space X is finitely dominated iff the $\mathbb{Z}[\pi_1(X)]$-chain complex $C_*(\widetilde{X})$ is finitely dominated.*

The following theorem gives a situation in which a projective $\mathbb{Z}[G]$-resolution naturally appears.

Theorem 6.103. *(Swan [631] Theorem 4.1) If G is a P-group with cohomological period n, then there is a periodic projective $\mathbb{Z}[G]$-resolution $C : 0 \to \mathbb{Z} \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$ of period n of G. In fact, there is an $(n-1)$-dimensional simplicial homotopy $(n-1)$-sphere on which G acts freely and simplicially.*

Notice that this theorem only gives the existence of a finite-dimensional homotopy sphere, but does *not* insist that space be a *finite* complex.

Definition 6.104. *Let G be a finite group. An $(n-1)$-dimensional CW complex Y with basepoint y_0 has a (G, n)-polarization if there is an isomorphism $\alpha_Y : \pi_1(Y, y_0) \to G$ along with a homotopy equivalence $f_Y : \widetilde{Y} \to \mathbb{S}^{n-1}$.*

Remark 6.105. *Note that we can discuss (G, n)-polarized complexes even if they are not finite.*

Definition 6.106. *Let (Y, y_0) and (Z, z_0) be (G, n)-polarized complexes equipped with isomorphisms $\alpha_Y : \pi_1(Y, y_0) \to G$ and $\alpha_Z : \pi_1(Z, z_0) \to G$ as well as homotopy equiv-*

alences $f_Y : \widetilde{Y} \to \mathbb{S}^{n-1}$ and $f_Z : \widetilde{Z} \to \mathbb{S}^{n-1}$. We say that Y and Z have the same polarization type if there is a homotopy equivalence $g : Y \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(Y) & & \\ & \searrow^{\alpha_Y} & \\ \downarrow g_* & & G \\ & \nearrow_{\alpha_Z} & \\ \pi_1(Z) & & \end{array}$$

We denote by $P(G,n)$ the collection of (G,n)-polarizations up to polarization type.

The following theorem tells us that polarizations immediately have the finite domination property.

Proposition 6.107. *(Swan [631]) The $\mathbb{Z}[G]$-chain complex $C_*(\widetilde{Y})$ associated to the universal cover $\widetilde{Y}$ of a (G,n)-polarized Y is $\mathbb{Z}[G]$-chain homotopic to a finite projective complex; i.e. $C_*(\widetilde{Y})$ is finitely dominated.*

If G is a P-group of cohomological period n with an additive generator $\alpha \in H^n(G;\mathbb{Z}) \cong \mathbb{Z}_{|G|}$, then $\cup\alpha : H^i(G;\mathbb{Z}) \to H^{i+n}(G;\mathbb{Z})$ is an isomorphism for all $i \geq 1$. A theorem by Roggenkamp-Wall [681] shows that α gives rise to an exact sequence $0 \to \mathbb{Z} \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$, where the P_i are finitely generated projective $\mathbb{Z}[G]$-modules. As in the free case, one can extend this sequence to the left. Note that $C^n(G;\mathbb{Z}) \cong \mathrm{Hom}(P_n,\mathbb{Z}) \cong \mathrm{Hom}(P_0,\mathbb{Z})$, which contains g. Now $C^{n-1}(G;\mathbb{Z}) \cong \mathrm{Hom}(P_{n-1},\mathbb{Z})$, and we have maps $C^{n-1}(G;\mathbb{Z}) \xrightarrow{\phi} C^n(G;\mathbb{Z}) \to 0$. Therefore $H^n(G;\mathbb{Z}) \cong C^n(G;\mathbb{Z})/\mathrm{im}(\phi)$ contains $[g]$, which by abuse of notation we simply denote by g.

Definition 6.108. *The $g \in H^n(G;\mathbb{Z})$ defined above is called the k-invariant of the sequence C^*. We write $k(C^*) = g$. For such k-invariants $g \in H^n(G;\mathbb{Z})$, we denote by $\chi(g)$ the associated Euler characteristic in $\widetilde{K}_0(\mathbb{Z}[G])$. (Recall our discussion of the Swan homomorphism in Section 1.4.)*

Remark 6.109. *The definition gives us a convenient way to describe the usual Postnikov k-invariant. It is the natural generalization of the product of the rotation numbers of a linear lens space, and plays the same role in homotopy classification.*

Proposition 6.110. *Two (G,n)-polarizations have the same polarized homotopy type iff they have the same Eilenberg-Maclane k-invariant.*

6.6.3 Swan's obstruction

We will be interested in knowing, if G is a P-group, whether there are elements of $P(G, n)$ which are homotopic equivalent to a finite complex. We will see that the Euler characteristic obstruction to finiteness can be linked to the k-invariant of a complex X.

Definition 6.111. *Let G be a group and let $C_* = \{C_n\}$ be a chain complex of projective $\mathbb{Z}[G]$-modules. We say that C_* is homotopy finite if it is $\mathbb{Z}[G]$-chain homotopic to a finite free complex $\{C_i\}_{i=0}^n$.*

Definition 6.112. *The space X is homotopy finite if X is homotopy equivalent to a finite complex, i.e. a complex constructible from finitely many simplices.*

Theorem 6.113. *(Wall [681]) The space X is homotopy finite iff $C_*(\widetilde{X})$ is homotopy finite, i.e. if $\chi(C_*(X))$ vanishes in $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$.*

In general, the cellular chain complex $C_*(\widetilde{X})$ on the universal cover $\widetilde{X}$ of a CW complex X will always be free, but might not be finite. Such $C_*(\widetilde{X})$ may still however be $\mathbb{Z}[\pi_1(X)]$-chain homotopic to a finite projective complex. The first examples of finitely dominated spaces which are not homotopy equivalent to a finite CW complex were given by de Lyra [198] in 1965. Wall's paper shows that all elements of $\widetilde{K}_0(\mathbb{Z}[G])$ arise.

To handle Swan's theorem, we ask whether there is a periodic free resolution over $\mathbb{Z}[G]$ of period k, i.e. whether there is a periodic projective resolution E_* of period k whose Euler characteristic $\chi(E_*)$ vanishes in $K_0(\mathbb{Z}[G])$.

Definition 6.114. *Let $\mathrm{Sw} : \mathbb{Z}_{|G|}^{\times} \to \widetilde{K}_0(\mathbb{Z}[G])$ be the usual Swan homomorphism. Let $q : \widetilde{K}_0(\mathbb{Z}[G]) \to \widetilde{K}_0(\mathbb{Z}[G])/\mathrm{Sw}(G)$ be the obvious projection map and let $C_* : 0 \to \mathbb{Z} \to C_{n-1} \to C_{n-1} \to \cdots \to C_2 \to C_1 \to \mathbb{Z} \to 0$ be any projective $\mathbb{Z}[G]$-resolution of period n of G. Define the Swan obstruction $\lambda_n(G) = q(\chi(C_*))$.*

The key properties of the Swan obstruction are as follows:

Lemma 6.115. *Let G be a P-group. Let s_1 and s_2 be additive generators of $H^n(G; \mathbb{Z}) \cong \mathbb{Z}_{|G|}$. Let g be as above. Then*

1. $\chi(C_{s_1}) - \chi(C_{s_2}) = \mathrm{Sw}(s_1/s_2)$,
2. $\chi(\underbrace{g \cup \cdots \cup g}_{n}) = n\chi(g)$.

The first is essentially Mislin's formula, which we discussed in Section 1.4.

Lemma 6.116. *(Swan [631] Lemma 7.3) Let E_* be a periodic projective resolution of period n and let X be a (G, n)-polarization. Denote by E_X the cellular chain complex on the universal cover $\widetilde{X}$ of X. Then $\chi(E) - \chi(E_X) = (-1)^n \mathrm{Sw}(k(X))$.*

Different polarizations can be compared to one another by a propagation map as we discussed for lens spaces.

Corollary 6.117. *(Swan [631]) The Swan obstruction satisfies the following properties:*

1. $\lambda_{kn}(G) = k\lambda_n(G)$ *for all positive integers* k;
2. $\lambda_n(G)$ *vanishes iff there is an exact sequence*

$$0 \to \mathbb{Z} \to C_{n-1} \to C_{n-2} \to \cdots \to C_1 \to C_0 \to \mathbb{Z}$$

   with all C_i *finitely generated free;*
3. $\lambda_n(G)$ *vanishes iff there is a finite CW complex* X^{n-1} *homotopy equivalent to* $\mathbb{S}^{n-1}$ *on which* G *acts freely.*

In fact, Swan proves that, when G is a P-group of period n and order N, then $\lambda_{dn}(G) = 0$ when $d = \gcd(N, \phi(N))$.

Since $\widetilde{K}_0(\mathbb{Z}[G])$ is finite for finite groups G, every P-group therefore acts on some finite-complex homotopy sphere. With the theorem above, we then know that there is a finite CW complex X^{dn-1} homotopy equivalent to $\mathbb{S}^{dn-1}$ on which G acts freely. The quotient Y^{dn-1}/G is then a (G, dn)-polarized complex that is homotopic to a finite complex. Wall improves these statements by showing that d can be replaced by 2. We write this result for completeness.

Theorem 6.118. *(Wall [681]) Suppose that* G *is a* P*-group with period* n*. Denote by* T_G *the image of the Swan homomorphism.*

1. *The class* $\lambda_{2n}(G) = 0$ *in* $\widetilde{K}_0(\mathbb{Z}[G])/T_G$.
2. *The group* G *acts freely and cellularly on a finite CW complex* Y *homotopy equivalent to the sphere* $\mathbb{S}^{2n-1}$.
3. *There is a* $(G, 2n)$*-polarized complex* Y^{2n-1} *that is homotopy equivalent to a finite CW complex.*

See Davis-Milgram [192] for the P-groups G for which $\lambda_n(G) = 0$, and examples where $\lambda_n(G) \neq 0$.

6.6.4 Exploiting orthogonal actions

Theorem 6.119. *(Wolf [712]) Let* G *be solvable. Then* G *has a fixed-point-free finite* $\mathbb{C}$*-representation iff* G *satisfies all* pq*-conditions.*

Proof. We explain necessity. Suppose that G is solvable and let $\pi : G \to GL(V)$ be a fixed-point-free representation of G on the complex vector space V. First we note that, if H is a non-trivial subgroup of G and $v \in V$ and $h' \in H$ is non-trivial, then

$\pi(h') \sum_{h\in H} \pi(h)v = \sum_{h\in H} \pi(h)v$. Hence $(\pi(h') - 1) \sum_{h\in H} \pi(h)v$. Since $\pi|_H$ is also fixed-point-free, we have $\pi(h') \neq 1$, so $\sum_{h\in H} \pi(h)v = 0$.

We proceed by contradiction. Suppose that there is some noncyclic subgroup $H \subseteq G$ of order pq, where p and q are primes with $p \leq q$. Let $\{S_1, \dots, S_k\}$ be the proper non-trivial subgroups of H. It is not hard to see that $k \geq 2$ (the assumption of noncyclic H is used here in the case $p = q$). Each S_i is cyclic of order p or q. If $h \in H$ is non-trivial, then h belongs to exactly one of the S_i. For all $v \in V$, we have $\sum_{i=1}^k \sum_{s\in S_i} \pi(s)v = 0$, or equivalently $kv + \sum_{h\in H\setminus\{1\}} \pi(h)v = 0$, so $kv - v = 0$. Since $k \geq 2$, we have $v = 0$ for all $v \in V$, i.e. $V = \{0\}$. Hence G does not act freely on V, a contradiction.

The reverse direction is done by a case-by-case examination. □

Corollary 6.120. *If K is solvable group and satisfies all pq-conditions, then K acts orthogonally and freely on some sphere.*

In the following, if Y has fundamental group G and G' is a subgroup of G, we use the notation $Y(G')$ to mean the cover of Y corresponding to G'. Note that $Y = Y(G)$ and that $\pi_1(Y(G')) \cong G'$.

Theorem 6.121. *(Madsen-Thomas-Wall [422] Lemma 2.1) Let G be a P-group with cohomological dimension n. There is a multiple m of n along with a (G, m)-polarized $Y = Y(G)$ such that, for each subgroup $G' \leq G$ admitting a fixed-point free orthogonal representation, the covering space $Y(G')$ of Y corresponding to G' is homotopy equivalent to a manifold.*

Proof. Let N be the order of G and let n be its cohomological period. Let $g' \in H^{2n}(G; \mathbb{Z}) \cong \mathbb{Z}_N$ be a generator. By Swan-Wall's Theorem 6.118 there is a $(G, 2n)$-polarized complex Y_0 homotopy equivalent to a finite complex whose k-invariant is g'. So there is a natural free action of G on the universal cover $\widetilde{Y_0}$ and a homotopy equivalence $\widetilde{Y_0} \to \mathbb{S}^{2n-1}$.

Let G' be a subgroup of G with a fixed-point-free orthogonal representation, acting on some sphere $\mathbb{S}^{\ell-1}$ for some positive integer ℓ. Then G' acts with a fixed-point-free orthogonal representation on the $2n$-fold join $\mathbb{S}^{\ell-1} * \cdots * \mathbb{S}^{\ell-1} = \mathbb{S}^{2n\ell-1}$. Iterating this process successively with all subgroups of G with a fixed-point-free orthogonal representation, we can find an even multiple k of n such that any such subgroup acts in a fixed-point-free manner on $\mathbb{S}^{k-1}$. Now let $W_0 = \widetilde{Y_0} * \cdots * \widetilde{Y_0}$ be the $(k/2n)$-fold join of $\widetilde{Y_0}$. Then W_0 is homotopy equivalent to $\mathbb{S}^{k-1}$ and admits a cellular and free action by G.

If r is the exponent of $\mathbb{Z}_N^\times$ and $m = rk$, the join W of r copies of W_0 is homotopy equivalent to $\mathbb{S}^{rk-1} = \mathbb{S}^{m-1}$ and is equipped with a free action of G. Since G acts cellularly on W, the orbit space $Y = W/G$ is a finite complex. Note that this Y is a (G, m)-polarized complex.

The quotient space W_0/G is a (G, k)-polarization and therefore corresponds to an additive generator $g_0 \in H^k(G; \mathbb{Z})$ via the k-invariant. Let g be the r-fold cup product

$g = g_0^r = g_0 \cup \cdots \cup g_0$, which lies in $H^m(G;\mathbb{Z})$. Note by Lemma 6.115 that this group is also cyclic and isomorphic to $\mathbb{Z}_N$.

We want to show that g is the unique r-th power that generates $H^m(G;\mathbb{Z})$. Suppose that $\widetilde{g}$ is a generator of $H^m(G;\mathbb{Z})$ and $\widetilde{g} = h^r$ for some $h \in H^k(G;\mathbb{Z})$. Then h is a generator of $H^k(G;\mathbb{Z})$ and $h = ug_0$ for some $u \in \mathbb{Z}_N^\times$. Then $\widetilde{g} = h^r = (ug_0)^r = u^r g_0^r = g_0^r = g$. Hence there is a unique r-th power that generates $H^m(G;\mathbb{Z})$.

Suppose that G' is a subgroup of G of order M. Then we know that k is also a period for G', and in particular $H^k(G';\mathbb{Z})$ is cyclic. For all $j \in \mathbb{Z}_{\geq 1}$, let the homomorphism $\psi_j : H^j(G;\mathbb{Z}) \to H^j(G';\mathbb{Z})$ be induced by the inclusion map, which is an epimorphism $\psi_j : \mathbb{Z}_N \to \mathbb{Z}_M$ whenever j is a period of G. In addition, any unit of $\mathbb{Z}_M$ has order dividing r.

Note that $\psi_m(g) = \psi_m(g_0^r) = \psi_k(g_0)^r$ is a generator of $H^m(G';\mathbb{Z})$ which is an r-th power. We would like to show that it is the unique generator of $H^m(G';\mathbb{Z})$ which is an r-th power. Suppose that $\xi \in H^m(G';\mathbb{Z})$ is also a generator which is an r-th power; say $\xi = h^r$ for some $h \in H^k(G',\mathbb{Z}) \cong \mathbb{Z}_M$. Now h and $\psi_k(g_0)$ are both generators of $H^k(G';\mathbb{Z})$, so there is $v \in \mathbb{Z}_M^\times$ such that $h = v\psi_k(g_0)$. Then $\xi = h^r = v^r\psi_k(g_0)^r = \psi_m(g)$, so $\psi_m(g)$ is the unique element of $H^m(G';\mathbb{Z})$ satisfying these properties.

Now suppose in addition that G' has a fixed-point-free orthogonal action on $\mathbb{S}^{k-1}$ via a representation $\chi_{G'} : G' \to O(k-1)$. Clearly the smooth quotient manifold $\mathbb{S}^{k-1}/G'$ has a (G',k)-polarization, so it corresponds to a generator $g_{G'} \in H^k(G';\mathbb{Z})$. The direct sum $r\chi_{G'}$ of r copies of this representation corresponds to a fixed-point-free action of G' on the r-fold join $\mathbb{S}^{n-1} * \cdots * \mathbb{S}^{n-1} = \mathbb{S}^{rn-1} = \mathbb{S}^{m-1}$. The corresponding k-invariant is $g_{G'}^r$ in $H^m(G';\mathbb{Z})$, which is an r-th power generator. By uniqueness, we know that $g_{G'}^r = \psi_m(g)$, the latter of which corresponds to the cover $Y(G')$ of Y determined by G'. Two (G',m)-polarizations are homotopy equivalent iff their k-invariants are the same by Corollary 6.110, so $Y(G')$ is homotopy equivalent to $\mathbb{S}^{m-1}/G'$, which is a smooth manifold. □

6.6.5 Finding a normal invariant

Now that we have finite polarized complexes, we can try to perform surgery. Our first step is to find a normal invariant.

Theorem 6.122. *(Existence of normal invariants) Let Y be a Poincaré complex with fundamental group G. If every cover of Y corresponding to a Sylow subgroup of G has a normal invariant, then Y itself has a normal invariant.*

The proof uses the fact that $B(F/Cat)$ is an infinite loop space. In particular, we are interested in knowing whether the composite of the Spivak fibration $Y \to BF \to B(F/Cat)$ is nullhomotopic. The transfer $Y(G_p) \to Y$ gives at p a stable splitting $\Sigma^\infty Y_+ \to \Sigma^\infty Y(G_p)_+$ (see Adams [6] 100-103). Therefore we can detect triviality by checking that the composite $Y(G_p) \to Y \to B(F/Cat)_{(p)}$ is trivial for all p, but it is trivial by assumption.

Unfortunately this simple argument does not suffice to prove the main theorem. We want a normal invariant whose surgery obstruction vanishes, and the above argument is not enough. The correct approach is to find a normal invariant that agrees with the ones given by the manifold structure on the subgroups of G that act freely and orthogonally.

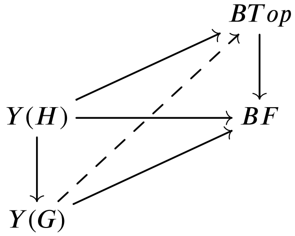
The next lemmas can be found in Madsen-Thomas-Wall [423] and involve a significant amount of case-by-case arguments about finite groups and their homology. We state them without proof.

Lemma 6.123. *If G is solvable, then G contains a subgroup H such that*

1. *H contains a Sylow 2-subgroup of G;*
2. *the only prime divisors of $|H|$ are 2 and 3;*
3. *the restriction homomorphism $H^2(G; \mathbb{Z}_2) \to H^2(H; \mathbb{Z}_2)$ is an isomorphism;*
4. *there is a fixed-point-free orthogonal action of H on a sphere.*

Since all the Sylow 2-subgroups of G are conjugate, we can select H to contain a prescribed Sylow 2-subgroup.

Lemma 6.124. *Let H be as above. Recall that $Y(H)$ denotes the lift of $Y = Y(G)$ corresponding to the subgroup H so that $\pi_1(Y(H)) \cong H$. Any topological normal invariant for $Y(H)$ extends to one for $Y(G)$; i.e. there is a dotted line making the following diagram commute:*



The non-solvable case is yet more technical and we merely quote the following lemma from Madsen-Thomas-Wall [422].

Lemma 6.125. *Suppose that G is not solvable. Let $G_{(2)}$ be a Sylow 2-subgroup of G and let $\mathcal{Q}$ be the collection of all quaternionic subgroups $Q_8 \subseteq G_{(2)} \cap T^*$ for some binary tetrahedral group T^*. Suppose that $Y(G_{(2)})$ admits a normal invariant $f : Y(G_{(2)}) \to BTop$ and denote by $f_{Y(Q_8)}$ the composition $Y(Q_8) \to Y(G_{(2)}) \to BTop$. Then C extends to a normal invariant for $Y(G)$ iff, for all $Q \in \mathcal{Q}$, the map $f_{Y(Q_8)}$ extends to a normal invariant for $Y(T^*)$.*

6.6.6 The surgery step

Let X be a Poincaré complex of dimension n with fundamental group G. Also let $\phi_G : M \to X$ be a degree one normal map between an n-manifold M and X with surgery obstruction $\sigma(\phi_G)$. If $H \leq G$ is a subgroup, then we denote by $\sigma(\phi_H)$ the surgery obstruction for the surgery problem of the lift $\phi_H : M(H) \to X(H)$. Clearly, if $\sigma(\phi_H) = 0$ and $H' \leq H$, then $\sigma(\phi_{H'}) = 0$.

The following lemma follows directly from the assembly map, using as always (1) that the domain is a homology theory and (2) that the L-groups here are 2-groups.

Lemma 6.126. *(Wall [679] and Madsen-Thomas-Wall [422] Lemma 4.3) Let G be a p-hyperelementary group with p odd. Then $G = G_1 \oplus H$ where G_1 has odd order and H is a cyclic 2-group. For n odd, we have $L_n(\mathbb{Z}[G]) \cong L_n(\mathbb{Z}[H])$.*

Lemma 6.127. *(Wall [679]) Let $\phi_G : M \to X$ be a degree one normal map between closed manifolds with finite fundamental group G. Surgery is possible on ϕ_G to produce a homotopy equivalence $\phi'_G : M' \to X$ iff surgery is possible for the covering map $\phi_{G_{(2)}} : M(G_{(2)}) \to X(G_{(2)})$, where $G_{(2)}$ is any Sylow 2-subgroup of G.*

The next theorem depends critically on Dress induction from Section 2.5. Dress asserts that $L^h_{2k+1}(\mathbb{Z}[G]) = \varprojlim L^h_{2k+1}(\mathbb{Z}[H])$, where the K range over all p-hyperelementary subgroups. Again, since $L^h_{2k+1}(\mathbb{Z}[H])$ only has 2-torsion, the surgery obstruction $\sigma(\phi)$ vanishes if $\sigma(r \circ \phi)$ vanishes for each restriction map $r : H \to G$ on a 2-hyperelementary subgroup.

Theorem 6.128. *(Madsen-Thomas-Wall [422] Theorem 4.1) Let M^n be a closed odd-dimensional manifold with $n \geq 5$ and let $\phi_G : M \to Y$ be a degree one normal map from M to a finite Poincaré complex Y with finite fundamental group G. Then surgery on ϕ_G to obtain a homotopy equivalence is possible iff*

1. *for each 2-hyperelementary subgroup $G' \leq G$, the covering space $Y(G')$ is homotopy equivalent to a manifold, and*
2. *surgery is possible for the covering normal map $\phi_{G_{(2)}} : M(G_{(2)}) \to Y(G_{(2)})$ to yield a homotopy equivalence, where $G_{(2)}$ is any Sylow 2-subgroup of G.*

Proof. Suppose that surgery on $\phi_G : M \to Y$ is possible to obtain a homotopy equivalence. Then (1) and (2) certainly hold.

Now assume both (1) and (2) hold. There is a single surgery obstruction $\sigma(\phi_G) \in L_n(\mathbb{Z}[G])$ to perform the surgery. By Dress, the natural restriction map $i : L_n(\mathbb{Z}[G]) \to \bigoplus_{G'} L_n(\mathbb{Z}[G'])$, where G' runs over all hyperelementary subgroups of G, is injective. Now $i(\sigma(\phi_G)) = \oplus \sigma(\phi_{G'})$, where $\sigma(\phi_{G'})$ is the surgery obstruction for the surgery problem $\phi_{G'} : M(G') \to X(G')$ for the covering map. Therefore it suffices to show that each $\sigma(\phi_{G'})$ is zero.

Suppose first that G' is p-hyperelementary, where p is an odd prime. By Lemma 6.126, we know that $G' \cong G'' \times H$, where G'' has odd order and H is a cyclic 2-group. Since n is odd, it follows that $L_n(\mathbb{Z}[G']) \cong L_n(\mathbb{Z}[H])$. Also $[G' : H]$ is odd and the torsion subgroup of $L_n(\mathbb{Z}[H])$ has exponent at most 2, and so $\sigma(\phi_{G'}) = \sigma(\phi_H)$. Let $G_{(2)}$ be a Sylow 2-subgroup of G that contains H. Then condition (2) gives $\sigma(\phi_{G_{(2)}}) = 0$. Therefore $\sigma(\phi_H) = 0$ by the remark preceding the theorem. Hence $\sigma(\phi_{G'}) = 0$, as required.

Now suppose that G' is 2-hyperelementary containing a Sylow 2-subgroup H'. By condition (1), the lift $Y(G')$ is homotopy equivalent to a manifold, so the normal map $\phi_{G'} : M(G') \to Y(G')$ can be considered a map of manifolds. By Lemma 6.127, it follows that $\sigma(\phi_{G'})$ vanishes iff $\sigma(\phi_{H'})$ vanishes. Let $G_{(2)}$ be a Sylow 2-subgroup of G that contains H'. Condition (2) indicates that $\sigma(\phi_{G'}) = 0$, and the remark above shows that $\sigma(\phi_{H'}) = 0$. Therefore $\sigma(\phi_{G'}) = 0$, as required.

Assembling this information, we see that $\sigma(\phi_{G'}) = 0$ for all hyperelementary subgroups $G' \leq G$. By Dress induction, we have $\sigma(\phi_G) = 0$, so surgery on ϕ_G can be done to produce a homotopy equivalence. □

Remark 6.129. *If the dimension of Y is even, then there is an additional requirement, that the equivariant signature of Y be a multiple of the regular representation of G.*

6.6.7 The final theorem

Theorem 6.130. *(Madsen-Thomas-Wall [422] Theorem 0.5) If G satisfies all p^2 and $2p$-conditions, then there is a free topological action of G on a sphere.*

Proof. By Theorem 6.121 we can pick a finite (G, n)-polarized complex $Y = Y(G)$ such that, for all subgroups $G' \leq G$ admitting a fixed-point-free orthogonal representation, the covering space $Y(G')$ of Y is homotopy equivalent to a manifold.

As a first case, suppose that G is solvable. Take H as in Theorem 6.123, so that, in addition to various other properties, it admits a free orthogonal action on a sphere. So therefore $Y(H)$ is homotopy equivalent to a manifold $Z(H)$ by the previous paragraph. This correspondence gives a normal invariant $c_H : Y(H) \to BTop$.

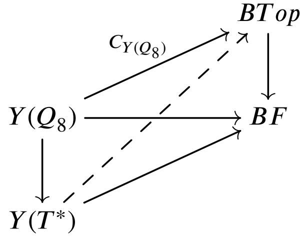
By Lemma 6.124, we know that c_H can be extended to a normal invariant $c_G : Y = Y(G) \to BTop$. Let $\phi_G : M \to Y$ be the associated degree one normal map from a closed manifold M to Y. Theorem 6.128 states that, in order to execute surgery on ϕ_G to form a homotopy equivalence, we have two conditions to show: (a) the covering space $Y(G')$ is homotopy equivalent to a manifold for each 2-hyperelementary subgroup $G' \leq G$; (b) surgery is possible for the covering map $\phi_{G_{(2)}} : M(G_{(2)}) \to Y(G_{(2)})$, where $G_{(2)}$ is the Sylow 2-subgroup of G.

Suppose first that $G' \leq G$ is a 2-hyperelementary subgroup. We know that G' is solvable. If a subgroup of G' has odd order, then it is cyclic because G' is 2-hyperelementary. Also, by our hypothesis on G, every subgroup of G' of order $2p$ is cyclic. Therefore G' satisfies all pq-conditions. Therefore G' has a fixed-point-free orthogonal action on some sphere by Wolf's Corollary 6.120. Hence $Y(G')$ is homotopy equivalent to a manifold again by the first sentence of this proof (Theorem 6.121).

Second, let $G_{(2)}$ be any Sylow 2-subgroup of G. Then $G_{(2)}$ is a subgroup of the H found in Theorem 6.121. As stated in the second paragraph of the proof, there is a homotopy equivalence $Y(H) \simeq Z(H)$, where $Z(H)$ is a manifold, so there is a homotopy equivalence $Y(G_{(2)}) \simeq Z(G_{(2)})$; i.e. surgery can be executed on $\phi_{G_{(2)}} : M(G_{(2)}) \to Y(G_{(2)})$ to yield a homotopy equivalence with a manifold.

So we have proved, in the solvable case, that surgery can be performed on $\phi : M \to Y$ to yield a homotopy equivalence between Y and some manifold M'.

In the second case, suppose that G is unsolvable. Note that $G_{(2)}$ is cyclic or generalized quaternionic, either of which has fixed-point-free orthogonal action χ on some sphere. Therefore the first paragraph of the proof of Theorem 6.121 implies that the covering space $Y(G_{(2)})$ is homotopy equivalent to some manifold $M(G_{(2)})$. This correspondence certainly gives a normal map $C_{Y(G_{(2)})} : M(G_{(2)}) \to Y(G_{(2)})$, which we would like to extend to $Y(G)$. By Lemma 6.125, it suffices to show that, for all quaternionic subgroups $Q_8 \subseteq G_{(2)} \cap T^*$ for some binary tetrahedral group T^*, the lift $C_{Y(Q_8)} : X(Q_8) \to Y(Q_8)$ extends to $Y(T^*)$:



Now Q_8 has a unique irreducible fixed-point free representation ψ, so the restriction $\chi|_{Q_8}$ of the representation χ of $G_{(2)}$ is then a multiple of ψ and therefore extends to a fixed-point free representation $\chi|_{T^*}$ (which is just a larger multiple of ψ) of T^*. By the choice of Y, the covering space $Y(T^*)$ is homotopy equivalent to a manifold, giving an extension $Y(T^*) \to BTop$ in the above diagram. Hence we have proved that there is a normal invariant of $Y = Y(G)$ extending the normal invariant of $Y(G_{(2)})$. Now we show that the associated degree one normal map $\rho : N \to Y$ from a closed manifold N to Y admits surgeries to form a homotopy equivalence using the two requirements stipulated in Theorem 6.128.

The first condition follows as in the solvable case: if $G' \leq G$ is 2-hyperelementary, then it satisfies all pq-conditions. By Corollary 6.120, there is a fixed-point-free orthogonal action admitted by G', so $Y(G')$ is homotopy equivalent to a manifold by Theorem 6.121. For the second condition, let $G_{(2)}$ be the Sylow 2-subgroup of G. We have noticed above that $Y(G_{(2)})$ is homotopy equivalent to a manifold. Therefore, surgery can be executed on $\phi_{G_{(2)}} : M(G_{(2)}) \to Y(G_{(2)})$ to yield a homotopy equivalence with a manifold. Therefore we have shown that Y is homotopy equivalent to a closed manifold.

We are now done, having shown that Y is homotopy equivalent to a manifold, say X, so

$\widetilde{Y}$ is homotopy equivalent to the manifold $\widetilde{X}$. But Y has a (G, n)-polarization, so $\widetilde{Y}$ is homotopy equivalent to $\mathbb{S}^{n-1}$. Since then $\widetilde{X}$ is homotopy equivalent to $\mathbb{S}^{n-1}$, it must be homeomorphic to $\mathbb{S}^{n-1}$ by the Poincaré conjecture. The action of $G = \pi_1(Y) \cong \pi_1(X)$ on $\widetilde{X} = \mathbb{S}^{n-1}$ gives a free action on a sphere. □

6.7 OOZING PROBLEM

Suppose that $n \geq 6$ and M^n is a closed Cat $(n-1)$-manifold with fundamental group G. Let $\sigma : \mathcal{N}^{Cat}(M \times I) \to L_n(\mathbb{Z}[G])$ be the surgery map for $M \times I$. Then Wall realization states that, for all $\gamma \in L_n(\mathbb{Z}[G])$ there is a degree one normal map

$$(F, B) \colon (W^n, \partial_0 W, \partial_1 W) \to (M \times I, M \times \{0\}, M \times \{1\})$$

such that $\sigma(F, B) = \gamma$. In other words, every element of $L_n(\mathbb{Z}[G])$ is the surgery obstruction of some degree one normal map from a Cat n-manifold W^n with boundary to another. We wish now to ask whether every element $L_n(\mathbb{Z}[G])$ is the surgery obstruction of some degree one normal map from a *closed* Cat n-manifold to another. This question is known as the *oozing problem*, a term coined by Cappell.[3] The reason for this nomenclature in the following paragraphs is intuitive and we hope to convey this intuition below. Let us state the problem more formally.

Definition 6.131. *Let G be a finitely presented group and $n \geq 5$. Denote by d the L-group decoration given by $p, h,$ or s. Let $C_n^d(G, w)$ be the subset of $L_n^d(\mathbb{Z}[G], w)$ consisting of those elements γ for which*

1. *there is a closed Cat n-manifold M with fundamental group G and orientation character w;*
2. *there is a degree one normal map $f : N^n \to M^n$ from a Cat n-manifold N to M;*
3. *the surgery map $\sigma : \mathcal{N}^{Cat}(M) \to L_n^d(\mathbb{Z}[G], w)$ satisfies $\sigma(f) = \gamma$.*

If w is trivial, we simply write $C_n^d(G)$.

The set $C_n^d(\mathbb{Z}[G], w)$ is clearly a subgroup of $L_n^d(\mathbb{Z}[G], w)$. We will refer to it as the *ooze subgroup of G*. This section is dedicated to the computation of $C_n^d(\mathbb{Z}[G], w)$. We begin with a few observations for $C_n^h(G, w)$.

In particular, suppose that X is an n-dimensional Poincaré complex with $\pi_1(X) = G$ endowed with a degree one normal map $g : M \to X$ from a closed Cat n-manifold M. If its surgery obstruction $\sigma(g)$ does *not* lie in $C_n^h(\mathbb{Z}[G])$, then X cannot be homotopy equivalent to a closed manifold.

[3] Cappell has said that it was coined by Morgan.

Proposition 6.132. *Suppose that M^n and N^n are normally cobordant via a cobordism W. Let there be a map $g : W \to M \times I$ restricting to homotopy equivalences $f : M \to M$ and $f' : N \to M$ on the boundaries. Let $G = \pi_1(M)$. If $[N, g]$ is the trivial element of $\mathcal{S}^{Cat}(M)$, then $\sigma(W) \in C_{n+1}(\mathbb{Z}[G])$.*

Proof. If $[N, f']$ is trivial in $\mathcal{S}^{Cat}(M)$, then N and M are homeomorphic. We can then modify the map g by glueing both the domain and range to itself along their boundaries to obtain a map $g' : W' \to M \times \mathbb{S}^1$. This construction is the basis of the ρ-invariant and related secondary invariants that were discussed in Section 6.5. By performing a surgery to eliminate the extra factor of $\mathbb{Z}$ in the fundamental group, we see that $\sigma(W)$ coincides with the surgery of a closed $(n+1)$-manifold in $L^h_{n+1}(\mathbb{Z}[G])$ and therefore lies in $C^h_{n+1}(\mathbb{Z}[G])$. □

Remark 6.133. *Therefore, if $C_*(\mathbb{Z}[G])$ is small, the structure sets tend to be large, and there is a greater possibility that invariants will obstruct a closed Poincaré complex from being a manifold.*

First we mention a relationship between ooze groups and the assembly map. For simplicity, we work with oriented manifolds.

Proposition 6.134. *Let G be a finitely generated group.*

1. *If M is closed with fundamental group G, then we have the diagram*

$$\begin{array}{ccccc} \mathcal{S}^{Top}(M) & \longrightarrow & H_n(M; \mathbb{L}_\bullet) & \longrightarrow & L_n(\mathbb{Z}[G]) \\ & & \downarrow & & \downarrow = \\ & & H_n(BG; \mathbb{L}_\bullet) & \xrightarrow[A]{} & L_n(\mathbb{Z}[G]) \end{array}$$

   Therefore, we have $C^h_n(G) \subseteq \operatorname{im}(H_n(BG; \mathbb{L}_\bullet) \xrightarrow{A} L^h_n(\mathbb{Z}[G]))$.

2. *Let $\widetilde{\sigma} : MSCat_n(BG \times F/Cat) \to L^h_n(\mathbb{Z}[G])$ be defined in the following way. Let $(M^n, f) \in MSCat_n(BG \times F/Cat)$. The projection $M \to F/Cat$ defines an oriented degree one normal map $g : N \to M$ from a Cat n-manifold N. The other projection $M \to BG$ induces a map $\pi_1(M) \to G$ on fundamental groups, which in turn induces a map $f_* : L^h_n(\mathbb{Z}[\pi_1(M)]) \to L^h_n(\mathbb{Z}[G])$ on L-groups. Define $\widetilde{\sigma}(M, f)$ to be $f_*(\sigma(N, g))$. Then*

$$C^h_n(G) = \operatorname{im}(MSCat_n(BG \times F/Cat) \xrightarrow{\widetilde{\sigma}} L^h_n(\mathbb{Z}[G])).$$

The result in (2) implies the following corollary and gives a formula for $C^h_n(G)$, while (1) gives just an upper bound. Of course, the study of $MSO_*(F/Cat)$ for Cat = PL or Top was used in the homotopy analysis of F/Cat.

We repeat results that we used in the previous section and in Chapter 4.

Corollary 6.135. *(Wall) Let G be a finite group and let $G_{(2)}$ be a Sylow 2-subgroup of G. Let $j_* : L_*(\mathbb{Z}[G_{(2)}]) \to L_*(\mathbb{Z}[G])$ be the map induced by inclusion, and let $j_* \otimes \mathbb{Q}$ be the rationalization of j_*. Then*

1. *$C_n^h(G)$ is the image of $C_n^h(G_{(2)})$ under j_n;*
2. *$C_n^h(G) \otimes \mathbb{Q}$ is the image of $C_n^h(e) \otimes \mathbb{Q}$ under $j_n \otimes \mathbb{Q}$.*

To explain the meaning of the word *ooze* in the oozing problem, we start with Sullivan's characteristic variety theorem.

Theorem 6.136. *(Characteristic Variety Theorem, Sullivan) Let M be a closed PL or Top manifold. Then M contains a characteristic variety $V = \{V_j\}$, i.e. a set of singular submanifolds of M (to be explained below) such that, if $f : N \to M$ is a degree one normal invariant, then the surgery obstruction $\sigma(f)$ is determined by the collection σ_j of simply connected surgery obstructions of $f|_{f^{-1}(V_j)}$ in $L_j(e; \mathbb{Z}_k)$.*

Definition 6.137. *For all $n \geq 0$, a $\mathbb{Z}_n$-manifold is a space V with singularities such that the singular set βV is a manifold V with a neighborhood identified with $\beta V \times C_n$, where C_n is the cone on n points. The rest of V is then a manifold with boundary $n\beta V$. A characteristic variety of a manifold M is the union of oriented $\mathbb{Z}_n$-submanifolds of $M \times \mathbb{R}^m$ for sufficiently large m.*

Example 6.138. *A $\mathbb{Z}_0$-manifold is a union of two oriented manifolds whose dimensions differ by one. A $\mathbb{Z}_1$-manifold is a manifold with boundary. A $\mathbb{Z}_2$-manifold is a manifold with a codimension one locally separating submanifold. It is harder to envision $\mathbb{Z}_n$-manifolds for larger n; they are much more exotic.*

A *pseudomanifold P* is a compact oriented polyhedron such that every simplex of codimension one is incident to two n-simplices with $P - P^{n-2}$ dense. One can define homology as the bordism of oriented pseudomanifolds. If one modifies the chains to be oriented $\mathbb{Z}_n$-pseudomanifolds, then we obtain homology with coefficients in $\mathbb{Z}_n$. The notation βV is related to the Bockstein cycle in the map $H_j(\,\cdot\,; \mathbb{Z}_n) \to H_{j-1}(\cdot\,; \mathbb{Z})$. Then $\mathbb{Z}_n$-manifolds define the cycles for homology with $\mathbb{Z}_n$-coefficients, and fit into the exact sequence

$$\cdots \to \Omega_j(\,\cdot\,; \mathbb{Z}_n) \to \Omega_{j-1}(\cdot) \xrightarrow{n} \Omega_{j-1}(\cdot) \to \Omega_{j-1}(\,\cdot\,; \mathbb{Z}_n) \to \cdots$$

where the middle arrow is multiplying by n, and the preceding one is given by the Bockstein $V \to \beta V$. With this information one can define L-groups for $\mathbb{Z}_n$-manifold pairs. In the particular case when V and βV are simply connected, we obtain groups $L_*(e; \mathbb{Z}_n)$ which hold simply connected surgery obstructions for $\mathbb{Z}_n$-manifolds in high dimensions. The groups can be computed using the universal coefficient theorem. For example, we have $L_j(e; \mathbb{Z}_2) = \mathbb{Z}_2$ for all $j \not\equiv 1 \bmod 4$.

Therefore, if M is a manifold, the characteristic variety theorem says that there is a family $\{V_j\}$ of subsets of M such that, to show that a PL or Top normal invariant on M

vanishes, one can merely prove that the simply connected surgery obstruction associated to each V_j vanishes in $L_j(e; \mathbb{Z}_n)$. The oozing problem describes how high-codimension simply connected surgery problems "ooze up" to determine the codimension 0 non-simply connected surgery obstructions.

When the group G is torsion-free, the ooze subgroup $C_n^d(G)$ is fairly large, at least conjecturally.

Theorem 6.139. *Suppose that G is a finitely generated torsion-free group that satisfies the Borel conjecture; i.e. the Eilenberg-Maclane space $K(G,1)$ has the homotopy type of a closed Top n-manifold M and is topologically rigid. Then the ooze group $C_k^h(G)$ is the entire L-group $L_k^h(\mathbb{Z}[G])$ for all $k > n$.*

Proof. We have the surgery exact sequence extending to the right:

$$\cdots \to \mathcal{S}_{n+1}^{Top}(M) \to \mathcal{N}_{n+1}^{Top}(M) \to L_{n+1}^h(\mathbb{Z}[G]) \to \mathcal{S}_n^{Top}(M) \to \mathcal{N}_n^{Top}(M) \to L_n^h(\mathbb{Z}[G]).$$

By assumption, the structure sets $\mathcal{S}_k^{Top}(M)$ are trivial for all $k \geq n$. Therefore, there is an isomorphism $\mathcal{N}_k^{Top}(M) \to L_k^h(\mathbb{Z}[G])$ for all $k \geq n+1$. Since $\mathcal{N}_k^{Top}(M)$ contains closed manifolds, then $C_k^h(G) = L_k^h(\mathbb{Z}[G])$ for all $k \geq n+1$. □

The validity of the Borel conjecture for G would imply that $L_n^d(\mathbb{Z}[G]) = \varinjlim C_{n+4k}^d(G)$, although unstably $C_n^d(G)$ and $L_n^d(\mathbb{Z}[G])$ are typically not equal. One can verify this equality when $G = \mathbb{Z}^n$.

Remark 6.140. *When $k \leq n$ we are not guaranteed that $C_k^h(G) = L_k^h(\mathbb{Z}[G])$. For example, consider $G = \mathbb{Z}^n$. Then*

$$L_k^h(\mathbb{Z}[\mathbb{Z}^n]) = \bigoplus_{j=0}^{n} \binom{n}{j} L_{k-j}^h(\mathbb{Z}[e]).$$

The summands corresponding to $j = 0, \ldots, k-1$ are represented by the simply connected surgery obstruction of the subtori of the torus $\mathbb{T}^n$. However, the summands corresponding to $j = k, \ldots, n$ contribute only trivially to $C_k^h(G)$. For example, when $k = n$, then the summand $L_0(\mathbb{Z}[e])$ does not contribute to the ooze subgroup, while the other summands do. These obstructions have oozed up from codimension 0 through $n-1$. In Chapter 8 we will see that this last $\mathbb{Z}$ arises from homology manifolds.

6.7.1 Finite groups

We have seen how, for finite groups G, the space $K(G,1)$ is infinite-dimensional and the L-groups can also be quite large, often relating to the representations of G. Surprisingly, the reality is that the ooze subgroups in the case of finite groups are actually quite small. We have already seen this phenomenon in our calculations of $\mathcal{S}^{Top}(M) \otimes \mathbb{Q}$ for

M with finite fundamental group. A rough intuition might be that, for finite groups, homology and L-theory behave very differently from each other. In general, it is difficult to find non-trivial homomorphisms between them. Since there are only few elements in $C_n^h(\mathbb{Z}[G])$, it follows that we need only a small number of invariants $\mathcal{N}^{Cat}(BG) \to L_n(\mathbb{Z}[G])$ to detect them.

We can now focus attention on just the prime 2. The next proposition follows from the fact that, for any CW complex X, the homology $H_n(\cdot\,; \mathbb{L}_\bullet)_{(2)}$ is a sum of ordinary homology groups:

$$H_n(X; \mathbb{L}_\bullet)_{(2)} \cong \bigoplus H_{n-4j}(X; \mathbb{Z}_{(2)}) \oplus \bigoplus H_{n-4j-2}(X; \mathbb{Z}_2).$$

By a straightforward transfer argument, we have the following.

Definition 6.141. *Let $f : N^n \to M^n$ be a degree one normal map between closed Cat n-manifolds. Let $k \in \mathbb{Z}_{\geq 0}$ such that $n - k \equiv 2 \bmod 4$. Let $K^{n-k} \subseteq M^n$ be a codimension k submanifold of M^n with fundamental group G. Let $f' : f_{\pitchfork}^{-1}(K) \to K$ be the restriction of f to the transverse inverse image of K. Then the simply connected surgery obstruction of f' is a higher Arf invariant of f of codimension k (associated to K). In other words, it is the image of the degree one normal map f' under the map*

$$\mathcal{N}^{Cat}(M) \to L_{n-k}^d(\mathbb{Z}[e]),$$

where d is the decoration p, h, or s.

We are interested in knowing when a high-codimension Arf invariant obstructs surgery, e.g. factors through $L_h^d(\mathbb{Z}[G])$.

Notation 6.142. *If $f : N^n \to M^n$ and k are given as above, we denote by $\mathrm{Arf}_k^d(f)$ the subset of all elements in $L_2^d(\mathbb{Z}[e])$ consisting of higher Arf invariants of f of codimension k.*

Theorem 6.143. *(Hambleton [282] and Taylor-Williams [637]) Let $f : N^n \to M^n$ be a map of degree one normal map of closed orientable n-dimensional Cat manifolds with finite $G = \pi_1(M)$.*

1. *When $n \equiv 0 \bmod 4$, then the surgery obstruction $\sigma(f)$ vanishes in the projective L-group $L_n^p(\mathbb{Z}[G])$ iff $\mathrm{sig}(M) = \mathrm{sig}(N)$ in $\mathbb{Z}$.*
2. *When $n \not\equiv 0 \bmod 4$, then the surgery obstruction $\sigma(f)$ vanishes in $L_n^p(\mathbb{Z}[G])$ iff $\mathrm{Arf}_k^p(f)$ is trivial in $L_2^p(\mathbb{Z}[e])$ for both $k = 0$ and $k = 1$.*

In general, the value of $\sigma(f)$ is determined by these collections of invariants.

The main tool used here is the Browder-Livesay-Wall obstruction theory for splitting homotopy equivalences of manifolds along codimension one submanifolds with non-trivial normal bundle, together with many interacting arguments that play group homology off Dress induction.

Remark 6.144. *One can ask how many submanifolds are needed to detect $\sigma(f)$. For each homomorphism of $G = \pi_1(M) \to \mathbb{Z}_2$, there is a submanifold associated to it. Moreover, one only needs a basis for $H^1(BG;\mathbb{Z}_2)$ to give a sufficient set of codimension one Arf invariants.*

Remark 6.145. *The projective ooze subgroup $C_n^p(G,w)$ is $\mathbb{Z}_2$, $\mathbb{Z}$, or $H_1(G;\mathbb{Z}_2)$, or trivial. In general the ooze subgroups are not 4-periodic, as was clear from our discussion of the torsion-free group case, but it is 4-periodic here.*

Remark 6.146. *It is an exercise to prove the injectivity of $S^p(M) \to S^p(BG)\times H_n(M;\mathbb{L}_\bullet)$ for finite G.*

The oozing problem is therefore fairly well understood in the projective case. Despite its value, relatively little in the literature is devoted to L^s-ooze. We therefore focus on the L^h case. See Wall [680], Pardon [498], Cappell-Shaneson [126], Taylor-Williams [636], Hambleton [282], and Milgram [441, 442] for more information.

We now give the simplest examples of elements in the kernel of $C_*^h \to C_*^p$. Let K^{4k+2} be the Kervaire manifold of dimension $4k+2$. For example, the manifold K^2 is the torus $\mathbb{T}^2$ with the Lie group framing. Let $\rho: K^{4k+2} \to \mathbb{S}^{4k+2}$ be a degree one normal map. The Kervaire manifold is used to construct examples of degree one normal invariants with nonzero surgery obstruction.

1. Wall's example $\mathbb{RP}^{4j+1}\times K^{4k+2} \to \mathbb{RP}^{4j+1}\times\mathbb{S}^{4k+2}$ whose surgery obstruction represents the non-trivial element in $L_3^h(\mathbb{Z}[\mathbb{Z}_2]) = L_3^p(\mathbb{Z}[\mathbb{Z}_2]) = \mathbb{Z}_2$;
2. the example of Morgan-Pardon, which has the form
$$\mathbb{RP}^{4k+1}\times L_4^{4j+1}\times K^{4k+2} \to \mathbb{RP}^{4i+1}\times L_4^{4j+1}\times\mathbb{S}^{4k+2}$$
whose surgery obstruction is the non-trivial torsion element in $L_0^h(\mathbb{Z}[G]) = \mathbb{Z}^6\oplus\mathbb{Z}_2$. Here $G = \mathbb{Z}_2\oplus\mathbb{Z}_4$ and L_4^{4j+1} is a Lens space of dimension $4j+1$ with fundamental group $\mathbb{Z}_4$;
3. the Cappell-Shaneson surgery problem, which has the form
$$(\mathbb{S}^3/Q_8)\times K^{4j+2} \to (\mathbb{S}^3/Q_8)\times\mathbb{S}^{4j+2},$$
whose surgery obstruction is non-trivial in $L_1^h(\mathbb{Z}[Q_8])$ and is detected on a codimension 3 submanifold.

It is not known whether codimension 4 arises; if not then nothing higher arises. If it does, then ooze can only come from codimensions that are powers of two, in ways that factor through the codimension 4 morphism.

Let us discuss examples (2) and (3). Morgan-Pardon identified non-trivial surgery obstructions among a family of manifold products. In this case the group $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ is fixed. We define a degree one normal map of closed manifolds given by

$$g = 1 \times \rho\colon\ \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{S}^2,$$

where $\rho\colon\ \mathbb{T}^2 \to \mathbb{S}^2$ is the usual Kervaire map. The L-groups of G are as follows.

Proposition 6.147. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. Then the L-groups $L^h_*(\mathbb{Z}[G])$ of G are given as follows:*

$$L^h_n(\mathbb{Z}[G]) = \begin{cases} \mathbb{Z}^6 \oplus \mathbb{Z}_2 & \text{if } n \equiv 0 \mod 4, \\ \mathbb{Z}_2 & \text{if } n \equiv 1 \mod 4, \\ \mathbb{Z}^2 \oplus \mathbb{Z}_2 & \text{if } n \equiv 2 \mod 4, \\ \mathbb{Z}_2^3 & \text{if } n \equiv 3 \mod 4. \end{cases}$$

Theorem 6.148. *(Morgan-Pardon) Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. The ooze subgroups $C^h_n(G)$, i.e. the elements in $L^h_n(\mathbb{Z}[G])$ which are detected by surgery problems on closed manifolds, are given by*

1. *$\mathbb{Z} \times \mathbb{Z}_2$ in $L^h_0(\mathbb{Z}[G])$, from the simply connected surgery obstructions;*
2. *$\mathbb{Z}_2$ in $L^h_2(\mathbb{Z}[G])$, the Morgan-Pardon simply connected Kervaire problem;*
3. *$\mathbb{Z}_2^2$ in $L^h_3(\mathbb{Z}[G])$.*

Morgan speculated that the simply connected surgery obstructions for codimension 1 and 2 submanifolds are sufficient for the determination of the ooze subgroup $C^h_n(G, w)$. This conjecture is true when G is abelian in both the oriented and nonoriented contexts.

Remark 6.149. *The proof relies on passing to the subgroup $\mathbb{Z}_4$ of Q_8 through the sequence $1 \to \mathbb{Z}_4 \to Q_8 \to \{\pm 1\} \to 1$. Also M is realized as the union $M \cong E \cup_{\mathbb{T}^2} (\mathbb{D}^2 \times \mathbb{S}^1)$, where E is an oriented unit interval bundle over the Klein bottle K, and using a Browder-Livesay-type analysis.*

6.7.2 Ooze invariants

When G is finite, the ooze for L^h is quite well understood. We need a little rephrasing to express the results. The submanifolds that contribute to the surgery obstruction calculation have an interpretation as characteristic classes in the normal invariant term, since $\mathcal{N}^{Top}(M) = [M : F/Top]$ is the direct sum of cohomology groups. We use the same notation σ_β as above.

Proposition 6.150. *Let G be a finite group and let $\rho\colon\ K^{4k+2} \to \mathbb{S}^{4k+2}$ be a degree one normal map from the Kervaire manifold to a sphere. The collection of all surgery obstructions of the form $\sigma_\beta(id \times \rho)$ in $L^h_3(\mathbb{Z}[G])$ is a group that is isomorphic to $H_1(BG; \mathbb{Z}_2)$.*

Theorem 6.151. *If G is a finite group, then the only surgery obstruction that comes from closed oriented manifolds in the reduced L-group $\widetilde{L}^p_n(\mathbb{Z}[G])$ is the image of the subgroup $H_1(BG;\mathbb{Z}_2) \to L^p_3(\mathbb{Z}[G])$.*

This part of the usual assembly map can be understood as follows. A non-trivial element of $H_1(BG;\mathbb{Z}_2)$ is represented by an element g of G. Consider g as a map $g : \mathbb{Z} \to G$. Then this element of $H_1(BG;\mathbb{Z}_2)$ goes to $g_*(1) \in L^p_3(\mathbb{Z}[G])$, where 1 is the non-trivial element of $L_3(\mathbb{Z}[\mathbb{Z}]) \cong \mathbb{Z}_2$.

Let G be a finite group. We recall that, for any L-group decoration $d = h, s, p$, the ooze subgroup $C^d_n(G)$ lies in the image of the assembly map $A^d : H_n(BG;\mathbb{L}_\bullet) \to L^d_n(\mathbb{Z}[G])$. We have the diagram below:

$$\begin{array}{ccc}
H_n(BG;\mathbb{L}_\bullet) & \xrightarrow{\quad A^d \quad} & L^d_n(\mathbb{Z}[G]) \otimes \mathbb{Z}_{(2)} \\
\downarrow & & \\
H_n(BG;\mathbb{L}_\bullet)_{(2)} & \nearrow{\scriptstyle (\oplus \iota^d_j, \oplus \kappa^d_j)} & \\
\cong \downarrow & & \\
\bigoplus H_{n-4j}(BG;\mathbb{Z}_{(2)}) \oplus \bigoplus H_{n-4j-2}(BG;\mathbb{Z}_2) & &
\end{array}$$

where $\iota^d_j : H_j(BG;\mathbb{Z}_{(2)}) \to L^d_j(\mathbb{Z}[G]) \otimes \mathbb{Z}_{(2)}$ and $\kappa^d_j : H_j(BG;\mathbb{Z}_2) \to L^d_{j+2}(\mathbb{Z}[G]) \otimes \mathbb{Z}_{(2)}$ are called the *oozing homomorphisms*.

Theorem 6.152. *We have the following results about ι^d_j and κ^d_j. Note that in (3) and (4) the homology group is twisted to indicate that α is represented by a nonorientable manifold. In this case we have the assembly $A^d : H^w_n(BG;\mathbb{L}_\bullet) \to L^d_n(\mathbb{Z}[G], w) \otimes \mathbb{Z}_{(2)}$.*

1. *Let G be a finite group. For all $\alpha \in H_n(BG;\mathbb{L}_\bullet)$, we have $\iota^h_j(\alpha) = 0$ for all $j \geq 1$.*
2. *Let G be a finite abelian group. For all $\alpha \in H_n(BG;\mathbb{L}_\bullet)$, the quantity $A^p(\alpha)$ vanishes iff $\kappa^p_1(\alpha) = 0$ (Hambleton, Taylor-Williams).*
3. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. There is $\alpha \in H^w_n(BG;\mathbb{L}_\bullet)$ such that $A^p(\alpha) \neq 0$ for which $\kappa^p_1(\alpha) = 0$ but $\kappa^p_2(\alpha) \neq 0$ (Morgan-Pardon).*
4. *Let G be a finite abelian group. For all $\alpha \in H_n(BG;\mathbb{L}_\bullet)$ and $\alpha \in H^w_n(BG;\mathbb{L}_\bullet)$ the quantity $A^p(\alpha)$ vanishes iff $\kappa^p_1(\alpha)$ and $\kappa^p_2(\alpha)$ both vanish.*
5. *If $G = Q_8$, there is $\alpha \in H_n(BG;\mathbb{L}_\bullet)$ with $A^h(\alpha) \neq 0$ for which $\kappa^h_1(\alpha)$ and $\kappa^h_2(\alpha)$ both vanish but $\kappa^h_3(\alpha) \neq 0$ (Cappell-Shaneson).*
6. *Let G be a finite group and let $\alpha \in H_n(BG;\mathbb{L}_\bullet)$. If $j \neq 2^r$ for all $r \geq 2$, then $\kappa^h_j(\alpha) = 0$. Moreover, if $\kappa^h_j(\alpha) = 0$ for all $j \leq 4$, then $\kappa^h_j(\alpha) = 0$ for all j (Hambleton-Milgram-Taylor-Williams [287]).*

7. *The following diagram commutes for* $r \geq 2$*:*

$$\begin{array}{ccc} H_{2^r}(G;\mathbb{Z}_2) & \longrightarrow & H_{2^{r-1}}(G;\mathbb{Z}_2) \\ & {\scriptstyle \kappa_{2^r}} \searrow \quad \swarrow {\scriptstyle \kappa_{2^r-1}} & \\ & L_2^h(\mathbb{Z}[G]) & \end{array}$$

where the top line is the dual to the Steenrod operation

$$\mathrm{Sq}^{2^{r-1}} : H^{2^{r-1}}(G;\mathbb{Z}_2) \to H^{2^r}(G;\mathbb{Z}_2)$$

given by cup square.

6.8 INTRODUCTION TO THE FARRELL-JONES CONJECTURE

For a torsion-free group π, the Borel conjecture might suggest that $H_n(B\pi;\mathbb{L}_\bullet(R)) \to L_n(R[\pi])$ is an isomorphism for any ring R with involution. A few comments are in order.

1. When R is itself a group ring, it is not difficult to modify our discussion of the assembly map to define this homomorphism. Using the algebraic theory of surgery, one can make a suitable definition for any ring.
2. Note that it is necessary to be careful about decorations even for $\pi = \mathbb{Z}_p$, e.g. for $R = \mathbb{Z}[\mathbb{Z}_p]$ for $p > 3$, so the speculation that this map is an isomorphism should only be made in the context of $L^{-\infty}$. When we discuss controlled L-theory in the next chapter, this idea might perhaps seem less *ad hoc*.
3. The group ring $R[\pi]$ does not need to be "untwisted," but in that case the $\mathbb{L}_\bullet(R)$ coefficients should be a coefficient system.

This route of generalizations is sometimes known as the *Borel package*. When the group π has torsion, however, these results are far from correct, and the richness of the theory of manifolds with finite fundamental group is a testament to this failure. Indeed, the calculation of L-groups of finite group rings is hard to summarize, and indeed involves a seemingly inseparable interaction between π and R.

One could perhaps optimistically hope that, for a general π, the theory might be the same as in the torsion-free case, except perturbed by the L-theory of finite subgroups. The extreme case of products of torsion-free with finite groups, or more generally groups containing a normal maximal torsion subgroup, can readily be handled in this way, as is indicated by (2) and (3) above.

One way to describe this idea would be to use proper discontinuous actions of the group π rather than free actions, i.e. imagining a contractible space $\underline{E}\pi$ on which π acts properly, where the fixed sets of all finite subgroups are contractible. In this case the left-hand side would become modified to $H_n(\underline{E\pi}/\pi; \mathbb{L}_\bullet(R[\pi_x]))$. Here we have a cosheaf of spectra on the quotient $\underline{E\pi}/\pi$ which, at a point downstairs corresponding to a point x upstairs, is isomorphic to $\mathbb{L}_\bullet(R[\pi_x])$.

We could then boldly conjecture that $H_n(\underline{E\pi}/\pi; \mathbb{L}_\bullet^{-\infty}(R[\pi_x])) \to L_n^{-\infty}(R[\pi])$ is an isomorphism for all π. For $R = \mathbb{Z}$ this statement would have some nice interpretation in terms of the rigidity of some non-manifold orbifolds, as will be apparent in the next chapter.

However, for $\pi = \mathbb{Z}_2 * \mathbb{Z}_2$ we have already seen Cappell's result that $L_3(\mathbb{Z}[\pi])$ is infinitely generated, while the left-hand side for the conjecture would be an interval with each endpoint marked by a conjugacy class of $\mathbb{Z}_2$. Here $\underline{E\pi}$ is simply R with each involution acting by flipping across a different fixed point. These generate a proper infinite dihedral action. The relevant homology group would be $L_*(\mathbb{Z}[e]) \oplus \widetilde{L}_*(\mathbb{Z}[\mathbb{Z}_2]) \oplus \widetilde{L}_*(\mathbb{Z}[\mathbb{Z}_2])$, which is indeed a summand of the correct answer, but it misses the UNil.

As we have seen, according to Cappell, the UNil group vanishes when $\frac{1}{2} \in R$. For these rings, the speculation above is, as far as we know, quite reasonable, and in fact is proven for many cases, e.g. Bartels-Lück [42], Bartels-Lück-Reich-Rüping [44], and Bartels-Bestvina [41], as a consequence of a much more subtle conjecture.

In the course of proving the Borel conjecture for hyperbolic manifolds M, Farrell and Jones [232] were led to consider variations on geodesic flow, and observed a special role for the geodesics in M. They interpreted the closed orbits of the flows as maximal copies of $\mathbb{Z}$ in the group $\pi_1(M)$. It led them to conjecture that, in general, one needs to consider all of the subgroups that contain a cyclic subgroup of finite index, instead of the groups that contain a trivial subgroup of finite index, and then to consider the universal space $E_{VC}\pi$ instead of $\underline{E\pi}$, where VC stands for "virtually cyclic." This $E_{VC}\pi$ is endowed with a simplicial action of π so that fixed sets of subgroups that contain cyclic subgroups of finite index are contractible, but all other fixed sets are empty. See Davis-Lück [191] for the relevant homology theory and assembly maps. This modification, together with an analogue in algebraic K-theory and non-trivial coefficient systems, is the content of the Farrell-Jones conjecture and verified in many important cases, as we stated above. It is a fantastic prediction about the K-theory and L-theory of group rings. We recommend the ICM surveys of Lück [409] and Bartels [40] for more recent reports. We hope that they write a book surveying these ideas in the near future.

Remark 6.153. *The condition that $\underline{E\pi}$ is contractible is, of course, redundant, since it is the fixed set of the trivial (finite) subgroup. Our discussion is here simplified by the analogue of imagining (the counterfactual) that all torsion-free groups act freely and properly on a simplicial complex. However, one can modify these false statements by appropriate limiting procedures, like defining homology as the limit of the homology of finite subcomplexes.*

6.9 PROPAGATION OF GROUP ACTIONS ON CLOSED MANIFOLDS

In Chapter 1 we began a study of homology propagation of group actions. For applications to group actions on the disk, those results were quite effective, since we were able to apply the π-π theorem. However, for many problems it is important to deal with closed manifolds. We shall largely focus on one typical case.

Definition 6.154. *Let $r \geq 2$ be an integer. We say that a Cat manifold Σ^m is an m-dimensional Cat $\mathbb{Z}_r$-homology sphere if $H_0(\Sigma; \mathbb{Z}_r) \cong \mathbb{Z}_r \cong H_m(\Sigma; \mathbb{Z}_r)$ and $H_i(\Sigma; \mathbb{Z}_r) = 0$ for all other i.*

A typical problem that homology propagation will solve is the following. If Σ is a simply connected $\mathbb{Z}_{|G|}$-homology sphere and G acts freely on the standard sphere (cf. Section 6.6), does G also act freely on Σ? We address this question in the following theorem. Recall that w_G is the Wall finiteness obstruction from Section 1.4 that takes its values in $\widetilde{K}_0(\mathbb{Z}[G])$.

Theorem 6.155. *(Cappell-Weinberger [131] and Davis-Löffler [190]) Suppose G is an odd-order group and acts freely and homologically trivially on M. Then one can propagate across a $\mathbb{Z}_{|G|}$-homology equivalence to a simply connected manifold N iff (1) the Swan condition and (2) the normal invariant condition both hold:*

1. *$w_G(M_f)$ is zero in $\widetilde{K}_0(\mathbb{Z}[G])$, and*
2. *the normal invariant $\nu_{(p)}(f)$ lies in the image of the map*

$$[N/G : F/Cat_{(|G|)}] \to [N : F/Cat_{(|G|)}].$$

We produce Poincaré spaces as before by mixing localized homotopy types and check the existence of normal invariants. We also need to know that the result of mixing is finite, so we must check the finiteness obstruction as well. For odd p-groups, Cappell-Weinberger gave a number-theoretic argument, but for all odd-order groups, Davis-Löffler [190] obtain a better direct Reidemeister torsion-theoretic argument, that these Poincaré complexes have simple duality, using Dress induction and the fact that the Whitehead groups of cyclic groups are torsion-free. As L^s-groups of odd-order groups vanish in odd dimensions, and in even dimensions are detected by the multisignature, it is easy to compute the surgery obstructions that arise. Indeed, the multisignature of the Poincaré complex is obvious, and for the domain of the surgery map, it is a multiple of the regular representation (e.g. using the ideas of the previous section). These results almost immediately imply the odd-order case of the following theorem of Davis-Weinberger [194]. We choose the degree of the map $\Sigma \to \mathbb{S}^n$ to make the Swan condition hold in Theorem 6.155.

Theorem 6.156. *If G is a finite group that acts freely and orientation preservingly on $\mathbb{S}^n$, and if Σ is a PL or Top simply connected $\mathbb{Z}_{|G|}$-homology n-sphere, the G acts freely on Σ as well.*

The last detail is provided by the following.

Proposition 6.157. *Let Cat = PL or Top and let $m \geq 4$. If G acts freely and homotopically trivially on a closed Cat manifold Σ' that is homotopy equivalent to a mod $|G|$ homology sphere Σ^m, then G acts in the same way on Σ^m.*

Proof. By the Lefschetz theorem, we only have to consider the case with m odd. It suffices to show that the homotopy equivalence $\Sigma \to \Sigma'$ in the structure set $\mathcal{S}^{Cat}(\Sigma')$ is in the image of the transfer map $\tau : \mathcal{S}^{Cat}(\Sigma'/G) \to \mathcal{S}^{Cat}(\Sigma')$. Consider the surgery exact sequence:

$$\begin{array}{ccccccc} L^h_{m+1}(\mathbb{Z}[e]) & \longrightarrow & \mathcal{S}^{Cat}(\Sigma') & \longrightarrow & [\Sigma' : F/Top] & \longrightarrow & L^h_m(\mathbb{Z}[e]) \\ \uparrow & & \uparrow \tau & & \uparrow & & \uparrow \\ L^h_{m+1}(\mathbb{Z}[G]) & \longrightarrow & \mathcal{S}^{Cat}(\Sigma'/G) & \longrightarrow & [\Sigma'/G : F/Top] & \longrightarrow & L^h_m(\mathbb{Z}[G]) \end{array}$$

Let $n = |G|$. To finish the proof we use three facts:

1. If m is odd, then the group $L^h_m(\mathbb{Z}[G])$ is 2-torsion for n even and is 0 for n odd (see Wall [679]).
2. The group $[\Sigma' : F/Top]$ is torsion prime to n, i.e. any finite-ordered element has order coprime to n, since
$$[\Sigma' : F/Top] \otimes \mathbb{Z}_{(n)} = [\mathbb{S}^m : F/Top] \otimes \mathbb{Z}_{(n)} = 0.$$
3. The map $[\Sigma'/\pi : F/Top] \otimes \mathbb{Z}[1/n] \to [\Sigma' : F/Top] \otimes \mathbb{Z}[1/n]$ is an isomorphism. Indeed, since F/Top is an infinite loop space, we can apply the Atiyah-Hirzebruch spectral sequence to prove it. □

Remark 6.158. *Here we need to use the fact that $[\Sigma'/\pi : F/Top] \to L^h_m(\mathbb{Z}[G])$ is a homomorphism.*

Remark 6.159. *Using information from the spaceform problem, one can finish the case $m \equiv 1 \bmod 4$ in the above theorem.*

Proof. Recall that, if G acts freely and homologically trivially on Σ^m or $\mathbb{S}^m$, then there is an isomorphism $H^{m+1}(BG;\mathbb{Z}) \cong \mathbb{Z}_{|G|}$. The classification of groups satisfying this condition (see Davis-Milgram [192]) shows that G is of the form $G = \mathbb{Z}_{2^s} \times H$, for some integer s and group H of odd order. By Wall [679], it can be shown that $L^h_1(\mathbb{Z}[G]) = 0$. The theorem follows from the previous results and the vanishing of all surgery obstructions. □

Remark 6.160. *There are counterexamples when the condition of homotopy triviality is removed. David Chase showed that there are simply connected mod 2 homology* *spheres without free involutions. We explain this phenomenon in Section 8.2.*

As an example for more general finite groups, we record the following theorem, which is an early example of the use of "numerical surgery obstructions" (see Section 2.4).

Theorem 6.161. *(Cappell-Weinberger [131]) Suppose that $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts freely and homologically trivially on M. In addition to the normal invariant condition, there is then one more condition in dimension 3 mod 4 that the alternating product of the homology torsion of the map is congruent to $\pm$ 1 mod 8. In this case, the Swan condition is automatically true, since $K_0(\mathbb{Z}_2 \times \mathbb{Z}_2) = 0$. Therefore there are generally more propagation obstructions.*

At this point we should say that the combination of numerical invariants of Davis-Weinberger [195] and results on the oozing problem give rise to extensions of the propagation theory and the above theorem to more general groups. See for example the survey of Weinberger [685] for more information.

Chapter Seven

Flat and almost flat manifolds

We have discussed the Borel conjecture on the topological rigidity of aspherical manifolds in a number of settings, such as in Section 3.6. In the case when the fundamental group of the manifold is $\mathbb{Z}^n$, the conjecture is affirmed by Shaneson's $G \times \mathbb{Z}$ formula and Bass-Heller-Swan's calculation that the Whitehead groups vanish. In other words, a Top homotopy torus is a Top torus in dimension at least 5.

There are many generalizations that can be made. We can consider manifolds whose fundamental group is poly-$\mathbb{Z}$. Bass-Heller-Swan's result can be generalized to twisted Laurent series by Farrell-Hsiang. The group $G \times \mathbb{Z}$ can be generalized to $G \rtimes \mathbb{Z}$. Lastly, the statement is true in lower dimensions but surgery methods do not suffice to show it.

The technology has advanced greatly since the late 1960s when those classical results were proved. We have chosen one key case representing the subsequent work; a comprehensive treatment would take another volume just as long as this one. This case shows the beginning of the use of controlled topology to prove topological rigidity theorems, which has indeed been key to all subsequent approaches.

The manifolds under consideration are flat or more generally almost flat. Flat manifolds have metrics whose Riemannian curvature is identically zero. By a theorem of Bieberbach, they are finitely covered by the torus, and there is a unique flat manifold up to diffeomorphism for each fundamental group. Almost flat manifolds have, in the sense of Gromov, metrics of arbitrarily small curvature whose diameter is less than 1. Gromov showed that almost flat manifolds are finitely covered by nilmanifolds.

The theorem that flat manifolds are Top rigid was first proved by Farrell and Hsiang and is a tour de force of technique. It synthesizes the ideas in the proof for tori and poly-$\mathbb{Z}$ groups with the space form problem, and additionally mixes in a new ingredient, the α-approximation theorem of Chapman and Ferry. The latter is a geometric criterion for a homotopy equivalence to be close to a PL homeomorphism in the C^0 sense. Roughly speaking, a homotopy equivalence that cannot be handled by a combination of induction on the "rank" of the group and on the size of the toral cover (or nilcover) is related to ones that are geometrically nicer, so that ultimately the Chapman-Ferry criterion holds.

The first section of the chapter discusses the α-approximation theorem. We do not give a proof; the original proof used Kirby's torus trick on handle smoothening, although there is another proof that avoids it, and instead uses Edwards's theorem on the recognition problem for topological manifolds. See the discussion of homology manifolds in Chapter 8.

In the second section, we explain why flat manifolds with odd-order holonomy are topologically rigid. The restriction on holonomy simplifies the group theory considerably. The more complicated discussion of almost flat manifolds in the last section requires a slight generalization of the α-approximation theorem.

7.1 THE α-APPROXIMATION THEOREM

The α-approximation theorem is an occasion for us to begin to discuss the ideas of controlled topology, which redoes classical topological problems keeping track of the "size" of constructions or solutions to problems. We give an example. The Poincaré conjecture states that any contractible manifold with spherical boundary is a ball. However, the following slightly stronger statement is correct:

Theorem 7.1. *Suppose that $f : M^n \to \mathbb{S}^n$ with $n \geq 5$ is a PL map that only has finitely many non-regular values. If the inverse image of every point is contractible, then F is the C^0-limit of Top homeomorphisms.*

If one knew the PL version of this statement in all dimensions, one could show that, if $f : M^n \to \mathbb{S}^n$ is an *arbitrary* PL map with contractible point-inverses, then F is a limit of PL homeomorphisms. Unfortunately, the PL category is too constrained for these purposes; there are seemingly insurmountable difficulties in low dimensions. It is generally much better to work in Top for these problems. Indeed, Top is filled with interesting maps that are approximable by homeomorphisms (ABH) for non-obvious reasons. Bing [55] developed an amazing technology for identifying ABH maps; we refer to Daverman [185] for an explanation of Bing's ideas, in addition to a high point in this theory: the aforementioned theorem of Edwards that gives a topological characterization of manifolds. We will discuss this theorem more in Section 8.6.

In the Top category, however, there is the difficulty that point-inverses of maps between manifolds can be pathological. Indeed, one can produce an ABH map from $\mathbb{S}^n$ to itself such that the inverse image of every point is a point, with one exceptional fiber given by the topologist's sine curve, which is not even path-connected, much less contractible. Additionally, some maps may not even have regular values, so point-inverses may not be the best approach.

However, the topologist's sine curve, although not contractible, is null-homotopic in any neighborhood of itself. This observation suggests that definitions in Top should perhaps be made in terms of inverse images of open sets, and not of point-inverses.

Before stating the first result of this section, we need some classical definitions.

Definition 7.2. *An absolute neighborhood retract (ANR) is a topological space X with* *the property that, whenever $i : X \to Y$ is an embedding into a normal topological space Y, there exists a neighborhood U of $i(X)$ in Y and a retraction of U onto $i(X)$.*

These spaces have very good properties. While they are more general than polyhedra, a good case can be made to regard them as topological analogues of polyhedra. A theorem of West [699] asserts that finite-dimensional ANRs are homotopy equivalent to finite complexes, and indeed have a canonical simple homotopy type. Indeed, every finite-dimensional ANR is a Hilbert cube manifold after crossing with the Hilbert cube $[0, 1]^\infty$, which is homeomorphic to a polyhedron crossed with the Hilbert cube, unique up to simple homotopy type (see Chapman [157]). Moreover, the controlled versions of these statements are also correct. Needless to say, ANRs provide a suitable setting for doing homotopy theory; they have good homotopy extension principles and a theory of cofibrations.

Definition 7.3. *The following definitions are standard.*

1. *A compact metric space X is cell-like if it can be embedded in an ANR Z in which, for each neighborhood U of X, there is a neighborhood V of X contained in U such that V contracts to a point in U; i.e. the inclusion map $V \hookrightarrow U$ is homotopic to a constant map. We can also say that X is cell-like in Z. This notion is also called property UV^∞.*
2. *Let X and Y be topological spaces. We say that the map $f : X \to Y$ is proper if $f^{-1}(K)$ is compact for all compact subsets $K \subseteq Y$.*
3. *Let X and Y be topological spaces. A proper map $f : X \to Y$ is cell-like or CE if $f^{-1}(y)$ is cell-like for all $y \in Y$.*
4. *A proper surjection $p : K \to L$ is a UV^1-map if, for all $\varepsilon > 0$ and map $\alpha : P^2 \to L$ of a two-complex P into L with a lift $\alpha_0 : P_0 \to K$ defined on a subcomplex P_0, there is a map $\widetilde{\alpha} : P \to K$ with $\widetilde{\alpha}|_{P_0} = \alpha_0$ and $d(p\circ\widetilde{\alpha}, \alpha) < \varepsilon$.*

We can now state the following theorem of Siebenmann, which is a genuine Top version of Theorem 7.1.

Theorem 7.4. *(Siebenmann) Suppose that $n \geq 5$. A map $f : M^n \to N^n$ between topological manifolds is ABH iff it is CE.*

Remark 7.5. *Siebenmann was motivated by an observation of Sullivan. We have mentioned that, if $f : M \to N^n$ is a homeomorphism between smooth manifolds, then $p_i(M) - f^* p_i(N)$ is a torsion cohomology class. Sullivan, on examining the proof, observed that f did not need to be a homeomorphism for the argument to work. Instead, it was only required that f be a hereditary homotopy equivalence, i.e. that $f|_{f^{-1}(\mathcal{O})}$ be a (proper) homotopy equivalence for every open subset $\mathcal{O}$ of N.*

Lacher [380] shows that such proper equivalences are precisely the CE maps for finite-dimensional ANR neighborhood retracts. Therefore, under these conditions, if M is a closed, simply connected manifold of dimension at least 5, then surgery shows that h is homotopic to a PL homeomorphism.

Theorem 7.6. *(Lacher [380]) Let $f: X \to Y$ be a map of finite-dimensional ANRs. If f is CE, then f is a proper homotopy equivalence when restricted to every open set of Y. Conversely, if a proper map $f: X \to Y$ is a hereditary homotopy equivalence, then f is CE.*

Remark 7.7. *Sullivan's observation sadly gives no new information beyond the content of the philosophy of Novikov's theorem. However, it perhaps should be viewed as a cornerstone of the philosophy that for Top one should work with conditions defined by open sets, as we stated earlier.* [1]

Remark 7.8. *Another example is the idea of an approximate fibration, which is actually more suitable for topological neighborhood theory than the PL idea of a block bundle. See for example Chapman [159], Hughes [326], and Hughes-Taylor-Weinberger-Williams [328].*

Having characterized the limits of PL homeomorphisms, we can reasonably ask whether one can measure the "closeness" of a map to a homeomorphism, i.e. whether one can know if a given map f is metrically within some given ε of a homeomorphism.

We describe the answer for N compact, for which we can use ε-δ terminology.

Definition 7.9. *Let $f: M^n \to N^n$ be a map. We say f is an ε-equivalence if there is a map $g: N^n \to M^n$ so that $f \circ g$ and $g \circ f$ are homotopic to the identity, with the added condition that the homotopies $H: N^n \times [0,1] \to N^n$ and $f \circ H: M^n \times [0,1] \to M^n \to N^n$ have tracks, i.e. images of $\{p\} \times [0,1]$, of diameter $< \varepsilon$.*

Theorem 7.10. *(α-approximation theorem, Chapman-Ferry [161]) Let N be a compact manifold of dimension at least 5. For all $\varepsilon > 0$, there is $\delta > 0$ such that, if $f: M \to N$ is a δ-equivalence, then f is within ε of a homeomorphism in the uniform metric.*

Siebenmann's theorem readily follows from the α-approximation theorem. The proof given in Chapman-Ferry [161] uses Kirby's torus trick, as did Siebenmann in the proof of his theorem.

Remark 7.11. *The theorem is called the α-approximation theorem rather than the ε-approximation theorem because Chapman and Ferry chose to allow N to be noncompact. To express "closeness in the compact open topology" one needs to allow arbitrary covers, so rather than using ε, their version has an open cover α. However, there is no additional conceptual difficulty in this fancier version. Indeed, for many noncompact manifolds such as a simply connected Hadamard manifold G/K, there is no additional difficulty to give an ε-δ version.*

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[1] At the very least, it is a beautiful metric theorem and it was historically (to view it within an American context) a 20th century step towards reconciling America's south (Bing topology, centered in Texas) with the north (surgery theory, centered at Princeton).

We mention a beautiful consequence of α-approximation.

Theorem 7.12. *(Ferry [235]) Let M^n be a compact manifold with a metric. Then there is $\varepsilon > 0$ so that, if $f : M \to N^n$ is a continuous map onto another connected manifold with* $\mathrm{diam}(f^{-1}(n)) < \varepsilon$ *for all $n \in N$, then f is homotopic to a homeomorphism.*

The paper of Ferry-Weinberger [245] verifies that the work of Freedman-Quinn [254], which implies that all the theorems of Siebenmann, Chapman-Ferry, and Ferry hold in dimension 4. Now that the three-dimensional Poincaré conjecture has been proved, these theorems are known to be true in all dimensions.

Remark 7.13. *Quinn [520] observed that the manifold N in the α-approximation theorem actually plays two different and separate roles. It is the manifold that we are trying to modify to be homeomorphic to M, and it is the space in which we measure distances. Given a map $\phi : N^n \to X$, one can ask the conditions under which a map $f : M^n \to N^n$ is ε-homotopic over X to a homeomorphism, so that one can connect metric topology with K-theory and L-theory. Indeed, when $X = N^n$, it is the setting for the metric topology, and when X is a point, it is the setting of h-cobordism theorems and surgery theorems. When ϕ is UV^1, as the identity map, we will see in Chapter 8 that the relevant surgery obstruction groups are in fact $H_n(X; \mathbb{L}_\bullet)$.*

This perspective considers α-approximation as a kind of metric Poincaré conjecture, in which case ϕ is CE. Suppose instead of having the homotopy type of a point, we can arrange a theorem to take advantage of the fact that homotopy Top tori are Top tori. The result is as follows.

Theorem 7.14. *Suppose that $\phi : N^n \to X$ is a torus bundle. Then for every $\varepsilon > 0$, there is $\delta > 0$ so that, if $f : M^n \to N^n$ is a δ-homotopy equivalence over X, then f is ε-homotopic to an ε-homeomorphism over X.*

The proof involves no essentially new ideas beyond those of the original α-approximation theorem.

7.2 FLAT MANIFOLDS

This section is devoted to explaining the proof that some flat manifolds, specifically those with odd-order holomony, are topologically rigid. According to Bieberbach, such manifolds are finitely covered by the torus.

7.2.1 Farrell-Hsiang's rigidity for some flat manifolds

Let $E(n)$ be the isometry group for $\mathbb{R}^n$, i.e. $E(n) \cong O(n) \rtimes \mathbb{R}^n$. If Γ is a torsion-free uniform discrete subgroup of $E(n)$, then its image under the projection $E(n) \to O(n)$ is called the *holonomy group* of Γ.

Let M^n be a closed flat Riemannian manifold; i.e. all of its sectional curvatures are zero. The universal cover of such a manifold is isometrically equivalent to Euclidean space since the exponential map is an isometry. In particular, the fundamental group $\pi_1(M)$ is isomorphic to a uniform discrete subgroup of $E(n)$. The *holonomy* of M is the holonomy of $\pi_1(M)$; i.e. it measures the amount of $\pi_1(M)$ that rotates the universal cover.

Theorem 7.15. *(Farrell-Hsiang [224]) Let $n \geq 5$ and let M^n be a closed flat Riemannian manifold. If the holonomy group of M^n has odd order, then M is topologically rigid.*

This theorem has the following equivalent formulation.

Theorem 7.16. *Let $n \geq 5$ and let M^n be a closed connected Top manifold. Then M has a flat Riemannian connection with odd-order holonomy group iff M^n is aspherical and $\pi_1(M)$ contains an abelian subgroup with odd finite index.*

Remark 7.17. *Of course, later work removes the condition on holonomy and on dimension.*

Definition 7.18. *A Bieberbach group is the fundamental group of a flat Riemannian manifold. An abstract group Γ is a Bieberbach group if*

1. *Γ is finitely generated torsion-free;*
2. *Γ has a normal abelian subgroup $A \cong \mathbb{Z}^n$ for some $n \geq 1$ such that Γ/A is finite, i.e. Γ is virtually abelian.*

In this case, there is a faithful action of quotient $G = \Gamma/A$ on A given by $(\Gamma A, a) \mapsto \Gamma a \Gamma^{-1}$. The quotient $G = \Gamma/A$ is called the holonomy group of Γ. The Bieberbach rank of Γ is given by the rank of A as an abelian group. The map $T : G \to \mathrm{Aut}(A)$ describing this action is called the holonomy representation of Γ. Oftentimes we will denote a Bieberbach group by the data (Γ, A) or (Γ, A, T).

Notation 7.19. *For any $s \in \mathbb{Z}_{\geq 1}$, let $\Gamma_s = \Gamma/sA$ and $A_s = A/sA$. Then there is a sequence $0 \to A_s \to \Gamma_s \to G \to 0$ and the sequence splits if $(s, |G|) = 1$. In this case we have a semidirect product $\Gamma_s = A_s \rtimes G$.*

We now explore some of the group theory of Bieberbach groups.

Recall that a group G is *hyperelementary* if there is a prime p and an integer n coprime to p such that G has an extension $1 \to \mathbb{Z}_n \to G \to P \to 1$ for some p-group P.

Theorem 7.20. *Suppose that Γ is a Bieberbach group of Bieberbach rank n and holonomy group $G = \Gamma/A$. If $|G|$ is odd, then one of these two conditions holds:*

(a) Γ *is the semidirect product $\Gamma = \Gamma' \rtimes \mathbb{Z}$, for some subgroup Γ' with Bieberbach rank $n-1$. Here Γ' is also Bieberbach.*

(b) there is an infinite sequence of positive integers $s \equiv 1 \bmod |G|$ such that, for any hyperelementary subgroup E_s of Γ_s, if the natural projection map $E_s \hookrightarrow \Gamma_s \to \Gamma_s/A_s = G$ is surjective, then it is bijective.

Remark 7.21. *In the case when $|G|$ is even, one other possibility may arise. It is possible that Γ is the amalgamated free product $\Gamma = B *_D C$ of subgroups B and C of rank $n-1$, where D has index 2 in both B and C.*

Definition 7.22. *Let (Γ, A, T) be a Bieberbach group and let s be a positive integer. We say that a monomorphism $f : \Gamma \to \Gamma$ is s-expansive if*

1. *the restriction of f to A is the map $f|_A : A \to A$ given by multiplication by s;*
2. *if $\eta : \Gamma \to \Gamma/A$ is the projection map, then $\eta \circ f : \Gamma \to \Gamma/A$ has kernel A; i.e. f induces the identity on $G = \Gamma/A$.*

Definition 7.23. *Let M be an n-dimensional Riemannian manifold with Riemannian metric norm $|\cdot|$. We say that a map $g : M \to M$ is an expanding endomorphism if there is $s > 1$ such that $|dg(X)| = s|X|$ for all tangent vectors X of M.*

Notation 7.24. *If $f : \Gamma \to \Gamma$ is s-expansive, then we may notate it by f_s. We will also notate the canonical projection maps by $p_s : \Gamma \to \Gamma_s$ and $q_s : \Gamma_s \to G$.*

Theorem 7.25. *(Epstein-Shub [216]) Let Γ be a Bieberbach group. For any positive integer $s \equiv 1 \bmod |G|$, there is an s-expansive endomorphism of Γ. In addition, if M^n is a closed flat Riemannian manifold with $\pi_1(M^n) = \Gamma$ and $f : \Gamma \to \Gamma$ is s-expansive, then there is an expanding endomorphism $g : M^n \to M^n$ such that $g_* = f$. In other words, the cover corresponding to $f(\Gamma)$ is homothetic to M.*

Suppose that M^n is a closed Riemannian manifold which is a smooth deformation retract of a compact manifold W^{n+k} with boundary. Let $h : W \times [0,1] \to W$ be the homotopy and define $h_t : W \to W$ by $h_t(x) = h(x,t)$ for all $x \in W$ and $t \in [0,1]$. Let $h_0 = id_W$ and h_1 be the retraction $W \to M$. For each $x \in W$, let $\alpha_x : [0,1] \to M$ be the path given by $\alpha_x(t) = h_1(h_t(x))$ for all $t \in [0,1]$. In other words, the function α_x is the projection of the x-trace of the homotopy onto M. Let $||\alpha_x||$ be the arclength of α_x with respect to the Riemannian metric on M^n, and let $m(h) = \max_{x\in W} ||\alpha_x||$.

The following theorem of Ferry gives a metric condition for which a given homotopy equivalence is actually a simple homotopy equivalence. This theorem is a predecessor of the α-approximation theorem. We use the notation from the discussion above.

Theorem 7.26. *(Ferry [234]) There is $\varepsilon > 0$ depending only on M^n such that, if $m(h) < \varepsilon$, then the retraction h_1 is simple; i.e. the Whitehead torsion $\tau(h_1)$ vanishes.*

If $f : \Gamma \to \Gamma$ is a monomorphism, then there are restriction maps $f_* : \mathrm{Wh}(\Gamma) \to \mathrm{Wh}(\Gamma)$ and $f_* : \widetilde{K}_0(\mathbb{Z}[\Gamma]) \to \widetilde{K}_0(\mathbb{Z}[\Gamma])$. There are the two following vanishing theorems.

Theorem 7.27. *Let Γ be a Bieberbach group. For each $b \in \mathrm{Wh}(\Gamma)$, there is an integer N_b such that, if $s > N_b$ and $f : \Gamma \to \Gamma$ is s-expansive, then $f_*(b) = 0$ in $\mathrm{Wh}(\Gamma)$.*

Proof. Let M^n be a closed flat Riemannian manifold with fundamental group Γ. Embed M^n in a higher-dimensional compact smooth manifold W with a deformation retraction $h : W \times [0,1] \to W$ for which b is the Whitehead torsion $\tau(h_1)$ of the retraction $h_1 : W \to W$. This construction is possible by the s-cobordism theory in the Diff category. Let $\varepsilon > 0$ be the constant guaranteed by the previous theorem and let $N_b = m(h)/\varepsilon$. Assume that $s > N_b$ and suppose that $f : \Gamma \to \Gamma$ is s-expansive. By Theorem 7.25 there is an expanding endomorphism $g : M \to M$ with $g_* = f$. By definition the map g is a covering projection that induces a map $\overline{g} : \overline{W} \to W$ such that $M \subseteq \overline{W}$ and $\overline{g}|_M = g$:

$$\begin{array}{ccc} \overline{W} & \xrightarrow{\overline{h}_1} & M \\ {\scriptstyle \overline{g}}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow[h_1]{} & M \end{array}$$

Now h_t lifts to a deformation retraction $\overline{h}_t$ of $\overline{W}$ onto M and $f_*(b) = \tau(\overline{h}_1)$. Let $x \in W$ and $y \in \overline{W}$. Define $\alpha_x, \overline{\alpha}_y : [0,1] \to M$ by $\alpha_x(t) = h_1(h_t(x))$ and $\overline{\alpha}_y(t) = \overline{h}_1(\overline{h}_t(y))$ for all $t \in [0,1]$. Then $g(\overline{\alpha}_y(t)) = \alpha_{\overline{g}(y)}(t)$ for all $t \in [0,1]$. The expansiveness of g gives $||\overline{\alpha}_y|| = \frac{1}{s}\,||\alpha_{\overline{g}(y)}|| \leq \frac{1}{s}\,m(h) < \varepsilon$. Therefore $m(\overline{h}) < \varepsilon$ and hence $\tau(\overline{h}_1)$ vanishes. Then $f_*(b) = 0$, as desired. □

By embedding $\widetilde{K}_0(\mathbb{Z}[\Gamma])$ into $\mathrm{Wh}(\Gamma \times \mathbb{Z})$, we can deduce the K-theoretic analogue of the above.

Theorem 7.28. *Let Γ be a Bieberbach group. For each $b \in \widetilde{K}_0(\mathbb{Z}[\Gamma])$, there is an integer N_b such that, if $s > N_b$ and $f : \Gamma \to \Gamma$ is s-expansive, then $f_*(b) = 0$ in $\widetilde{K}_0(\mathbb{Z}[\Gamma])$.*

Theorem 7.29. *For any Bieberbach group Γ, we have $\mathrm{Wh}(\Gamma) = 0$ and $\widetilde{K}_0(\mathbb{Z}[\Gamma]) = 0$.*

Proof. Proceed by induction on the Bieberbach rank of Γ and then on the order of G; i.e. assume that $\mathrm{Wh}(H) = 0$ and $\widetilde{K}_0(\mathbb{Z}H) = 0$ for all Bieberbach groups H where either H has Bieberbach rank less than n or its holonomy group has order less than $|G|$. We use the characterization of Bieberbach groups from Theorem 7.20.

We separate into cases. First, if $\Gamma = \Gamma' \times_\alpha \mathbb{Z}$, then, since $\mathbb{Z}[\Gamma]$ is right regular, we have the exact sequence $\mathrm{Wh}(\Gamma') \to \mathrm{Wh}(\Gamma) \to \widetilde{K}_0(\mathbb{Z}[\Gamma'])$. By the induction hypothesis, we have $\mathrm{Wh}(\Gamma) = 0$. Furthermore, there is an epimorphism $\widetilde{K}_0(\mathbb{Z}[\Gamma']) \to \widetilde{K}_0(\mathbb{Z}[\Gamma])$ by Farrell-Hsiang [221], so by induction we have $\widetilde{K}_0(\mathbb{Z}[\Gamma]) = 0$.

In the second case, let $b \in \mathrm{Wh}(\Gamma)$ be arbitrary and let N_b be the integer guaranteed by Theorem 7.27. Let s be an integer guaranteed by Theorem 7.20 with the added condition that $s > N_b$. If S is a subgroup of Γ_s, then $\mathrm{Wh}(p_s^{-1}S)$ is a Frobenius module over Swan's Frobenius functor $G_0(S)$. Therefore it suffices to prove that b vanishes under the transfer maps to the hyperelementary subgroups E of Γ_s. By Dress induction it also suffices to consider the case in which G is hyperelementary.

If E projects to a proper subgroup of G, then the holonomy group of $p_s^{-1}(E)$ is precisely $q_s(E)$, so has order less than $|G|$. Therefore $\mathrm{Wh}(p_s^{-1}(E)) = 0$ by the induction hypothesis.

Otherwise E projects isomorphically onto G, but all such subgroups of Γ_s are conjugate since $(|G|, s) = 1$ and therefore $H^1(G; A_s) = 0$. Hence it suffices to consider one of them. Choose an s-expansive map $f_s : \Gamma \to \Gamma$ and let $E = p_s \circ f_s(\Gamma)$, where $f_s : \Gamma \to \Gamma$ is s-expansive. Theorem 7.25 guarantees the existence of such f_s. Recall that $\Gamma_s \cong A_s \rtimes G$ in this case. Therefore $E \cong G$ and it is hyperelementary. But we know that $f_{s*}(b) = 0$ by Theorem 7.27, so b vanishes under all appropriate transfer maps. Therefore $b = 0$, as required. The proof that $\widetilde{K}_0(\mathbb{Z}[\Gamma]) = 0$ is done similarly. □

Let M^n be a closed flat Riemannian manifold with fundamental group Γ and holonomy group G. Since the Whitehead group of Γ vanishes, there is no difference between the structure set of M and the simple structure set of M. We recall that the structure set $\mathcal{S}^{Top}(M^n \times I^k)$ is an abelian group when $n+k \geq 5$. Our goal is to show that this group is trivial. Recall that the notation suggests that the structure sets are given relative to the boundary.

Lemma 7.30. *Suppose that M^n is a closed flat Riemannian manifold. If $n + k \geq 5$ and $k \geq 1$, then $\mathcal{S}^{Top}(M^n \times I^k) \cong \mathcal{S}^{Top}(M^n \times I^{k+4})$. If in addition $n \geq 5$ and $\mathcal{S}^{Top}(M^n \times I^4) = 0$, then $\mathcal{S}^{Top}(M^n) = 0$.*

Proof. This result is of course part of the periodicity explained in Chapter 5. □

Lemma 7.31. *Let M^n be a closed flat Riemannian manifold with fundamental group Γ of Bieberbach rank m. Suppose that Γ is a semidirect product $\Gamma \cong \Gamma' \rtimes \mathbb{Z}$, where Γ' has Bieberbach rank $m-1$ (i.e. Γ is of Type (a) in Theorem 7.20). If $n + k \geq 6$, then there is a closed flat Riemannian manifold N^{n-1} of dimension $n-1$ together with an exact sequence*

$$\mathcal{S}^{Top}(N^{n-1} \times I^{k+1}) \to \mathcal{S}^{Top}(M^n \times I^k) \to \mathcal{S}^{Top}(N^{n-1} \times I^k).$$

Proof. By Calabi [108] there is a flat Riemannian manifold N^{n-1} such that M fibers over a circle with fiber N^{n-1}. □

Definition 7.32. *Suppose that M^n is a closed flat Riemannian manifold with fundamental group Γ. Let Λ be a subgroup of Γ of finite index and let $p_\Lambda : M_\Lambda \to M$ be the covering space corresponding to Λ. Denote by $i_\Lambda : \Lambda \to \Gamma$ the inclusion map. Let*

$(M', f) \in \mathcal{S}^{Top}(M^n \times I^k)$, and let the closed manifold $\overline{M}'$ and the map $f_\Lambda : \overline{M}' \to M_\Lambda \times I^k$ be constructed as the pullback of the diagram

$$\begin{array}{ccc} \overline{M}' & \xrightarrow{f_\Lambda} & M_\Lambda \times I^k \\ \Big\downarrow & & \Big\downarrow {\scriptstyle p_\Lambda \times id} \\ M' & \xrightarrow[f]{} & M \times I^k \end{array}$$

Let $i_\Lambda^* : \mathcal{S}^{Top}(M \times I^k) \to \mathcal{S}^{Top}(M_\Lambda \times I^k)$ be the transfer given by $i_\Lambda^*(M', f) = (\overline{M}', f_\Lambda)$.

The following theorem gives a vanishing condition for elements in $\mathcal{S}^{Top}(M^n \times I^k)$. It follows from the local contractibility for Homeo$(M^n \times \mathbb{D}^k)_{\text{rel}}$, i.e. that homeomorphisms sufficiently close to the identity can be isotoped to the identity. See Edwards-Kirby [211] and Černavskiĭ [147].

Lemma 7.33. *Suppose that M^n is a closed flat Riemannian manifold. Let $k \geq 1$ and $n+k \geq 5$. For each $N \in \mathcal{S}^{Top}(M^n \times I^k)$ there is an integer K_N such that, if $s > K_N$ and $\phi : \Gamma \to \Gamma$ is s-expansive, then $i_{\phi(\Gamma)}^*(N) = 0$ in $\mathcal{S}^{Top}(M_{\phi(\Gamma)} \times I^k)$. Here $i_{\phi(\Gamma)} : \phi(\Gamma) \to \Gamma$ is the inclusion map and $i_{\phi(\Gamma)}^*$ denotes the transfer map defined above.*

Lemma 7.34. *Let M be a closed flat Riemannian manifold with fundamental group Γ and let $i_\Lambda : \Lambda \to \Gamma$ be an inclusion map for a subgroup of finite index. Then the transfer maps i_Λ^* for $L_*(\mathbb{Z}[\Gamma])$, $\mathcal{S}^{Top}(M^n \times I^k)_{\text{rel}}$ and $[M^n \times I^k, F/Top]_{\text{rel}}$ form a commutative diagram with the maps of the surgery exact sequence as follows:*

$$\begin{array}{ccccccccc} L_{n+k+1}(\mathbb{Z}[\Gamma]) & \longrightarrow & \mathcal{S}^{Top}(M^n \times I^k)_{\text{rel}} & \longrightarrow & [M^n \times I^k : F/Top]_{\text{rel}} & \xrightarrow{\sigma} & L_{n+k}(\mathbb{Z}[\Gamma]) & \longrightarrow & \cdots \\ {\scriptstyle i_*^\Lambda} \uparrow \downarrow {\scriptstyle i_\Lambda^*} & & \Big\downarrow & & \Big\downarrow {\scriptstyle i_\Lambda^*} & & {\scriptstyle i_*^\Lambda} \uparrow \downarrow {\scriptstyle i_\Lambda^*} & & \\ L_{n+k+1}(\mathbb{Z}[\Lambda]) & \longrightarrow & \mathcal{S}^{Top}(M_\Lambda^n \times I^k)_{\text{rel}} & \longrightarrow & [M_\Lambda^n \times I^k : F/Top]_{\text{rel}} & \xrightarrow[\sigma_\Lambda]{} & L_{n+k}(\mathbb{Z}[\Lambda]) & \longrightarrow & \cdots \end{array}$$

Proof. The result follows from the definitions. □

The following is a homological statement:

Lemma 7.35. *If $n+k \geq 5$ and $k \geq 1$ and $i : \Lambda \to \Gamma$ denotes the inclusion of a subgroup Λ of finite index in Γ, then*

$$i_*^\Lambda \sigma_\Lambda [M_\Lambda \times I^k : F/Top][1/2] \subseteq \sigma[M \times I^k : F/Top][1/2].$$

Lemma 7.36. *Let M^n be a closed flat Riemannian manifold with fundamental group Γ with holonomy group Γ/A. Let s be a positive integer and let $p : \Gamma \to \Gamma_s$ be the canonical projection, where $\Gamma_s = \Gamma/sA$. For all subgroups $E \subseteq \Gamma_s$, let $\Lambda = p^{-1}(E)$,* *and denote by* i^*_Λ *the transfer map*

$$i^*_\Lambda : [M^n \times I^k : F/Top]_{\text{rel}} \to [M_\Lambda \times I^k : F/Top]_{\text{rel}} .$$

If $b \in [M^n \times I^k : F/Top]_{\text{rel}}$ *such that* $i^*_\Lambda(b) = 0$ *for all hyperelementary subgroups* $E \subseteq \Gamma_s$, *then* $b = 0$.

Proof. Let $N = M^n \times I^k$. The homotopy type of F/Top as ascertained by Sullivan (see Theorem 3.97) shows that $[N : F/Top][{}^1\!/\!_2] \cong KO(N)[{}^1\!/\!_2]$ and $[N : F/Top]_{(2)}$ is isomorphic to the direct sum of $\bigoplus_{i\geq 0} H^{4i}(N; \mathbb{Z}_{(2)})$ and $\bigoplus_{i\geq 0} H^{4i+2}(N; \mathbb{Z}_2)$. Dress induction is valid for all homology theories involved. For example, the map

$$KO_*(M \times I^k) \xrightarrow{\oplus i^*_\Lambda} KO_*(M_\Lambda \times I^k)$$

is injective. So b only depends on its restrictions to the hyperelementary subgroups of Γ_s; i.e. if $i^*_\Lambda(b) = 0$ for all Λ hyperelementary in Γ_s, then $b = 0$. □

Recall that a group is *poly-*$\mathbb{Z}$ if it possesses a normal series with infinite cyclic factor groups.

Lemma 7.37. *(Farrell-Hsiang [223]) Let* $m \geq 2$. *If* Γ *is poly-*$\mathbb{Z}$ *of finite rank and* A *is a normal subgroup of* Γ *with finite quotient* $G = \Gamma/A$, *then*

$$i^*_A \otimes id : L_k(\mathbb{Z}[\Gamma], w) \otimes \mathbb{Z}[{}^1\!/\!_m] \to (L_k(\mathbb{Z}[A], w) \otimes \mathbb{Z}[{}^1\!/\!_m])^G$$

is an isomorphism for $k = 0, 1, 2, 3$.

This result is a consequence of induction, and, by inverting m, one only deals with cyclic groups.

Theorem 7.38. *Let* M^n *be a closed flat Riemannian manifold with fundamental group* Γ *with holonomy group* $G = \Gamma/A$. *If* $n+k \geq 5$ *and* $|G|$ *is odd, then* $S^{Top}(M^n \times I^k)_{(2)} = 0$.

Proof. It suffices to consider the case in which $|G|$ is hyperelementary. Let $i : A \to \Gamma$ be the inclusion map and i^*_A be the transfer maps. Recall that by construction the cover M_A is an n-torus, so it follows that $S^{Top}(M^n_A \times I^k) = 0$ by Hsiang-Shaneson and Wall. Also $i^*_A : L_*(\mathbb{Z}[\Gamma])_{(2)} \to L_*(\mathbb{Z}[A])^G_{(2)}$ is an isomorphism by results of Farrell-Hsiang (Corollary 2.4, Theorem 3.1, Lemmas 2.8 and 4.1 of [223]). See the previous lemma. Using Sullivan's description of $[\cdot : F/Top]$ in terms of cohomology and the fact that $|G|$ is odd, we have an induced isomorphism

$$i^*_A : [M^n \times I^k, F/Top]_{(2)} \to [M^n_A \times I^k, F/Top]^G_{(2)}.$$

Using the five-lemma in the diagram

$$\begin{array}{ccc}
[M^n \times I^{k+1} : F/Top]_{(2)} & \xrightarrow{\cong} & [M^n_A \times I^{k+1} : F/Top]^G_{(2)} \\
\downarrow & & \downarrow \\
L_{n+k+1}(\mathbb{Z}[\Gamma])_{(2)} & \xrightarrow{\cong} & L_{n+k+1}(\mathbb{Z}[A])^G_{(2)} \\
\downarrow & & \downarrow \\
\mathcal{S}^{Top}(M^n \times I^k)_{(2)} & \longrightarrow & \mathcal{S}^{Top}(M^n_A \times I^k)^G_{(2)} \\
\downarrow & & \downarrow \\
[M^n \times I^k : F/Top]_{(2)} & \xrightarrow{\cong} & [M^n_A \times I^k : F/Top]^G_{(2)} \\
\downarrow & & \downarrow \\
L_{n+k}(\mathbb{Z}[\Gamma])_{(2)} & \xrightarrow{\cong} & L_{n+k}(\mathbb{Z}[A])^G_{(2)}
\end{array}$$

we conclude that $\mathcal{S}^{Top}(M \times I^k)_{(2)} = 0$. □

Theorem 7.39. *If $n + k \geq 5$ and $|G|$ is odd, then $\mathcal{S}^{Top}(M^n \times I^k)[1/2] = 0$.*

Proof. Recall that we can retain exactness by localizing since localization is flat. We proceed by induction on n and $|G|$. Assume that $\mathcal{S}^{Top}(N^m \times I^k)[1/2] = 0$ for all closed flat Riemannian manifolds N^m with $m+k \geq 5$ such that either $m < n$ or the order of the holonomy group of N is less than $|G|$. Since $|G|$ is odd, one of two possibilities arises.

Suppose that Γ satisfies condition (a) in Theorem 7.20; i.e. Γ is the semidirect product $\Gamma' \rtimes \mathbb{Z}$. Then by the induction hypothesis applied to the exact sequence in Lemma 7.31 we have $\mathcal{S}^{Top}(M^n \times I^k)[1/2] = 0$.

Suppose now that Γ satisfies condition (b) in Theorem 7.20. As before denote by S the set of all positive integers s guaranteed by this condition.

Claim 1: The map $\tau : \mathcal{S}^{Top}(M^n \times I^k)[1/2] \to [M^n \times I^k, F/Top][1/2]$ is the zero map.

Proof of Claim 1: Let $N \in \mathcal{S}^{Top}(M \times I^k)_{\text{rel}}$. Choose s as in Lemma 7.33 and denote by $p : \Gamma \to \Gamma_s = \Gamma/sA$ the projection map. Let E be hyperelementary in Γ_s and let $\Lambda = p^{-1}(E)$. If E projects via the canonical map to a proper subgroup of G, then the holonomy group of $p^{-1}(E)$ has order less than $|G|$. By the induction hypothesis, we know that $\mathcal{S}^{Top}(M_\Lambda \times I^k)[1/2] = 0$. Denote by $i_\Lambda : \Lambda \to \Gamma$ the inclusion map. The commutative diagram given by

$$\begin{array}{ccc} \mathcal{S}^{Top}(M \times I^k)[{}^1\!/\!{}_2] & \xrightarrow{\ \tau\ } & [M \times I^k : F/Top][{}^1\!/\!{}_2] \\ \downarrow{\scriptstyle i^*_\Lambda} & & \downarrow{\scriptstyle i^*} \\ \mathcal{S}^{Top}(M_\Lambda \times I^k)[{}^1\!/\!{}_2] & \xrightarrow[\tau_\Lambda]{} & [M_\Lambda \times I^k : F/Top][{}^1\!/\!{}_2] \end{array}$$

implies that $i^*(\tau(M)) = 0$.

Otherwise E projects isomorphically onto G. As in the proof of Theorem 7.29, consider the particular hyperelementary group $E = p_s \circ f(\Gamma)$. By construction we have $\Lambda = p^{-1}(E) = f(\Gamma)$. By the preceding diagram, we know that $i^*_\Lambda(N) = 0$, so $i^*(\tau(b)) = 0$. By Dress induction, it follows that $\tau(N) = 0$, as required.

For the next claim, we will consult the following diagram:

$$\begin{array}{ccccccc} [M^n \times I^{k+1} : F/Top][{}^1\!/\!{}_2] & \xrightarrow{\ \sigma\ } & L_{n+k+1}(\mathbb{Z}[\Gamma])[{}^1\!/\!{}_2] & \xrightarrow{\ d\ } & \mathcal{S}^{Top}(M \times I^k)[{}^1\!/\!{}_2] & \xrightarrow{\ \tau\ } & \cdots \\ {\scriptstyle i_*}\uparrow\ \downarrow{\scriptstyle i^*} & & {\scriptstyle i_*}\uparrow\ \downarrow{\scriptstyle i^*} & & {\scriptstyle i_*}\uparrow\ \downarrow{\scriptstyle i^*} & & \\ [M^n_\Lambda \times I^{k+1} : F/Top][{}^1\!/\!{}_2] & \xrightarrow[\sigma_\Lambda]{} & L_{n+k+1}(\mathbb{Z}[\Lambda])[{}^1\!/\!{}_2] & \xrightarrow[d_\Lambda]{} & \mathcal{S}^{Top}(M_\Lambda \times I^k)[{}^1\!/\!{}_2] & \longrightarrow & \cdots \end{array}$$

Claim 2: Let $b \in L_{n+k+1}(\mathbb{Z}[\Gamma])[{}^1\!/\!{}_2]$. There is $s \in S$ along with an s-expansive map $\phi_s : \Gamma \to \Gamma$ such that, for all $r \in \mathrm{GW}(G, \mathbb{Z})$, the class $i_* \circ i^*(q_s^\#(r)b)$ belongs to the image of σ. Here $q_s^\# : \mathrm{GW}(G, \mathbb{Z}) \to \mathrm{GW}(\Gamma_s, \mathbb{Z})$ is induced by the projection $q_s : \Gamma_s \to G$.

Proof of Claim 2: As in the diagram above, let d be the action of the L-group on the structure set. Recall that the L-group $L_*(\mathbb{Z}[\Gamma])$ is a module over the Dress ring $\mathrm{GW}(\Gamma_s, \mathbb{Z})$. The product $q_s^\#(r)b$ is defined by this module structure. Let $N = d(q_s^\#(r)b)$ in $\mathcal{S}^{Top}(M \times I^k)[{}^1\!/\!{}_2]$. Choose s as in Lemma 7.33. By Theorem 7.25 we know that there is an s-expansive $\phi_s : \Gamma \to \Gamma$. Let $\Lambda = \phi_s(\Gamma)$. Then $i^*(N) = 0$ in $\mathcal{S}^{Top}(M_\Lambda \times I^k)[{}^1\!/\!{}_2]$. From Claim 1, we know that

$$\tau : \mathcal{S}^{Top}(M^n \times I^k)[{}^1\!/\!{}_2] \to [M^n \times I^k, F/Top][{}^1\!/\!{}_2]$$

is the zero map.

Commutativity shows that $d_\Lambda(i^*(q_s^\#(r)b)) = 0$ in $\mathcal{S}^{Top}(M_\Lambda \times I^k)[{}^1\!/\!{}_2]$, i.e. $i^*(q_s^\#(r)b) \in \ker(d_\Lambda)$. By exactness, there is $v \in [M^n_\Lambda \times I^k : F/Top][{}^1\!/\!{}_2]$ such that $\sigma_\Lambda(v) = i^*(q_s^\#(r)b)$. Clearly $i_*(i^* q_s^\#(r)b)$ lies in $\mathrm{im}\,(i_* \circ \sigma_\Lambda)$. By Lemma 7.35, it follows that $i_*(i^* q_s^\#(r)b)$ also lies in $\mathrm{im}\,(\sigma)$.

Claim 3: Let $b \in L_{n+k+1}(\mathbb{Z}[\Gamma])[{}^1\!/\!{}_2]$. Let $s \in S$ and $\phi_s : \Gamma \to \Gamma$ be as given in Claim 2. Let $q : \Gamma_s \to G$ be the projection map. For all $r \in \mathrm{GW}(\Gamma_s, \mathbb{Z})$ induced from a subgroup $D_s \subseteq \Gamma_s$ such that $q(D_s) \neq G$, the product rb lies in $\mathrm{im}\,(\sigma)$.

Proof of Claim 3: Let G' be a proper subgroup of G and let q be the composition $q : \Gamma \to \Gamma_s \to G$. Let Λ be the subgroup of Γ given by $\Lambda = q^{-1}(G')$, and let $i : \Lambda \to \Gamma$ be the inclusion map. We have a diagram

$$\begin{array}{ccc} \Lambda = q^{-1}(G') & \longrightarrow & G' \\ {\scriptstyle i}\downarrow & & \downarrow \\ \Gamma & \xrightarrow[f_s]{} & G \end{array}$$

We also have the following diagram:

$$\begin{array}{ccccc} [M^n \times I^{k+1} : F/Top][{}^1\!/\!_2] & \xrightarrow{\sigma} & L_{n+k+1}(\mathbb{Z}[\Gamma])[{}^1\!/\!_2] & \xrightarrow{d} & \mathcal{S}^{Top}(M \times I^k)[{}^1\!/\!_2] \\ {\scriptstyle i_*}\uparrow\ \downarrow{\scriptstyle i^*} & & {\scriptstyle i_*}\uparrow\ \downarrow{\scriptstyle i^*} & & {\scriptstyle i_*}\uparrow\ \downarrow{\scriptstyle i^*} \\ [M^n_\Lambda \times I^{k+1} : F/Top][{}^1\!/\!_2] & \xrightarrow[\sigma_\Lambda]{} & L_{n+k+1}(\mathbb{Z}[\Lambda])[{}^1\!/\!_2] & \xrightarrow[d_\Lambda]{} & \mathcal{S}^{Top}(M_\Lambda \times I^k)[{}^1\!/\!_2] \end{array}$$

Let $b \in L_{n+k+1}(\mathbb{Z}[\Gamma])[{}^1\!/\!_2]$ and $r \in \mathrm{GW}(\Gamma_s, \mathbb{Z})$. By the induction hypothesis, the group $\mathcal{S}^{Top}(M_\Lambda \times I^k)[{}^1\!/\!_2]$ is trivial, so that σ_Λ is surjective. Therefore there is $u \in [M^n_\Lambda \times I^{k+1} : F/Top][{}^1\!/\!_2]$ such that $\sigma_\Lambda(u) = i^*(rb)$. So $i_*(\sigma_\Lambda(u)) = i_*(i^*(rb))$. By Lemma 7.35 we know that $i_*(i^*(rb))$ lies in $\mathrm{im}\,(\sigma)$. By Frobenius reciprocity, we have $i_*(\sigma_\Lambda(u)) = i_*(i^*(rb)) = q_s^\#(r)b$, where $q_s^\# : \mathrm{GW}(\Gamma_s, \mathbb{Z}) \to \mathrm{GW}(G, \mathbb{Z})$ is induced by the projection $q_s : \Gamma_s \to G$.

Let $\phi_s : \Gamma \to \Gamma$ be s-expansive and $p : \Gamma \to \Gamma_s$ be the projection map. Let $i : \phi_s(\Gamma) \to \Gamma$ and $k : p(\phi_s(\Gamma)) \to \Gamma_s$ be the obvious maps. The composite $q_s \circ k : p(\phi_s(\Gamma)) \to G$ is an isomorphism, and $r \in \mathrm{im}\,(k_\# : \mathrm{GW}(\Gamma_s, \mathbb{Z}) \to \mathrm{GW}(\Gamma_s, \mathbb{Z}))$ by assumption:

$$\begin{array}{ccc} \mathrm{GW}(G, \mathbb{Z}) & \xrightarrow{(q_s \circ k)^\#} & \mathrm{GW}(\Gamma_s, \mathbb{Z}) \\ & {\scriptstyle q_s^\#}\searrow & {\scriptstyle k_\#}\downarrow\ \uparrow{\scriptstyle k^\#} \\ & & \mathrm{GW}(\Gamma_s, \mathbb{Z}) \end{array}$$

Therefore there is $r' \in \mathrm{GW}(G, \mathbb{Z})$ such that $r = k_\# k^\# q_s^\#(r')$, where $q_s^\# : \mathrm{GW}(G, \mathbb{Z}) \to \mathrm{GW}(\Gamma_s, \mathbb{Z})$ is induced by the projection $q_s : \Gamma_s \to G$. By Frobenius reciprocity and the fact that $L_*(p^{-1}(\cdot))$ is a Frobenius module over $\mathrm{GW}(\,\cdot\,, \mathbb{Z})$, we have

$$rb = (k_\# k^\# q_s^\#(r'))b = i_*((k^\# q_s^\#(r'))i^*(b)) = i_* i^*(q_s^\#(r')b).$$

The latter expression however lies in $\mathrm{im}\,(\sigma)$ by Claim 2, so $rb \in \mathrm{im}\,(\sigma)$.

Claim 4: The class b lies in $\mathrm{im}\,(\sigma)$.

Proof of Claim 4: Let s and $\phi_s : \Gamma \to \Gamma$ be as given in Claim 1. Let $\mathcal{P}_s$ be the collection of subgroups D_s of Γ_s for which $q_s(D_s) \neq G$. Let $p : \Gamma \to \Gamma_s$ be the canonical projection. Let $r \in \mathrm{GW}(\Gamma_s, \mathbb{Z})$ be the element induced by $p \circ \phi_s(\Gamma)$. Since $\mathrm{GW}(\bullet, \mathbb{Z})$ is a Frobenius functor, the condition on Γ implies that the unit 1 in $\mathrm{GW}(\Gamma_s, \mathbb{Z})$ satisfies the equation $1 = r + \sum r_j$, where $\sum r_j$ is a finite sum of elements induced by subgroups in $\mathcal{P}_s$. Hence $b = rb + \sum r_j b$. By Claim 3, we know that $\sum r_j b \in \mathrm{im}(\sigma)$. But rb also lies in $\mathrm{im}\,\sigma$ by Claim 3. Therefore b lies in $\mathrm{im}\,\sigma$.

Claim 5: If $n + k \geq 5$ and $|G|$ is odd, then $\mathcal{S}^{Top}(M^n \times I^k)[{}^1\!/\!_2] = 0$.

Proof of Claim 5: By Claim 1, we have the sequence

$$0 \to [M^n \times I^k : F/Top][{}^1\!/\!_2] \xrightarrow{\sigma} L_{n+k+1}(\mathbb{Z}[\Gamma])[{}^1\!/\!_2] \xrightarrow{d} \mathcal{S}^{Top}(M^n \times I^k)[{}^1\!/\!_2] \to 0.$$

We want to show that every element $N \in \mathcal{S}^{Top}(M^n \times I^k)[{}^1\!/\!_2]$ is trivial. Since d is surjective from Claim 1, every N has a preimage $b \in L_{n+k+1}(\mathbb{Z}[\Gamma])[{}^1\!/\!_2]$ with $d(N) = b$. It suffices to show that b lies in the kernel of d. By exactness, it suffices to show that b lies in $\mathrm{im}\,(\sigma)$. This statement is proved in Claim 4. □

7.3 ALMOST FLAT MANIFOLDS

Almost flat manifolds arise very naturally in differential geometry. They are related to groups of polynomial growth and ends of complete negatively curved manifolds. As a consequence of Farrell-Hsiang [230] and Gromov [269], they are topologically identical to aspherical manifolds with virtually nilpotent fundamental group.

Definition 7.40. *An infranilmanifold is a closed manifold of the form $\Gamma \backslash G/K$ with $G = L \rtimes K$, where L is a simply connected nilpotent Lie group, K is a compact Lie group, and Γ is a discrete cocompact subgroup of G.*

Such a manifold is clearly aspherical. The following theorem of Gromov and Ruh characterizes infranilmanifolds as precisely those that support a sequence of "flatter and flatter" metrics.

Theorem 7.41. *(Gromov [269], Ruh [567]) The smooth manifold M is a closed infranilmanifold iff it supports an almost flat structure; i.e. there is a sequence g_n of Riemannian metrics such that the sectional curvatures of (M, g_n) converge uniformly to 0 while the diameters of (M, g_n) remain uniformly bounded from above.*

The main goal of this section is to prove Farrell and Hsiang's result that infranilmanifolds of dimension at least 5 are topologically rigid. We follow their discussion closely with a few small simplifications.

Theorem 7.42. *(Farrell-Hsiang [230]) Suppose that $n \neq 3, 4$ and let M^n be a closed connected manifold that supports an almost flat smooth structure. Then $\mathcal{S}^{Top}(M^n) = 0$.*

As a corollary of this theorem (and Freedman-Quinn and Perelman for dimensions 3 and 4), a closed connected manifold M^n supports an almost flat smooth structure iff M is aspherical and $\pi_1(M^n)$ is virtually nilpotent.

Definition 7.43. *A finitely generated group Γ is an n-dimensional crystallographic group if it contains a torsion-free maximum abelian group A of rank n of finite index. In other words, there is a short exact sequence*

$$0 \to A \to \Gamma \to G \to 0$$

where $G \leq \mathrm{GL}_n(\mathbb{Z}) \cong \mathrm{Aut}\,(A)$ is a finite group acting faithfully on A. The quotient group G is the point group or the holonomy group of Γ. The group A is called the translation subgroup of A.

Note that a torsion-free crystallographic group is Bieberbach. By maximality, we know that A is characteristic in G, and so it is normal. As before, we call $G = \Gamma/A$ the *holonomy group of* Γ. It acts faithfully on A, and with respect to this action this A is called the *holonomy representation of* Γ. The Bieberbach rank of Γ is defined to be the rank of A as an abelian group. If s is a positive integer, we define $\Gamma_s = \Gamma/sA$ and $A_s = A/sA$.

Theorem 7.44. *Let Γ be a crystallographic group of Bieberbach rank n and holonomy group $G = \Gamma/A$. Then one of the following holds:*

1. *Γ is the semidirect product $\Gamma = \Gamma' \rtimes \mathbb{Z}$ for some subgroup Γ' which is crystallographic with Bieberbach rank $n-1$;*
2. *there is an epimorphism $\Gamma \to \widehat{\Gamma}$ from Γ to a non-trivial crystallographic group $\widehat{\Gamma}$ with holonomy $\widehat{G}$ along with an infinite set $\widehat{S}$ of positive integers $s \equiv 1 \bmod |\widehat{G}|$ such that, for any hyperelementary subgroup $\widehat{E}_s$ of $\widehat{\Gamma}_s$, if the natural projection map $\widehat{E}_s \hookrightarrow \widehat{\Gamma}_s \to \widehat{G}$ is surjective, then it is bijective;*
3. *$G = \Gamma/A$ is an elementary abelian 2-group and either*
   a) *$\Gamma = A \rtimes \mathbb{Z}_2$ and $G = \mathbb{Z}_2$ act on A via multiplication by -1, or*
   b) *there is an epimorphism $\Gamma \to \widehat{\Gamma}$ from Γ to a crystallographic group $\widehat{\Gamma}$ with holonomy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and translation group $\mathbb{Z} \oplus \mathbb{Z}$ such that the image of the holonomy representation in $\mathrm{GL}_2(\mathbb{Z})$ is either*

$$\left\{ \begin{pmatrix} j & 0 \\ 0 & k \end{pmatrix} \middle|\, j,k = \pm 1 \right\} \text{ or } \left\{ \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \text{ and } \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \middle|\, j = \pm 1 \right\}.$$

Definition 7.45. *Denote by $E(n)$ the isometry group of $\mathbb{R}^n$. Let Γ be a finitely generated, torsion-free, virtually nilpotent group in $E(n)$. Then an m-fibering apparatus $\mathcal{A}_m = (\widehat{\Gamma}, \phi, f)$ for Γ consists of the following data:*

1. *a crystallographic group $\widehat{\Gamma} \subseteq E(m)$;*

2. *an epimorphism* $\phi: \Gamma \to \widehat{\Gamma}$*;*
3. *a properly discontinuous (hence free) action of* Γ *on* $\mathbb{R}^n$ *with compact orbit space;*
4. *a* ϕ*-equivariant fiber bundle map* $f: \mathbb{R}^n \to \mathbb{R}^m$ *with fiber diffeomorphic to* $\mathbb{R}^{n-m}$*, i.e.* $f(\gamma.x) = \phi(\gamma).f(x)$ *for all* $\gamma \in \Gamma$ *and* $x \in \mathbb{R}^n$*.*

If m is not specified, we omit it from the notation.

Definition 7.46. *Suppose that* $\mathcal{A} = (\widehat{\Gamma}, \phi, f)$ *is a fibering apparatus for* Γ*. Let* $\widehat{G} = \widehat{\Gamma}/\widehat{A}$ *be the holonomy group for* $\widehat{\Gamma}$*. Then we say that* $\mathcal{A}$ *is special if*

1. $\widehat{\Gamma}$ *is a type 1, 2, or 3 crystallographic group;*
2. *there is an infinite set* $\widehat{S}$ *of positive integers* $s \equiv 1 \bmod |\widehat{G}|$ *such that, for any hyperelementary subgroup* $\widehat{E}_s$ *of* $\widehat{\Gamma}_s$*, if the natural projection map* $\widehat{E}_s \hookrightarrow \widehat{\Gamma}_s \to \widehat{G}$ *is surjective, then it is bijective.*

The following lemmas are proved in Farrell-Hsiang [230].

Lemma 7.47. *Let* $\phi: \Gamma \to \widehat{\Gamma}$ *be an epimorphism between crystallographic groups* $\Gamma \subseteq E(n)$ *and* $\widehat{\Gamma} \subseteq E(m)$*. There is a* ϕ*-equivariant affine surjection* $f: \mathbb{R}^n \to \mathbb{R}^m$*; i.e. there are* $\gamma \in \Gamma$ *and* $a \in \mathbb{R}^n$ *and* $b \in \mathbb{R}^m$ *such that* $f(\gamma x + a) = \phi(\gamma)f(x) + b$*.*

Note that, if Γ is torsion-free and $\phi: \Gamma \to \widehat{\Gamma}$ is an epimorphism between crystallographic groups, then $(\widehat{\Gamma}, \phi, f)$ is a fibering apparatus for Γ, where f is the map guaranteed by the previous lemma.

Notation 7.48. *If* Γ *is a group, denote by* $\mathrm{cd}(\Gamma)$ *its cohomological dimension of* Γ*. For example, if* Γ *is a Bieberbach group, then* $\mathrm{cd}(\Gamma) = \mathrm{rank}(\Gamma)$*.*

Lemma 7.49. *Let* Γ *be a finitely generated, torsion-free, virtually nilpotent group in* $E(n)$ *with* $\mathrm{cd}(\Gamma) \geq 2$*. Then there is a fibering apparatus* $\mathcal{A} = (\widehat{\Gamma}, \phi, f)$ *for* Γ *with* $\mathrm{rank}(\widehat{\Gamma}) \geq 2$*.*

Corollary 7.50. *Let* Γ *be a finitely generated, torsion-free, virtually nilpotent group in* $E(n)$ *and* $\mathcal{A} = (\widehat{\Gamma}, \phi, f)$ *be a fibering apparatus for* Γ*. If* $\widehat{\Gamma} = \Gamma' \rtimes \mathbb{Z}$*, then* $\mathbb{R}^n/\Gamma$ *fibers over a circle whose fiber is a closed Eilenberg-Maclane space* $K(\phi^{-1}(\Gamma'), 1)$*. Note that* $\phi^{-1}(\Gamma')$ *is also finitely generated, torsion-free, and virtually nilpotent with* $\mathrm{cd}(\phi^{-1}(\Gamma')) = \mathrm{cd}(\Gamma) - 1$*.*

Let M^n be a closed aspherical manifold for which $\mathrm{Wh}(\pi_1(M^n) \times \mathbb{Z}^m) = 0$ for all $m \geq 0$. Let E^{n+k} be the total space of an I^k-bundle over M^n. We will be interested in computing the structure set $\mathcal{S}^{Top}(E^{n+k})$, which is an abelian group when $n + k \geq 5$.

We describe the transfer maps for $\mathcal{S}^{Top}(E^{n+k})$ in the following. Let Λ be a subgroup of finite index in $\pi_1(M^n) \cong \pi_1(E^{n+k})$ and let $p_\Lambda: E_\Lambda \to E$ be the covering space corresponding to Λ. Note that E_Λ is the total space of an I^k-bundle with base space

M_Λ, which was constructed in Definition 7.32 of the last section. If (N^{n+k}, f) is an element in $\mathcal{S}^{Top}(E^{n+k})$, then we define the transfer map $i^*_\Lambda : \mathcal{S}^{Top}(E^{n+k}) \to \mathcal{S}^{Top}(E^{n+k}_\Lambda)$ by $i^*_\Lambda(N^{n+k}, f) = (N', f')$, where (N', f') is the top row of the following pullback diagram:

$$\begin{array}{ccc} N' & \xrightarrow{f'} & E_\Lambda \\ \downarrow & & \downarrow p_\Lambda \\ N & \xrightarrow[f]{} & E \end{array}$$

We have a surgery exact sequence with respect to Λ given by

$$\cdots \to L_{n+k+1}(\mathbb{Z}[\Lambda], \omega_\Lambda) \xrightarrow{d_\Lambda} \mathcal{S}^{Top}(E^{n+k}_\Lambda) \xrightarrow{\tau_\Lambda} [E^{n+k}_\Lambda : F/Top] \xrightarrow{\sigma^\Lambda_*} L_{n+k}(\mathbb{Z}[\Lambda], \omega_\Lambda).$$

Here $\omega_\Lambda : \pi_1(E_\Lambda) \to \mathbb{Z}_2$ is the first Stiefel-Whitney class of E_Λ.

The following theorems are due to Nicas [482]: they are direct consequences of Dress induction once one knows that structure sets are L-groups, i.e. have a "Chapter 9" type definition.

Theorem 7.51. *Let $k \geq 1$ and $(N, f) \in \mathcal{S}^{Top}(E^{n+k})$. Suppose that, for all finite factor groups G' of $\Gamma = \pi_1(M^n)$ and every hyperelementary subgroup H of G', the image of (N, f) under the transfer map $i^*_{p^{-1}(H)} : \mathcal{S}^{Top}(E^{n+k}) \to \mathcal{S}^{Top}(E^{n+k}_{p^{-1}(H)})$ is the zero class, where $p : \Gamma \to G = \Gamma/A$ is the canonical projection. Then $\tau_\Gamma(N, f) = 0$.*

Theorem 7.52. *Let $k \geq 1$. Suppose that, for all finite factor groups G' of $\Gamma = \pi_1(M^n)$ and every hyperelementary subgroup H of G', the map $\tau_{p^{-1}(H)} : \mathcal{S}^{Top}(E^{n+k}_{p^{-1}(H)}) \to [E^{n+k}_{p^{-1}(H)}$ rel $\partial : F/Top]$ is the zero homomorphism. Let $(N, f) \in \mathcal{S}^{Top}(E^{n+k})$ be a class for which $i^*_{p^{-1}(H)}(N, f) = 0$ in $\mathcal{S}^{Top}(E^{n+k}_{p^{-1}(H)})$ for all hyperelementary subgroups H of $G = \Gamma/A$. Then (N, f) is the zero class.*

Definition 7.53. *A poly-$\mathbb{Z}$ group is a group with a normal series, each of whose factor groups is infinite cyclic.*

Remark 7.54. *The Borel conjecture in this case is in Wall's book. It is essentially Shaneson's thesis together with Farrell fibering.*

Since every finite generated nilpotent group has this property, it then follows that every finitely generated torsion-free virtually nilpotent group is torsion-free virtually poly-$\mathbb{Z}$.

We will recall without proof from Farrell-Hsiang [227] the relevant Whitehead group calculation.

Lemma 7.55. *Let G be a torsion-free virtually poly-$\mathbb{Z}$ group. Then the Whitehead group* Wh(G) *is zero. In particular, if Γ is a finitely generated torsion-free virtually nilpotent group, then* Wh$(\Gamma \times \mathbb{Z}^n)$ *is trivial for all $n \geq 0$.*

Of course, the second statement follows from the first. The following is the usual Farrell fibering surgery fibration.

Lemma 7.56. *Let M^n be a closed aspherical manifold that fibers over the circle with fiber N^{n-1}. If $\pi_1(M)$ is a virtually poly-$\mathbb{Z}$ and E^{n+k} is an I^k-bundle over M^n and $n+k \geq 6$, then there is an exact sequence*

$$\mathcal{S}^{Top}(E_{(N)}^{n+k-1} \times I) \to \mathcal{S}^{Top}(E^{n+k}) \to \mathcal{S}^{Top}(E_{(N)}^{n+k-1})$$

where $E_{(N)}^{n+k-1}$ is the restriction of E to N.

In particular, we have the Shaneson formula with no decorations necessary.

Lemma 7.57. *Let M^n be a closed aspherical manifold whose fundamental group is a virtually poly-$\mathbb{Z}$ group. Suppose that E^{n+k} is an I^k-bundle over M^n with $n+k \geq 1$. Then there is an exact sequence*

$$\mathcal{S}^{Top}(E^{n+k} \times I) \to \mathcal{S}^{Top}(E^{n+k} \times \mathbb{S}^1) \to \mathcal{S}^{Top}(E^{n+k}).$$

The general theory of assembly maps from Chapter 5 includes the following as a special case.

Proposition 7.58. *Let M and M' be two closed aspherical n-manifolds with isomorphic fundamental groups. Suppose that $\pi_1(M)$ is virtually nilpotent and $\mathcal{S}^{Top}(E^{n+k}) = 0$ for every I^k-bundle E^{n+k} over M with $n+k \geq 5$. Then $\mathcal{S}^{Top}(\overline{E}^{n+k}) = 0$ for every I^k-bundle $\overline{E}$ over M' with $n+k \geq 5$.*

Definition 7.59. *If $\widehat{\Gamma} \subseteq E(m)$ is a crystallographic group, we say that m is the dimension of $\widehat{\Gamma}$, denoted by $\dim(\widehat{\Gamma})$. If $\widehat{\Gamma}$ is torsion-free, then $\dim(\widehat{\Gamma}) = \mathrm{cd}(\widehat{\Gamma})$. In general $\dim(\widehat{\Gamma})$ is the virtual cohomological dimension $\mathrm{vcd}(\widehat{\Gamma})$.*

Definition 7.60. *Let Γ be a finitely generated, torsion-free, virtually nilpotent group acting freely properly, discontinuously, and cocompactly on $\mathbb{R}^n$. Consider the collection $\mathcal{G}$ of all crystallographic groups $\widehat{\Gamma}$ that can occur in a fibering apparatus $\mathcal{A} = (\widehat{\Gamma}, \phi, f)$ for Γ with $\mathrm{vcd}(\widehat{\Gamma}) \geq 2$.*

1. *Let the holonomy number $h(\mathbb{R}^n, \Gamma)$ be the minimum order of the holonomy group $\widehat{\Gamma}/\widehat{A}$ as $\widehat{\Gamma}$ ranges over $\mathcal{G}$.*
2. *Define the holonomy number $h(\Gamma)$ of Γ to be the minimum of $h(\mathbb{R}^n, \Gamma)$ as $(\mathbb{R}^n, \Gamma)$ ranges over all free, properly discontinuous, cocompact actions of Γ on $\mathbb{R}^n$.*

The following theorem is an analogue of the key lemma in the proof for flat manifolds with odd holonomy.

Theorem 7.61. *Let $\mathcal{A} = (\widehat{\Gamma}, \phi, f)$ be a special fibering apparatus for Γ and let E^{n+k} be an I^k-bundle (where $n+k \geq 5$) over $M^n = \mathbb{R}^n/\Gamma$ and let $y \in \mathcal{S}^{Top}(E^{n+k} \times I)$.*

Then there is an infinite set $\mathcal{P}(y)$ of positive integers each of which is relatively prime to $|\widehat{G}|$ and satisfying the following property: for each $s \in \mathcal{P}(y)$ and each hyperelementary subgroup H of $\widehat{\Gamma}_s$ such that $|\widehat{\Gamma}_s|$ divides $|H|$, the element y vanishes under transfer to $\mathcal{S}^{Top}(E^{n+k}_{\Lambda} \times I)$, where $\Lambda = \phi^{-1}(q^{-1}(H))$ and $q : \widehat{\Gamma} \to \widehat{\Gamma}_s$ denotes the canonical map.

Theorem 7.62. *Let $k \geq 0$ and let M^n be a closed aspherical manifold whose fundamental group is virtually nilpotent and let E^{n+k} be the total space of an I^k-bundle with base space M^n. If $n + k \geq 5$, then $\mathcal{S}^{Top}(E^{n+k}) = 0$. In particular, the structure set $\mathcal{S}^{Top}(M^n) = 0$ if $n \geq 5$.*

Proof. Let $\Gamma = \pi_1(M^n)$. We will proceed by induction on the dimension on M and then on the holonomy number $h(\Gamma)$. Assume that $\mathcal{S}^{Top}(E^{m+k}_{(N)}) = 0$ for any I^k-bundle $E^{m+k}_{(N)}$ whose base space is a closed aspherical manifold N^m with $\pi_1(N^m)$ virtually nilpotent and either (a) $m < n$ or (b) $n = m$ and $h(\pi_1(N^m)) < h(\Gamma)$.

If $n = 0$, then M^n is a point and $E^{n+k} = I^k$. The proof of the generalized Poincaré conjecture proves that $\mathcal{S}^{Top}(E^{n+k}) = 0$.

If $n = 1$, then M^n is a circle. Now assume that $n \geq 2$.

The case $h(\Gamma) = 1$ is the result of Wall mentioned above. Then there is a fibering apparatus $\mathcal{A} = (\widehat{\Gamma}, f, \phi)$ for Γ such that $\widehat{\Gamma}$ is free abelian of rank 2 (since $\widehat{\Gamma}$ is equal to its maximal torsion-free abelian subgroup). Let $(\mathbb{R}^n, \Gamma)$ denote the associated Γ-action on $\mathbb{R}^n$. By Corollary 7.50, since $\widehat{\Gamma}$ is a product, the quotient $\mathbb{R}^n/\Gamma$ fibers over a circle with a closed aspherical manifold N^{n-1} as the fiber. Also N^{n-1} is virtually nilpotent. If $\overline{E}^{n+k}$ is the total space of an I^k-bundle with base space $\mathbb{R}^n/\Gamma$, then by the Farrell fibering Lemma 7.56,we have the fibration $\mathcal{S}^{Top}(\overline{E}^{n+k-1}_{(N)} \times I) \to \mathcal{S}^{Top}(\overline{E}^{n+k}) \to \mathcal{S}^{Top}(\overline{E}^{n+k-1}_{(N)})$. Using the induction hypothesis for $n-1$, we know that $\mathcal{S}^{Top}(\overline{E}^{n+k-1}_{(N)} \times I)$ and $\mathcal{S}^{Top}(\overline{E}^{n+k-1}_{(N)})$ are both contractible, so $\mathcal{S}^{Top}(\overline{E}^{n+k})$ is also trivial. If E^{n+k} is an I^k-bundle for M, then Corollary 7.58 shows that $\mathcal{S}^{Top}(E^{n+k})$ is also trivial.

Now assume that $h(\Gamma) \geq 2$.

Suppose that $\overline{E}^{n+k}$ is an I^k-bundle (with $n+k \geq 5$) with base space $\mathbb{R}^n/\Gamma$ and $h(\mathbb{R}^n, \Gamma) = h(\Gamma)$. Let $\mathcal{A} = (\widehat{\Gamma}, \phi, f)$ be a fibering apparatus for Γ with Γ-action $(\mathbb{R}^n, \Gamma)$ and $\operatorname{vcd}(\widehat{\Gamma}) \geq 2$ and $|\widehat{G}| = |\widehat{\Gamma}/\widehat{A}| = h(\Gamma) < \infty$. We claim that the map

$$\tau : \mathcal{S}^{Top}(\overline{E}^{n+k} \times I^4) \to [\overline{E}^{n+k} \times I^4 : F/Top]$$

is the zero map. There are three possibilities for $\widehat{\Gamma}$ as given in Theorem 7.44.

Suppose that (1) holds; i.e. Γ is the semidirect product $\Gamma = \Gamma' \rtimes \mathbb{Z}$ for some Γ' crystallographic with Bieberbach rank $n - 1$. Then again by Corollary 7.50 we know that $\mathbb{R}^n/\Gamma$ fibers over the circle. Then $\mathcal{S}^{Top}(\overline{E}^{n+k}) = 0$ by the same argument as above. By Lemma 7.30 we have $\mathcal{S}^{Top}(\overline{E}^{n+k} \times I^4) = 0$. Clearly τ is the zero map.

Suppose that (2) or (3) holds. In this case, we will first reason that the fibering apparatus is special. If (3) holds, then there is an epimorphism $\psi : \widehat{\Gamma} \to \Gamma'$, where Γ' is of type 1, 2 or 3. By Lemma 7.47 there is a ψ-equivariant affine surjection $F : \mathbb{R}^m \to \mathbb{R}^2$, where $m = \dim(\widehat{\Gamma})$. Then the apparatus $(\Gamma', \psi\circ\phi, F\circ f)$ will be special. If (2) holds, a similar construction yields a special fibering apparatus. Note that the only two crystallographic groups with virtual cohomological dimension 1 are $\mathbb{Z}$ and $\mathbb{Z}_2 * \mathbb{Z}_2$. The case $\Gamma' = \mathbb{Z}$ falls under (1), which we have already discussed, while $\mathbb{Z}_2 * \mathbb{Z}_2$ does not satisfy condition (2) of Definition 7.46. Therefore we can assume that $\mathrm{vcd}(\Gamma') \geq 2$.

Returning to the claim that τ is the zero map, let $\widehat{E} = \overline{E}^{n+k} \times I^3$ and let $y \in \mathcal{S}^{Top}(\widehat{E}\times I)$. By Theorem 7.61 there is a positive integer s coprime to $|\widehat{\Gamma}|$ such that, for each hyperelementary subgroup $H \leq \widehat{\Gamma}_s$ with order divisible by $|\widehat{G}|$, the class y vanishes under the transfer map $\mathcal{S}^{Top}(\widehat{E}\times I) \to \mathcal{S}^{Top}(\widehat{E}_\Lambda\times I)$, where $\Lambda = \phi^{-1}(q^{-1}(H)) \leq \Gamma$ and $q : \widehat{\Gamma} \to \widehat{\Gamma}_s$ is the canonical projection. Keeping this same s, we now consider a hyperelementary group $H \leq \widehat{\Gamma}_s$ with order not divisible by $|\widehat{G}|$. Again let $\Lambda = \phi^{-1}(q^{-1}(H))$ and consider the fibering apparatus $\mathcal{A}_H = (q^{-1}(H), \phi|_\Lambda, f)$ with $\mathrm{vcd}(q^{-1}(H)) = \mathrm{vcd}(\widehat{\Gamma}) \geq 2$. The holonomy group of $q^{-1}(H)$ is a proper subgroup of $\widehat{G}$, so $h(\Lambda) < h(\Gamma)$. By the induction hypothesis, we have $\mathcal{S}^{Top}(\widehat{E}_\Lambda \times I) = 0$. Therefore the hypotheses of Theorem 7.51 are satisfied. So $\tau = 0$.

Remark 7.63. *This vanishing of τ is the integral Novikov conjecture. The paper of Ferry-Weinberger [245] gives an approach to prove such results directly from the α-approximation theorem for many fundamental groups with geometric interpretation. Almost flat fundamental groups easily fit in this framework.*

Now we wish to prove that, if $\overline{E}^{n+k}$ is an I^k-bundle (with $n+k \geq 5$) over $\mathbb{R}^n/\Gamma$, where $h(\mathbb{R}^n, \Gamma) = h(\Gamma)$, then $\mathcal{S}^{Top}(\overline{E}^{n+k} \times I^4) = 0$. We keep the same fibering apparatus $\mathcal{A} = (\widehat{\Gamma}, \phi, f)$ as before, using the special one in the cases of (2) or (3). In the case of (1), we argue as we did earlier and conclude that $\mathcal{S}^{Top}(\overline{E}^{n+k} \times I^k) = 0$. In the case of (2) or (3), let $y \in \mathcal{S}^{Top}(\widehat{E}_\Lambda \times I)$ and let s be one of the positive integers guaranteed by Theorem 7.61. Using the same notation as above, we also proved using the induction hypothesis that y vanishes under the transfer map $\mathcal{S}^{Top}(\widehat{E} \times I) \to \mathcal{S}^{Top}(\widehat{E}_\Lambda \times I)$ since the latter structure set is trivial.

We now wish to show that the hypotheses in Theorem 7.52 holds, i.e. that the map

$$\tau_\Lambda : \mathcal{S}^{Top}(\widehat{E}_\Lambda \times I) \to [\widehat{E}_\Lambda \times I; F/Top]$$

is the zero homomorphism for all hyperelementary $H \leq \widehat{\Gamma}_s$ and $\Lambda = \phi^{-1}(q^{-1}(H))$. If $|\widehat{G}|$ does not divide $|H|$, then $\mathcal{S}^{Top}(\widehat{E}_\Lambda \times I) = 0$ by the discussion above. So clearly $\tau_\Lambda = 0$. If $|\widehat{G}|$ divides $|H|$, then take $\mathcal{A}_H = (q^{-1}(H), \phi|_\Lambda, f)$ as a fibering apparatus for Λ. Here $\mathrm{vcd}(q^{-1}(H)) = \mathrm{vcd}(\widehat{\Gamma}) \geq 2$ and the holonomy group for $q^{-1}(H)$ is isomorphic to $\widehat{G}$. Then either $h(\Lambda) < h(\Gamma)$ or $h(\mathbb{R}^n, \Lambda) = h(\Lambda) = h(\Gamma)$. In the former case, the induction hypothesis implies that $\mathcal{S}^{Top}(\widehat{E}_\Lambda \times I)$ vanishes, so τ_Λ is the zero map. In the latter case, we note that Λ is a finitely generated, virtually nilpotent group with the same cohomological dimension as Γ. We can repeat the argument above using the pair $(\mathbb{R}^n, \Lambda)$ to show that $\tau_\Lambda \equiv 0$. Hence we have shown that $\tau_\Lambda \equiv 0$ in all cases.

By Theorem 7.52, it follows that $\mathcal{S}^{Top}(\widehat{E} \times I)$ is trivial. By the periodicity asserted in Theorem 7.30, we conclude that $\mathcal{S}^{Top}(E^{n+k})$ is trivial as well for all k. $\square$

Chapter Eight

Other surgery theories

The flexibility of the surgery exact sequence lends itself to numerous generalizations and modifications. This chapter gives an overview of some of them.

The first step is to weaken the homotopy data necessary to begin surgery, which allows for classifications in some setting, in which homotopy equivalence and Cat isomorphism are equally difficult to determine. We then move to various kinds of homological surgery. The first variation is local surgery in the sense of localization theory. It has numerous applications to group actions. The surgery groups are no longer associated to integral group rings, but the real change is to the idea of normal invariants, where we follow the distillation of Quinn's work by Taylor and Williams. In this case, the second variation is general homology surgery, due to Cappell and Shaneson, which has applications to knot and link theory, and to nonlocally flat codimension 2 embeddings, as well as to the classification of homology spheres with a given fundamental group (according to Hausmann).

We then turn to noncompact manifolds and explain proper surgery, where the main differences (after one sets it up) are in the algebra. Much more interesting is bounded surgery and its close cousins: continuously controlled at infinity surgery and controlled surgery. The fundamental results here were initiated by Quinn, building on the work of Chapman, Ferry, Kirby, and Siebenmann, and were vigorously pursued by many others. This theory has applications to basic properties of the Top category and can be used to give new approaches to the results of Kirby-Siebenmann. We explain their utility in proving Novikov's theorem on rational Pontrjagin classes, as well as simple cases of the Novikov conjecture. We do not push this theory nearly as far as it deserves, nor use it for proving the Borel conjecture (although some such ideas were foreshadowed in the previous chapter). It is clear that this subject needs a monograph of its own.

In the succeeding two sections, we apply controlled surgery to homology manifolds where surgery assumes its seemingly most perfect form, and to stratified spaces, for which algebraicization and functoriality is very sorely lacking.

The early sections in this chapter can be read after the most classical ideas from the first three chapters, but once we arrive at bounded and controlled surgery, the spacified version is critical, at the least for expressing results. The controlled point of view and the blocked points of view are quite close in spirit, although the latter is considerably more elementary. The blocked version is adequate for the PL stratified theory, although the fully topological version depends of course on the whole controlled theory.

8.1 SMOOTH SURGERY WITHOUT USUAL NORMAL DATA

Kreck [366] developed a modified theory of surgery which is applicable under weaker conditions than classical surgery. The idea is that, instead of fixing a homotopy type, one first considers weaker information. One compares n-dimensional compact manifolds with topological spaces whose k-skeleta are fixed, where $k \geq \left[\frac{n}{2}\right]$. The hope is that Poincaré duality can then fill in the missing homotopy theory.

A particularly attractive example which illustrates the idea is given by complete intersections. According to the Lefschetz hyperplane theorem, a complete intersection of complex dimension n has the same $(n-1)$-skeleton as $\mathbb{CP}^n$, and one can use Kreck's theory to obtain information about its diffeomorphism type, even though the homotopy classification is not known. The theory reduces this classification result to the determination of complete intersections in a certain bordism group. See Kreck [366] and Sixt [592]. A novel feature is the appearance of surgery obstruction monoids instead of the usual Wall groups whose calculation is generally quite difficult, but we still obtain valuable information that does not seem to be available to more traditional methods.

Kreck's theory replaces normal maps by a considerably weaker notion called *normal smoothings*, which are lifts of the stable normal bundle $M \to BO$ to a fibration $B \to BO$. Normal maps are a special case of this concept. He also allows maps that are $\left[\frac{n}{2}\right]$-equivalences on the boundary of a normal smoothing, whereas in Wall's theory we started with full homotopy equivalences. The theory assigns to any cobordism W of normal smoothings an element in the monoid $l_{2q}(\Lambda)$ which is *elementary* if and only if W is cobordant rel boundary to an s-cobordism. Sometimes one can determine stable information, and sometimes one can achieve precise information.

Definition 8.1. *Let $B \to BO$ be a fibration. If M^n is an n-dimensional manifold with a stable normal map $\nu : M \to BO$, we say that M has a normal B-structure if there is a lift $\widetilde{\nu} : M \to B$ fitting into the diagram:*

$$\begin{array}{ccc} & & B \\ & \overset{\widetilde{\nu}}{\nearrow} & \downarrow \\ M & \xrightarrow[\nu]{} & BO \end{array}$$

Definition 8.2. *Let B be a fibration over BO and $k \geq 0$.*

1. *A normal B-structure $\widetilde{\nu} : M \to B$ of a manifold M is a normal k-smoothing if it is a $(k+1)$-equivalence.*

2. *We say that* $\boldsymbol{B}$ *is k-universal if the fiber of* $\boldsymbol{B} \to \boldsymbol{BO}$ *is connected and its homotopy groups vanish above dimension k.*

By obstruction theory, if two fibrations $\boldsymbol{B}$ and $\boldsymbol{B'}$ over $\boldsymbol{BO}$ are both k-universal and admit a normal k-smoothing of the same manifold M, then the two fibrations are fiber homotopy equivalent. The theory of Moore-Postnikov decompositions implies that, for each manifold M, there is a k-universal fibration $\boldsymbol{B}_k(M)$ over $\boldsymbol{BO}$ admitting a normal k-smoothing of M. This $\boldsymbol{B}_k(M)$ is also called that *normal k-type of M*, which one can classify using obstruction theory.

There is an obvious bordism relation on closed n-manifolds with normal $\boldsymbol{B}$-structures. Their corresponding bordism group is denoted by $\Omega_n^{\boldsymbol{B}}$.

Theorem 8.3. *Let* M_0 *and* M_1 *be connected* $(2q-1)$*-dimensional manifolds with* $q \geq 3$ *and let* $f : \partial M_0 \to \partial M_1$ *be a diffeomorphism. Suppose that the following conditions hold.*

1. *There are normal* $(q-2)$*-smoothings in a fibration* $\boldsymbol{B}$ *over* $\boldsymbol{BO}$ *compatible with* f.
2. *There is a* $\boldsymbol{B}$*-zero bordism* W *of* $M_0 \cup_f M_1$*, where* W *is endowed with a normal* $\boldsymbol{B}$*-structure* $\overline{\nu}$.

Then $(W, \overline{\nu})$ *is* $\boldsymbol{B}$*-bordant rel boundary to a relative s-cobordism iff a certain well-defined obstruction* $\theta(W, \overline{\nu}) \in \ell_{2q}(\pi_1(\boldsymbol{B}), w_1(\boldsymbol{B}))$ *is elementary. (A slightly different condition holds when* $q = 3, 7$*.)*

One application of Kreck surgery is the classification of complete intersections in $\mathbb{CP}^n$, i.e. algebraic varieties V in projective space of codimension k for which the ideal of V is generated by exactly k elements. See Kreck [366]. A complete intersection is topologically determined by its multidegree, i.e. the degrees of the various transverse hypersurfaces that intersect to produce it. The topological structure of these spaces is quite non-trivial to unravel, and there are many diffeomorphic complete intersections with different multidegrees, as was first discovered by Libgober and Wood [400] using ordinary surgery theory. Kreck's work as extended and completed by Fang [217] shows that, when the multidegree has no small prime factors, the whole diffeomorphism classification is just determined by Euler characteristic classes and Pontrjagin classes. The homotopy classification is determined by the Euler characteristic and signature. Fang only claims a classification up to homeomorphism under these conditions, but attributes to Kreck and Stolz the statement that these conditions suffice for diffeomorphism. Other applications of this type of surgery to 4-manifolds and to homogeneous manifolds can be found in Kreck [368], Kreck-Stolz [371], and Davis [188].

8.2 LOCAL SURGERY

Suppose that, instead of wanting to understand manifolds within a given homotopy type, we wanted to understand $\mathbb{Z}_{(2)}$-homology spheres, i.e. manifolds M such that $H_*(M;\mathbb{Z}_{(2)}) \cong H_*(\mathbb{S}^n;\mathbb{Z}_{(2)})$.

Ordinary surgery theory would not be able to solve this problem; we lack the basic necessary homotopy data to start the process. But, the ideas naturally lead to the theory of *local surgery*, which we will sketch here. We follow the treatment of Taylor-Williams [635], which in turn is a reformulation of ideas of Quinn [518] (but see Anderson [13] for an early attempt). The surgery obstruction groups are only a minor modification of the ordinary surgery groups, where the group ring $\mathbb{Z}[\pi]$ is replaced by $\mathbb{Z}_{(2)}[\pi]$, or $\mathbb{Z}_{(P)}[\pi]$ for a different set of primes than 2. However, the normal invariants are more complicated and mix bordism with coefficients with a localization of ordinary normal invariants.

Definition 8.4. *Let P be a collection of primes. We say that a space X is a P-local Poincaré complex if*

1. *X has a fundamental class $[X]$;*
2. *the universal cover $\widetilde{X}$ of X is a $\mathbb{Z}_{(P)}$-local space, i.e. it is its own $\mathbb{Z}_{(P)}$-localization;*
3. *the fundamental class $[X]$ induces Poincaré duality on $H_*^{lf}(X;\mathbb{Z}_{(P)})$.*

We are interested in "classifying" maps $M \to X$ which are degree one and $\mathbb{Z}_{(P)}$-equivalences on their universal covers, i.e. for which $H_*(\widetilde{M};\mathbb{Z}_{(P)}) \to H_*(\widetilde{X};\mathbb{Z}_{(P)})$ is an isomorphism. In classical surgery theory, the degree one condition was automatic. Here it needs to be stated explicitly, since the degree could be a priori any unit in $\mathbb{Z}_{(P)}$. Unlike our earlier descriptions, there is at present no useful h-cobordism theorem for homology h-cobordisms, and therefore we must be content with a classification of P-local Poincaré complexes that is defined up to homology h- or s-cobordism. There is a definable torsion in this setting that lies in $K_1(\mathbb{Z}_{(P)}[G])/(\pm G)$. With this interpretation, our goal is to study the set $S^{Cat}(X;\mathbb{Z}_{(P)})$.

A degree one normal map is a degree one map $M \to X$ and a map $X \to BSCat_{(P)}$ so that the natural diagram

$$\begin{array}{ccc} M & \longrightarrow & X \\ \downarrow & & \downarrow \\ BSCat & \longrightarrow & BSCat_{(P)} \end{array}$$

commutes. For now, let $\mathcal{N}^{Cat}(X)$ be the set of normal cobordism classes of them.

Using the same reasoning as in the usual setting, a surgery exact sequence can be achieved: $L_{n+1}(\mathbb{Z}_{(P)}[G]) \to S^{Cat}(X;\mathbb{Z}_{(P)}) \to \mathcal{N}^{Cat}(X) \to L_n(\mathbb{Z}_{(P)}[G])$. One executes surgery until the middle dimension. In the middle dimension, the L-groups measure the obstruction to surgery, except now up to multiplying a homology class by a number n, coprime to the primes in P. The relevant variant of the Hurewicz theorem is provided by Serre's mod C theory. Serre introduced the idea of working in homotopy theory modulo some class C of abelian groups. In this theory, two groups A and B were treated as "isomorphic" if there is an appropriate map comparing A and B whose kernel and cokernel lie in C.

Let us return to the classification of normal invariants. Recall that, in the classical case, normal invariants are exactly the lifts of Spivak fibration of the Poincaré complex X to $BCat$. Local Poincaré complexes have local Spivak fibrations $X \to BF_{(P)}$ so the beginning of their theory is the same as the ordinary case. We need a lift to $BCat_{(P)}$.

Away from P, there is at least a bordism class associated $M \to X$ and one can map further to $B\pi$, twisting if required to respect the orientation, to produce for us an element in $MSCat_n(B\pi)$. With the commutative diagram

$$\begin{array}{ccc} \mathcal{N}^{Cat}(X) & \longrightarrow & [X : F/Cat_{(P)}] \\ \downarrow & & \downarrow \\ MSCat_n(B\pi) & \longrightarrow & MSCat_n(B\pi; \mathbb{Z}_{(P)}) \end{array}$$

the pullback square

$$\begin{array}{ccc} MSCat_n(B\pi) & \longrightarrow & MSCat_n(B\pi; \mathbb{Z}_{(P)}) \\ \downarrow & & \downarrow \\ MSCat_n(B\pi; \mathbb{Z}[{}^1/_P]) & \longrightarrow & MSCat_n(B\pi; \mathbb{Q}) \end{array}$$

can be replaced by a square

$$\begin{array}{ccc} \mathcal{N}^{Cat}(X) & \longrightarrow & [X : F/Cat_{(P)}] \\ \downarrow & & \downarrow \\ MSCat_n(B\pi; \mathbb{Z}[{}^1/_P]) & \longrightarrow & MSCat_n(B\pi; \mathbb{Q}) \end{array}$$

which removes some of the excess information that is encoded in the previous diagram. Moreover, this diagram is a pullback; i.e. there is an exact sequence

$$\begin{aligned} \cdots \to MSCat_{n+1}(B\pi; \mathbb{Q}) \to & \mathcal{N}^{Cat}(X) \\ \to [X : F/Cat_{(P)}] \oplus MSCat_n(B\pi; \mathbb{Z}[{}^1/_P]) \to & MSCat_n(B\pi; \mathbb{Q}). \end{aligned}$$

In particular, the classification of these topological $\mathbb{Z}_{(2)}$-homology spheres $\theta_n^{Cat}(\mathbb{Z}_2)$ would have contributions from the odd torsion in Top bordism $MSTop_n$ and would also contain copies of $MSTop_{n+1} \otimes (\mathbb{Z}_{(2)}/\mathbb{Z})$.

Remark 8.5. *Although not explained in the literature, the normal invariant set $\mathcal{N}^{Top}(X)$ is an abelian group and the surgery exact sequence is a sequence of groups and homomorphisms, as follows from all the techniques of Chapter 5.*

We will make computations for $\mathbb{Z}_2$-homology spheres, for which the relevant L-groups are $L_*(\mathbb{Z}[\mathbb{Z}_{(2)}])$. These vanish in odd dimensions, and are the usual L-groups $L_*(\mathbb{Z}[e])$ except in dimension 0 mod 4, when there is an additional infinite sum of copies of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ coming from the terms Witt($\mathbb{F}_p$) for odd primes. See our discussion of Witt rings in Chapter 2. They contribute to infinitely generated 2-torsion in $\theta^{Cat}_{4k+3}(\mathbb{Z}_2)$ in all the categories.

Remark 8.6. *The elements of the Witt group $L_0(\mathbb{Z}[\mathbb{Z}_{(2)}])$ do not arise as surgery obstructions since the quadratic forms arising from closed manifolds are $\mathbb{Z}$-nonsingular.*

At the prime 2, the story is exactly the same as for θ^{Cat}_n, so for Cat = Top or PL there is no other 2-torsion. At odd primes, the analysis is the same as for $\theta^{Cat}_n(\mathbb{Z}_2)$. We can now use this calculation to prove a theorem of Chase mentioned when we discussed homology propagation which depends on θ^{Cat}_{2n}. Let Σ^{2n} be a simply connected Cat manifold with a Cat degree one map $\phi : \Sigma^{2n} \to \mathbb{S}^{2n}$ to the standard sphere. Assume that ϕ is a $\mathbb{Z}_{(2)}$-homology equivalence. Then one cannot always propagate the free involution from the sphere.

Proposition 8.7. *(Chase [162]) There are even-dimensional simply connected mod 2 homology spheres that do not have free involutions.*

Proof. If there were, then by the Lefschetz fixed-point theorem, these actions would be orientation-reversing, so would represent elements of order 2 in the group of mod 2 homology spheres. Since $MSTop_{2i}$ can have non-trivial odd torsion, the result follows from this proposition below. □

Proposition 8.8. *Every $\mathbb{Z}_{(2)}$-homology sphere of dimension at least 5 is $\mathbb{Z}_{(2)}$-homology cobordant to a simply connected $\mathbb{Z}_{(2)}$-homology sphere.*

See Barge-Lannes-Latour-Vogel [37] for a general calculation of $\mathbb{Z}_{(P)}$-homology spheres by a more direct approach.

8.3 HOMOLOGY SURGERY

In classical surgery theory one begins with a degree one normal map $f : M \to X$ and then attempts by surgery to modify f into equivalence from the point of view of $H^*(\,\cdot\,; \mathbb{Z}[\pi])$.

However, knot theory presents us with a situation in which we try to achieve much less. Suppose that $K \hookrightarrow \mathbb{S}^{n+2}$ is a Cat locally flat knot, i.e. an embedding of $\mathbb{S}^n$ in $\mathbb{S}^{n+2}$. Fox and Milnor asked whether K bounds a locally flat disk in $\mathbb{D}^{n+3}$. Kervaire [353, 354] answers this question in the affirmative when n is even, but that in odd dimensions these knots form an infinitely generated abelian group, called the *knot cobordism group*. Here two knots are viewed as equivalent if they bound an annulus in the cylinder $\mathbb{S}^{n+2} \times I$. See also Levine [396] for more about the structure of this group, and the definitive paper of Stoltzfus [619] on its fine structure.

Kervaire proved his result by doing ambient surgery. In other words, one considers the Seifert surface, i.e. the orientable hypersurface in $\mathbb{S}^{n+2}$ that is bounded by K, and pushes it into $\mathbb{D}^{n+3}$. Then he executes a surgery by handle attachments on this manifold, inside the disk to make the Seifert surface contractible.

Cappell and Shaneson [122] suggested an abstract approach that takes advantage of the Poincaré conjecture, avoiding the need to work within a given manifold. As a result, the methods apply to the study of smooth or locally flat PL or Top knots invariant under a free action of a cyclic group or fixed under a semifree action, the so-called "counterexamples to the Smith conjecture," and to many other applications. They proceed as follows.

Definition 8.9. *Suppose that $f : X \to Y$ is a map of finite connected CW complexes with $\pi = \pi_1(Y)$. Assume also that f induces isomorphisms on all homology groups with local coefficients in Λ with a map $\mathbb{Z}[\pi] \to \Lambda$. Let C_f be the mapping cylinder of f with universal cover $\widetilde{C}_f$. Let C_* be the cellular chains of $(\widetilde{C}_f, \widetilde{A})$. Then $C_* \otimes_{\mathbb{Z}[\pi]} \Lambda$ is acyclic and its torsion element $\Delta_{\mathcal{F}}(f)$ in $\mathrm{Wh}(\mathcal{F})$ is a well-defined invariant of the homotopy class of f. If $\Delta_{\mathcal{F}}(f) = 0$, we say that f is a simple homology equivalence over Λ. Here $\mathrm{Wh}(\mathcal{F})$ is simply $K_1(\Lambda)/\langle \pm g \rangle$, where g runs over the elements of G.*

Let $\mathcal{F} : \mathbb{Z}[\pi] \to \Lambda$ be a surjective ring homomorphism, where G is a finitely presented group. Consider the following generalized surgery data of Cappell and Shaneson [122]:

1. a simple Poincaré pair (X, Y) over Λ of dimension n with $\pi_1(X) = G$;
2. a stable vector bundle ξ over X and a degree one normal map (f, b)

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & X \end{array}$$

   from a manifold pair $(M^n, \partial M) \to (X, Y)$, where ν_M is the stable normal bundle of M and $f|_{\partial M}$ is a simple homology equivalence over Λ.

Remark 8.10. *A Poincaré complex over Λ, of course, satisfies Poincaré duality with coefficients in Λ.*

Remark 8.11. *If $\partial Y = Y_1 \cup \cdots \cup Y_r$ is the decomposition of ∂Y into multiple connected components, then let $\rho_i : \pi_1(Y_i) \to \mathbb{Z}[\pi]$ be the induced map of group rings for each i. Let $N_i = (f|_{\partial M})^{-1}(Y_i)$. In this situation, we will require that*

1. *$f|_{N_i} : N_i \to Y_i$ induces an isomorphism of homology groups with local coefficients in Λ for all i, and*
2. *$\Delta_{\mathcal{F}\circ\rho_i}(f|_{N_i}) = 0$ for all i.*

Henceforth we shall assume an augmentation $\Lambda \twoheadrightarrow \mathbb{Z}$, so that there is no need to consider the kinds of normal invariants arising in local surgery, as we considered in the previous section.

Definition 8.12. *Let (X, Y) be a simple Poincaré pair of dimension n over Λ. Suppose that $\pi = \pi_1(X)$ and $\mathcal{F} : \mathbb{Z}[\pi] \to \Lambda$ is a ring epimorphism. A homology Cat manifold structure on (X, Y) is a simple homology Λ-equivalence $(M^n, \partial M) \to (X, Y)$ from a Cat manifold pair $(M, \partial M)$. We say that two homology Cat manifolds are equivalent if there is a homology s-cobordism between them. The equivalence classes of homology Cat manifold structures on (X, Y) form the simple homology structure set $\mathcal{S}^{\mathcal{F},s}(X)$ of X. The structure set will be denoted by $\mathcal{S}^{\mathcal{F},h}(X)$ if simplicity is not required. In order to analyze this set, Cappell and Shaneson define Γ-groups as follows:*

Let π be a finitely presented group and $w : \pi \to \mathbb{Z}_2$ be a homomorphism. Endow the group ring $\mathbb{Z}[\pi]$ with the involution given by $\overline{g} = w(g)g^{-1}$ for all $g \in G$. Let $\mathcal{F} : \mathbb{Z}[\pi] \to \Lambda$ be an epimorphism of rings.

Definition 8.13. *Let $\eta = \pm 1$ and let $I_\eta = \{a - \eta\overline{a} : a \in \mathbb{Z}[\pi]\}$. A special η-form over $\mathcal{F}$ is a triple (H, ϕ, μ), where H is a finitely generated right $\mathbb{Z}[\pi]$-module, the map $\phi : H \times H \to \mathbb{Z}[\pi]$ is $\mathbb{Z}$-bilinear, and $\mu : H \to \mathbb{Z}[\pi]/I_\eta$, satisfying the following conditions for all $x, y \in H$ and $a \in \mathbb{Z}[\pi]$:*

1. *$\phi(x, ya) = \phi(x, y)\, a$,*
2. *$\phi(x, y) = \eta\, \overline{\phi(y, x)}$,*
3. *$\phi(x, x) = \mu(x) + \eta\, \overline{\mu(x)} \bmod I_\eta$,*
4. *$\mu(x + y) - \mu(x) - \mu(y) \equiv \phi(x, y) \bmod I_\eta$,*
5. *$\mu(xa) = \overline{a}\, \mu(x)\, a$,*
6. *$H_\Lambda = H \otimes_{\mathbb{Z}[\pi]} \Lambda$ is stably based,*
7. *the map $A\phi_\Lambda : H_\Lambda \to \mathrm{Hom}_\Lambda(H_\Lambda, \Lambda)$, given by $A\phi_\Lambda(x)(y) = \phi_\Lambda(x, y)$ for all ϕ_Λ induced by ϕ, is a simple isomorphism with respect to a preferred class of stable bases and its dual.*

Definition 8.14. *We say that the special η-form $\alpha = (H, \phi, \mu)$ is strongly equivalent to zero, denoted by $\alpha \approx 0$, if there is a submodule $K \subseteq H$ such that*

1. *$\phi(x, y) = 0$ and $\mu(x) = 0$ for all $x, y \in K$;*
2. *the image of K_Λ in H_Λ is a subkernel, where subkernel is as given in Wall's definition of the L-groups.*

The special η-forms form a semigroup under orthogonal direct sum $\perp$. If $\alpha = (H, \phi, \mu)$ is a special η-form, then define $-\alpha = (H, -\phi, -\mu)$. Consider the relation $\sim$ on the collection of η-forms such that $\alpha \sim \beta$ iff $\alpha \perp (-\beta) \approx 0$. Let $\Gamma_\eta(\mathcal{F})$ be the set of equivalence classes of special η-forms under the equivalence relation generated by $\sim$. The direct sum $\perp$ induces the structure of an abelian group on $\Gamma_\eta(\mathcal{F})$. For all $k \in \mathbb{Z}_{\geq 0}$, define $\Gamma_{2k}(\mathcal{F}) = \Gamma_{(-1)^k}(\mathcal{F})$. These groups are the *homology surgery obstruction* Γ-*groups* of Cappell-Shaneson.

Remark 8.15. *When $n = 2k + 1$ is odd, the Γ-groups are subgroups of the usual L-groups; i.e. there is an injection $\Gamma_{2k+1}(\mathbb{Z}[\pi] \to \Lambda) \to L_{2k+1}(\Lambda)$. Therefore the homology surgery obstructions in odd dimensions are the usual L-theory surgery obstructions. However, the subgroup can be a proper subgroup.*

Theorem 8.16. *The normal map (f, b) determines an element $\sigma(f, b)$ in the surgery obstruction group $\Gamma^s_n(\mathcal{F})$ that only depends on the normal cobordism class of (f, b) with the property that $\sigma(f, b) = 0$ iff (f, b) is normally cobordant rel boundary to a simple Λ-homology equivalence of pairs. In fact, if $\sigma(f, b) = 0$, then f is normally cobordant to a $(k - 1)$-connected simple homology equivalence over Λ. There is an analogous result for $\Gamma^h_n(\mathcal{F})$ and homology equivalences that are not necessarily simple.*

Theorem 8.17. *(Homology surgery exact sequence) There is an exact sequence of pointed sets given by*

$$\cdots \to \Gamma^s_{n+1}(\mathcal{F}) \to \mathcal{S}^{\mathcal{F},s}(X) \to [X : F/Cat] \to \Gamma^s_n(\mathcal{F}).$$

The groups $\Gamma^s_n(\mathcal{F})$ and $\Gamma^h_n(\mathcal{F})$ generalize the Wall L-groups. If $\Lambda = \mathbb{Z}[\pi]$ and $\mathcal{F} = id$, then $\Gamma^s_n(\mathcal{F}) = L^s_n(\mathbb{Z}[\pi])$ and $\Gamma^h_n(\mathcal{F}) = L^h_n(\mathbb{Z}[\pi])$. Also $\sigma(f, b)$ is then the usual Wall surgery obstruction.

It is an exercise to show that there are manifolds $\mathbb{Z}$-homology equivalent to $\mathbb{T}^n$ with nonvanishing higher signatures.

The π-π theorem now takes the following form. Let $(X^n; Y_1, Y_2)$ be a simple Poincaré triad over $\mathcal{F} : \mathbb{Z}[\pi] \to \Lambda$, where X is connected and $\pi = \pi_1(X)$. Assume that Y_2 is connected and the map $\pi_1(Y_2) \to \pi$ is an isomorphism. Let $(f, b) : (M^n; \partial_1 M, \partial_2 M) \to (X, Y_1, Y_2)$ be a degree one normal map such that $f|_{\partial_1 M} : (\partial_1 M, \partial(\partial_1 M)) \to (Y_1, \partial Y_1)$ is a simple homology equivalence over Λ. We are now in a position to make definitions of relative groups in the manner of Wall Chapter 9.

Theorem 8.18. *Suppose that $n \geq 6$. The normal map (f, b) is normally cobordant rel $\partial_1 M$ to a simple homology equivalence of triads over Λ. A similar result holds in the non-simple case.*

Consider finitely generated groups π and π' and rings Λ and Λ' with maps that fit into the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[\pi] & \xrightarrow{\mathcal{F}} & \Lambda \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ \mathbb{Z}[\pi'] & \xrightarrow[\mathcal{G}]{} & \Lambda' \end{array}$$

and let $\phi = (\alpha, \alpha')\colon (\mathbb{Z}[\pi], \Lambda) \to (\mathbb{Z}[\pi'], \Lambda')$. One can define $\Gamma_n^s(\phi)$ and $\Gamma_n^h(\phi)$ for all $n \geq 6$. There are natural maps $\Gamma_n^s(\phi) \to \Gamma_n^h(\phi)$ for all such n.

Proposition 8.19. *For $n \geq 6$ there is a natural exact sequence*

$$\cdots \longrightarrow \Gamma_{n+1}^s(\phi) \xrightarrow{\partial} \Gamma_n^s(\mathcal{F}) \xrightarrow{\phi_*} \Gamma_n^s(\mathcal{G}) \longrightarrow \Gamma_n^s(\phi) \longrightarrow \cdots$$

A similar sequence holds for s replaced with h.

When $\Lambda = \mathbb{Z}[\pi]$ and $\Lambda' = \mathbb{Z}[\pi']$ and the maps are induced by group homomorphisms, there are natural maps $j_*\colon \Gamma_n^s(\phi) \to L_n^s(\alpha')$ so that the diagram

$$\begin{array}{ccccccc} \Gamma_{n+1}^s(\phi) & \xrightarrow{\partial} & \Gamma_n^s(\mathcal{F}) & \xrightarrow{\phi_*} & \Gamma_n^s(\mathcal{G}) & \longrightarrow & \Gamma_n^s(\phi) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\ L_{n+1}^s(\alpha') & \xrightarrow[\partial]{} & L_n^s(\mathbb{Z}[\pi]) & \xrightarrow[\alpha'_*]{} & L_n^s(\mathbb{Z}[\pi']) & \longrightarrow & L_n^s(\alpha') \end{array}$$

commutes, with a similar diagram when s replaced with h.

These relative groups solve a relative surgery problem. Let

$$(f, b)\colon (M^n; \partial_1 M, \partial_2 M) \to (X, Y_1, Y_2)$$

be a degree one normal map into the simple Poincaré triad (X, Y_1, Y_2) over $\mathcal{G}$ with $\pi' = \pi_1(Y_2)$ and $\pi = \pi_1(X)$. Let α the natural map induced by inclusion. Assume that $(Y_2, \partial Y_2)$ is a simple Poincaré pair over $\mathcal{F}$. Assume that $f_1 : \partial_1 M \to Y_1$ is a simple homology equivalence over Λ' and $f_1|_{\partial(\partial_1 M)} : \partial(\partial_1 M) \to \partial Y_1$ is a simple homology equivalence over Λ'.

Theorem 8.20. *Let $n \geq 6$. The element $\sigma(f, b) \in \Gamma^s(\phi)$ vanishes iff (f, b) is normally cobordant rel $\partial_1 M$ to a pair $(g, c)\colon (N, \partial_1 N, \partial_2 N) \to (X, Y_1, Y_2)$, where g is a simple homology equivalence over Λ' and $g|_{\partial_2 N} : \partial_2 N \to Y_2$ is a simple homology equivalence over Λ.*

Theorem 8.21. *Let $n \geq 5$ and let $(f, b)\colon (M^n, \partial M) \to (X, Y)$ be a normal map to* *a simple Poincaré pair* (X, Y) *over* Λ. *Let* $\mathcal{F} : \mathbb{Z}[\pi] \to \Lambda$ *be a homomorphism with* $\pi = \pi_1(X)$. *Assume that* f *induces a simple homology equivalence over* Λ. *Then* $\Gamma^s_n(\mathcal{F}) = \Gamma^s_{n+4}(\mathcal{F})$ *and* $\sigma(f, b) = \sigma((f, b) \times \mathbb{CP}^2)$.

Using the five-lemma, we have the following corollary.

Corollary 8.22. *If* $n \geq 7$, *then we have the isomorphism* $\Gamma^s_n(\phi) \to \Gamma^s_{n+4}(\phi)$ *by taking products with* $\mathbb{CP}^2$.

Remark 8.23. *Homology surgery fails in dimension 4; see Casson-Gordon [145], Hausmann-Weinberger [299], and Cochran-Orr-Teichner [166] for different aspects of this issue that are far from understood, even in very restrictive contexts.*

8.3.1 A few applications

Here we return to the codimension two problem of Cappell and Shaneson, which is solved by methods of surgery to obtain homology equivalences. Let X be a Poincaré complex with fundamental group π. If we are given a manifold Y^{n+2} and a submanifold Y^n, the problem of making a homotopy equivalence $f : W \to Y$ transverse regular to X, with $f|_{f^{-1}(X)} : f^{-1}(X) \to X$ a homotopy equivalence, is called the *ambient surgery problem*. There is always an *abstract surgery obstruction* in $L_n(\mathbb{Z}[\pi])$ to resolve this problem. If the abstract surgery obstruction vanishes, the problem can be solved and all the manifolds homotopy equivalent to X in one normal cobordism class can occur as $f^{-1}(X)$. In even dimensions, there is an additional obstruction lying in the Γ-groups described earlier. This obstruction can be interpreted in terms of knot cobordism groups. In the case of Cappell and Shaneson, instead of executing surgery on the knot or its Seifert surface, they work on the knot complement $\mathbb{S}^{n+2} \setminus \mathrm{int}(K \times \mathbb{D}^2)$. This complement is a manifold with boundary $(X, \partial X)$, where ∂X is identified with $\mathbb{S}^n \times \mathbb{S}^1$. With a bit of homological effort one can extend the composite map $\partial X \to \mathbb{S}^n \times \mathbb{S}^1 \to \mathbb{S}^1$ to X. Indeed we obtain a map of pairs $(X, \partial X)$ to $(\mathbb{D}^{n+1} \times \mathbb{S}^1, \mathbb{S}^n \times \mathbb{S}^1)$ which is a $\mathbb{Z}$-homology equivalence and a homeomorphism on the boundary.

We first make some general observations about Γ-groups and their computation.

Proposition 8.24. *There are natural maps* $L_n(\mathbb{Z}[\pi]) \to \Gamma_n(\mathcal{F}) \to L_n(\Lambda)$ *such that,*

1. *when n is odd, the map* $\Gamma_n(\mathcal{F}) \to L_n(\Lambda)$ *is injective;*
2. *when n is even, the map* $\Gamma_n(\mathcal{F}) \to L_n(\Lambda)$ *is surjective.*

These results generality imply Kervaire's theorem on the vanishing of even-dimensional knot cobordism groups. We begin with the following square ϕ:

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}] & \xrightarrow{\mathcal{F}} & \mathbb{Z}[\mathbb{Z}] \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ \mathbb{Z}[\mathbb{Z}] & \xrightarrow[\mathcal{G}]{} & \mathbb{Z}[e] \end{array}$$

For n even, the exact sequence of the pair for computing $\Gamma_{n+3}(\phi)$ is the following:

$$\Gamma_{n+3}(\mathcal{G}) \to \Gamma_{n+3}(\phi) \to \Gamma_{n+2}(\mathcal{F}) \to \Gamma_{n+2}(\mathcal{G}).$$

The leftmost group is trivial, and $\Gamma_{n+2}(\mathcal{F}) \cong L_{n+2}(\mathbb{Z}[\mathbb{Z}]) \cong L_{n+2}(\mathbb{Z}[e])$, which injects into $\Gamma_{n+2}(\mathcal{G})$, where we make use of the map $\Gamma_{n+2}(\mathcal{G}) \to L_{n+2}(\mathbb{Z}[e])$.

There are exact sequences

$$0 \to \mathbb{Z}_2 \to \Gamma^s_{n+2}(\phi) \to \mathcal{S}^{\mathcal{G}}(\mathbb{S}^1 \times \mathbb{D}^{n+1}) \to \pi_1(F/Cat) = 0$$

and

$$L^s_{n+2}(\mathbb{Z}[e]) \to \Gamma_{n+2}(\mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}) \to \Gamma^s_{n+2}(\phi) \to L_{n+1}(\mathbb{Z}[\mathbb{Z}]) \cong \Gamma_{n+1}(\mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}]).$$

Remark 8.25. *Note that the exact same reasoning would show that the group of concordance classes of fixed knots of $\mathbb{Z}_p$-actions is trivial in even dimensions. If we have maps $\mathcal{G}\colon\ \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[e]$ and $\mathcal{G}_p\colon\ \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}_p]$, then we would merely need to replace each occurrence of $\Gamma_n(\mathcal{G})$ with $\Gamma_n(\mathcal{G}_p)$.*

We can also define the relative structure set $\mathcal{S}^{\mathcal{G}}(\mathbb{D}^{n+1} \times \mathbb{S}^1 \text{ rel } \mathbb{S}^n \times \mathbb{S}^1)$ to be the class of such manifolds up to cobordisms. The superscript $\mathcal{G}$ indicates that the manifolds have fundamental group $\mathbb{Z}$, but that we are only trying to obtain homology isomorphisms in the smaller quotient ring $\mathbb{Z} = \mathbb{Z}[e]$. However, we can work not relative to the boundary since it incorporates the fact that the structure sets of $\mathbb{S}^n \times \mathbb{S}^1$ vanish, enabling us to change the normal invariants to $[\mathbb{D}^{n+1} \times \mathbb{S}^1 : F/Cat]$, which actually vanishes. By doing so, we can obtain an isomorphism of the relevant structure set, i.e. the knot cobordism group, with $\Gamma_{n+3}(\phi)$.

We obtain, in this case, a sequence rel boundary:

$$\cdots \to \Gamma_{n+3}(\mathcal{G}) \to \mathcal{S}^{\mathcal{G}}(\mathbb{D}^{n+1} \times \mathbb{S}^1)_{\text{rel}} \to [(\mathbb{D}^{n+1} \times \mathbb{S}^1)/(\mathbb{S}^n \times \mathbb{S}^1) \colon F/Cat] \to \Gamma_{n+2}(\mathcal{G}).$$

The calculation of knot cobordism groups or equivariant knot cobordism groups requires us to understand the maps $\Gamma_n(\mathcal{G})$ or $\Gamma_n(\mathcal{G}_p)$. The works of Cappell-Shaneson [122] on nonsingularity properties of subkernels and Vogel [656] demonstrate the following result, originally due to Kervaire.

Theorem 8.26. *The knot cobordism groups are infinitely generated in odd dimensions* *and vanish in even dimensions.*

The papers of Cappell-Shaneson [122, 123, 125] give a number of applications to a number of problems about codimension two embeddings: locally flat and nonlocally flat of connected and disconnected manifolds, i.e. generalized knot and link theory. In fact, their work gives versions in codimension 2 of the embedding theory of Browder-Casson-Haefliger-Sullivan-Wall for codimension at least 3 (see Wall [672] Corollary 11.3.1).

A completely different application of these ideas was given by Hausmann [295, 297] to the analysis of homology spheres with fundamental group π.

An *(integral) m-dimensional Cat homology sphere* Σ^m is a closed Cat manifold that shares the same $\mathbb{Z}$-homology groups as the standard sphere $\mathbb{S}^m$. In other words, the homology of Σ^m satisfies $H_0(\Sigma;\mathbb{Z}) \cong \mathbb{Z} \cong H_m(\Sigma;\mathbb{Z})$ and $H_i(\Sigma;\mathbb{Z}) = 0$ for all other i. A homology sphere Σ might not be simply connected, but clearly the abelianization of its fundamental group is trivial. Indeed, the Poincaré conjecture asserts that, if $\pi_1(\Sigma) = 0$, then Σ is homeomorphic to the standard sphere $\mathbb{S}^n$. The most well-known example of a non-trivial homology sphere is the three-dimensional Poincaré dodecahedral sphere, whose fundamental group is the binary icosahedral group of order 120. Integral homology spheres can be used to construct many interesting examples.

The following classical theorem of Kervaire tells us the possibilities for π.

Theorem 8.27. *If $n \geq 5$, a group π is the fundamental group of some closed homology n-sphere iff the following three conditions all hold:*

1. *π is finitely presented;*
2. *$H_1(B\pi;\mathbb{Z}) = 0$;*
3. *$H_2(B\pi;\mathbb{Z}) = 0$.*

The necessity of the first two are obvious. For the third, note that, if X is a space with fundamental group π, then an Eilenberg-Maclane space $K(\pi, 1)$ can be obtained by attaching 3-cells and higher, giving a surjection $H_2(X, \mathbb{Z}) \to H_2(B\pi;\mathbb{Z})$. Sufficiency is a simple application of surgery below the middle dimension. A homology sphere can be built as the boundary of a regular neighborhood of an immersion of an acyclic 3-complex X with fundamental group π into Euclidean space of dimension 6 or higher.

A group satisfying condition (1) is called a *perfect group*, and a group satisfying conditions (1) and (2) is called *superperfect*.

Remark 8.28. *Note that the homology sphere constructed here bounds an acyclic manifold with fundamental group π. Not every homology sphere has this property, e.g. non-simply connected three-dimensional homology spheres, as they do not even bound manifolds with fundamental group π.*

We now define a group $\Theta_n^{Cat}(\pi)$ of oriented homology n-spheres whose fundamental group admits a surjection to π; two elements are identified if their disjoint union is the boundary of a homology h-cobordism with the same condition.

In the following, we describe the Quillen plus-construction on an Eilenberg-Maclane space of the form $K(\pi, 1)$. We define the space $K(\pi, 1)^+$ to be the unique homotopy type such that, for any map f from $K(\pi, 1)$ to a simply connected space T, there is a unique factorization of f through $K(\pi, 1) \to K(\pi, 1)^+$. The map $K(\pi, 1) \to K(\pi, 1)^+$ induces an isomorphism on integral homology, but the space $K(\pi, 1)^+$ is simply connected. This $K(\pi, 1)^+$ can be built by attaching 2-cells to kill π_1, and then by attaching 3-cells for each of the new generators of the second homology group that are newly produced. One easily verifies that the result has the desired properties. For more information about the plus construction, see Quillen [515]. Hausmann proves that, under the operation of connected sum, we can relate $\Theta_n(\pi)$ to the Eilenberg-Maclane space $K(\pi, 1)$.

The following is a theorem of Hausmann that determines an isomorphism between Γ-groups and L-groups under a concrete condition.

Theorem 8.29. *(Hausmann [295]) Let $\psi : \pi \to \pi'$ be an epimorphism and $w_{\pi'} : \pi' \to \mathbb{Z}_2$ be a homomorphism. Denote by $\mathcal{F} : \mathbb{Z}[\pi] \to \mathbb{Z}[\pi']$ the induced map. Suppose that $N = \ker \psi$ is the normal closure of a finitely generated perfect group $N_0 \subseteq \pi$. There are isomorphisms $j_n^s : \Gamma_n^s(\mathcal{F}) \to L_n^s(\mathbb{Z}[\pi'], w_H)$ and $j_n^h : \Gamma_n^h(\mathcal{F}) \to L_n^h(\mathbb{Z}[\pi'], w_H)$ for all n.*

Theorem 8.30. *(Hausmann [296]) Let π be the fundamental group of a closed homology sphere. For $n \geq 5$ we have $\Theta_n^{Cat}(\mathbb{Z}[\pi]) = \Theta_n^{Cat} \oplus \pi_n(K(\pi, 1)^+)$. In particular, if Cat is PL or Top, we have $\Theta_n^{Cat}(\mathbb{Z}[\pi]) = \pi_n(K(\pi, 1)^+)$.*

Proof. (Sketch) The proof has two parts. First one proves that $\Gamma_n(\mathbb{Z}[\pi] \to \mathbb{Z}) \cong L_n(\mathbb{Z}[e])$. It is trivial in odd dimensions, and in even dimensions one can prove the isomorphism by executing surgery below the middle dimension, and finally embed all the spheres in the kernel using a homology sphere with fundamental group π rather than a standard sphere. The obstruction to embedding these homology spheres disjointly is no different than the obstruction in simply connected surgery: indeed, there are enough loops in the submanifold to enable Whitney tricks to amalgamate elements from all over π. Recall that, in defining intersection numbers, one needs paths on the various submanifolds from the intersection point to a basepoint. If the submanifolds are not simply connected, there is an indeterminacy that makes the construction of Whitney disks more possible.

For the second part, note that the relevant "Poincaré complexes" that initiate the surgery process are $\mathbb{Z}$-homology spheres Σ with fundamental group π. Every such sphere admits a degree one normal map, since its suspension is a homotopy sphere. The normal invariants are then in a bijective correspondence with $\pi_n(F/Cat)$, and the $\mathbb{Z}$-structures on X are then just Θ_n^{Cat} by comparison to the surgery exact sequence for homotopy spheres.

Let X be a homology sphere with fundamental group π. Since the plus construction is functorial, the natural map $X \to K(\pi, 1)$ induces a map $X^+ \to K(\pi, 1)^+$. However, using the orientation, there is a canonical homotopy equivalence $\mathbb{S}^n \to X^+$. Composing these maps, we can assign to X an element of $\pi_n(K(\pi,1)^+)$. Conversely, given a homotopy equivalence $\mathbb{S}^n \to K(\pi,1)^+$, one can define a $\mathbb{Z}$-homology sphere with fundamental group π by taking the pullback

$$\begin{array}{ccc} X & \longrightarrow & K(\pi,1) \\ \downarrow & & \downarrow \\ \mathbb{S}^n & \longrightarrow & K(\pi,1)^+ \end{array}$$

to serve as the relevant X. Therefore the homotopy group acts as the relevant parametrization for the underlying Poincaré homotopy theory. □

Remark 8.31. *Le Dimet [391] uses similar principles to develop a theoretical description of concordance classes of the union of disk links with $\mathbb{D}^n$ in $\mathbb{D}^{n+2}$ that are trivial on the boundary. Quillen's plus construction is replaced by a "Vogel localization" of a wedge of circles, which is a variation of the construction of Bousfield-Kan, which is, in turn, a variant of the Quillen plus construction.*

8.4 PROPER AND BOUNDED SURGERY

Up to this point in the text, the surgery that we have explicitly discussed has been on compact manifolds. The exception was our discussion of nonuniform arithmetic manifolds, where we were able to avoid a systematic analysis by resorting to the π-π theorem. However, noncompact manifolds are ubiquitous in mathematics, and cannot be avoided, even if one is only interested in closed manifolds. In this section, we first discuss the basic "uncontrolled" theory of noncompact manifolds, called *proper surgery*, i.e. surgery to obtain a proper homotopy equivalence. We will then discuss *bounded surgery*, where maps are not allowed to move points very far during homotopies, which has been invaluable since the 1980s. See Siebenmann [587] and Taylor [638].

The main sources of this proper theory are the works of (1) Siebenmann for the Whitehead theory, and (2) Taylor for the surgery. Just as for manifolds with boundary, the classification theory assumes that comparison maps are given as maps of pairs; in the noncompact theory, one works in a fixed proper homotopy type.

Definition 8.32. *A continuous map $f : X \to Y$ is proper if, for all compact $K \subseteq Y$, the inverse image $f^{-1}(K) \subseteq X$ is compact.*

Remark 8.33. *Paracompact manifolds with proper maps form a category.*

Note, for example, that degree theory makes sense for proper maps between (potentially noncompact) manifolds, but not for continuous maps. A map f is a *proper homotopy equivalence* if there is $g : Y \to X$ so that both $f \circ g$ and $g \circ f$ are properly homotopic to the identity. A proper homotopy between maps is of course a proper map from $X \times [0, 1]$ that restricts on the ends to the given maps.

Unlike compact manifolds, there are uncountably many noncompact manifolds, so this theory is much richer and much more complicated. However, it does become more manageable when restricted to manifolds satisfying Siebenmann's tameness condition, which will be discussed in the next few pages. Let us work by analogy with manifolds M with boundary ∂M. For these manifolds, the h-cobordism theorem asserts that $\mathrm{Wh}(M, \partial M) = \mathrm{Wh}(\pi_1(M)) \times \mathrm{Wh}(\pi_1(\partial M))$. For noncompact manifolds, we will have to interpret ∂M and $\mathrm{Wh}(\pi_1(\partial M))$ appropriately to describe the answer; even then the Whitehead groups of the pair will not break into a sum and will be more complicated and interesting.

Following earlier work of Browder-Levine-Livesay [91], Siebenmann [587] suggested the following. Let M be a noncompact space endowed with a complete metric d. Fix a point $p \in M$ and consider the function $f : M \to [0, \infty)$ given by $f(x) = d(x, p)$ for all $x \in M$. Using sublevel sets, we obtain an exhaustion of M by compact sets $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ and therefore an inverse system $M - K_1 \supseteq M - K_2 \supseteq M - K_3 \supseteq \cdots$.

Of course, the exhaustion will depend strongly on the metric, so constructions should occur in a category in which we eliminate such indeterminacy. For example, we can consider the components of these complementary regions, but it will not quite be well-defined. If we were in a Euclidean space and defined K_n to be the union of the ball of radius n around the origin and an annular shell from $2n$ to $3n$, then each complement would have 2 components, but this sequence of sets would be "pro-equivalent" to the constant system with just one component: one component is there from all the K_m with $m > n$ but one is not. The elements of the inverse limits of the sequence are the ends of M. It can be shown that, for one-ended manifolds, one can increase the size of any exhaustion so that the complementary regions are all connected. Then we can consider the inverse sequence of fundamental groups. In the many-ended case, one would require the category of groupoids.

Let us concentrate on the group case, i.e. the oriented case. We say that an inverse sequence $\pi_1 \leftarrow \pi_2 \leftarrow \pi_3 \leftarrow \cdots$ of groups is *pro-trivial*, if for each index i, there is n_i for which the composition $\pi_{n_i} \to \cdots \to \pi_i$ is trivial. The inverse system $\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \cdots$, where the arrows are all multiplication by 2, has trivial inverse limit, but it is not pro-trivial.

Siebenmann's thesis studies the condition for which M is isomorphic to the interior of a manifold with boundary. The fundamental group(oid) condition would be that the above pro-system is equivalent, i.e. using an appropriate commutative diagram, to $\pi \leftarrow \pi \leftarrow \pi \leftarrow \cdots$ for some group(oid) π, where each map $\pi \leftarrow \pi$ in the system is the identity. This condition is called *fundamental group tameness*. In dimensions at least 5, a noncompact manifold satisfies fundamental group tameness iff there is a sequence $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ of compact manifolds with boundary as above, so that all the annular regions $K_{i+1} \backslash K_i$ and the separating hypersurfaces ∂K_i have fundamental group π. If there is a manifold W with boundary such that $\mathrm{int}(W) = M$, then π is the fundamental group of the boundary ∂W.

While the general theory in Siebenmann and Taylor does not require fundamental group tameness, the situation is much more complicated without this condition, and the calculations of Whitehead and L-groups will involve $\lim^1$ of the induced inverse limit system on Whitehead and L-groups of the fundamental groups of the neighborhoods of ∞. It should not be surprising; cohomology and locally finite homology for noncompact spaces also involve a $\lim^1$ term in this calculation from finite approximations. When the end is fundamental group tame, this sequence is constant, and then the inverse limit system has no surprises, and $\lim^1$ vanishes.

Theorem 8.34. *(Siebenmann) Let M be a noncompact manifold, which is fundamental group tame. Then there is a group $\mathrm{Wh}^p(M)$, called the proper Whitehead group of M, which is in bijective correspondence with proper h-cobordisms with boundary M. There is an exact sequence*

$$\cdots \to \mathrm{Wh}(\pi_1^\infty(M)) \to \mathrm{Wh}(\pi_1(M)) \to \mathrm{Wh}^p(M) \to \widetilde{K}_0(\pi_1^\infty(M)) \to \widetilde{K}_0(\pi_1(M))$$

that calculates $\mathrm{Wh}^p(M)$.

Some special cases are worth noting.

1. If M is simply connected at ∞, then $\mathrm{Wh}(\pi_1(M)) \to \mathrm{Wh}^p(M)$ is an isomorphism.
2. If M is simply connected, then $\mathrm{Wh}^p(M) \to \widetilde{K}_0(\pi_1^\infty(M))$ is an isomorphism. Assuming tameness, this map takes the (relative) Siebenmann end obstruction of a neighborhood of ∞. The reason for projective modules is that Wall finiteness theory arises.
3. If M is π-π, i.e. $\pi_1^\infty(M) \to \pi_1(M)$ is an isomorphism, then $\mathrm{Wh}^p(M) = 0$.
4. If $M = N \times \mathbb{R}$ is the product of a compact manifold N and the reals line $\mathbb{R}$, then $\mathrm{Wh}^p(M) \to \widetilde{K}_0(\pi_1(M))$ is an isomorphism.

In any case, the proper Whitehead theory is naturally a mixture of K_0 and Wh phenomena. We can consider Wh^p as a relative K-group.

There is a similar L-theory situation giving a surgery exact sequence due to Taylor [638]. In other words, there are *proper Wall groups* $L_n^p(M)$ such that the following is exact:

$$\cdots \to H_{n+1}^{lf}(M;\mathbb{L}_\bullet) \to L_{n+1}^{pr}(M) \to \mathcal{S}^{pr}(M) \to H_n^{lf}(M;\mathbb{L}_\bullet) \to L_n^{pr}(M).$$

The definition of *proper structure set* $\mathcal{S}^{pr}(M)$ above should be clear.

The corresponding calculations of the proper L-groups in the situations above for proper homotopy equivalence are:

(a) $L_n^{pr}(M) \cong L_n^h(\pi_1(M))$,

(b) $L_n^{pr}(M) \cong L_n^p((e, \pi_1(M)) \cong \widetilde{L}_{n-1}^p(\pi_1(M))$,

(c) $L_n^{pr}(M) = 0$,
(d) $L_n^{pr}(M) \cong L_{n-1}^{p}(\pi_1(M))$,
(e) $L_n^{pr}(M) \cong L_n(\pi_1(M), \pi_1^{\infty}(M))$ modulo decoration, if M is tame.

We now shift gears and move onto the bounded category. Let us now consider an element $f : W \to M$ that represents the trivial element in the proper structure set $\mathcal{S}^{pr}(M)$. Since f is proper, the diameter of all point-inverses, or even the inverse image of a compact set, is finite, although it will typically not be uniformly bounded. Because it represents the trivial element, there is a proper homotopy $F : W \times [0,1] \to M$ of f to a homeomorphism. For each $w \in W$, the tracks $F(w \times [0,1])$ of the homotopy will also be compact. The function $g : W \to \mathbb{R}$ given by $g(w) = \operatorname{diam} F(w \times [0,1])$ is then a continuous function.

There are circumstances in which one can ask for more. We have already considered the sizes of tracks of homotopies when we studied the α-approximation theorem. Now we will ask that the tracks are bounded in size in terms of a specific metric on M. This idea leads us to the bounded category, bounded homotopy equivalences, and bounded surgery.

As an example, suppose that $f : W \to M$ is proper for the simple reason that it commutes with a cocompact action of a discrete group. Let us assume invariant metrics are chosen on all spaces; i.e. one takes a regular cover of a homotopy equivalence between compact manifolds. Then the sizes of point inverses are actually bounded uniformly, rather that just locally uniformly, and we are naturally in the bounded category. Here is an example.

Example 8.35. *If one takes a compact surgery problem $f : M \to X$ and crosses with $\mathbb{R}^3$, one can properly solve the problem by case (3) above. The proper homotopy equivalence extends as a homotopy equivalence between one-point compactifications. However, if one keeps track of the projection to $\mathbb{R}^3$ with the desire that the final homotopy equivalence be bounded over $\mathbb{R}^3$, one might encounter an obstruction. Such attention to the projection would be necessary if we wanted to compactify $X \times \mathbb{R}^3$ by glueing on a 2-sphere at infinity. In fact, we only need smallness in the "angular sense," and the bounded category is somewhat excessive for this purpose. This type of control gives the "continuously controlled at infinity" category. It is the case, however, that bounded and continuous control give the same invariants in this case.*

Definition 8.36. *Let $f : X \to Y$ be a function between metric spaces. Then the oscillation or modulus of discontinuity of f at x is $\lim_{r \to 0} \sup_y (d(f(x), f(y))$, where the supremum is taken over the ball of radius r around x. The coarse modulus will just take a fixed radius for measuring the local oscillation.*

Remark 8.37. *Continuous functions have 0 modulus of discontinuity at every point.*

Definition 8.38. *Let X be a metric space. A space over X is a space M, equipped with a function $f_M : M \to X$, perhaps discontinuous, called the reference map. We* *assume that f_M has bounded oscillation or equivalently a bounded coarse modulus of discontinuity.*

Definition 8.39. *The objects of the bounded category $C^b(X)$ of X are spaces M over X, and the morphisms are continuous functions $g : M \to N$ such that $g \circ f_N$ and f_M are uniformly bounded distance apart; i.e. there is c such that $d(g \circ f_N, f_M) < c$.*

Remark 8.40. *Note that, if two reference maps are uniformly bounded distance apart, then the identity map defines an isomorphism between these objects, and they define isomorphic objects in $C^b(X)$.*

The following notion is fundamental. If (X, d_X) and (Y, d_Y) are metric spaces, then we say they are *coarse quasi-isometric* if there is a map $f : X \to Y$ along with positive constants A and B such that

$$\frac{1}{A} d_X(x_1, x_2) - B < d_Y(f(x_1), f(x_2)) < A d_X(x_1, x_2) + B$$

for all $x_1, x_2 \in X$, and for some C every point of Y is within C of $f(X)$. The map f is called a *coarse quasi-isometry between X and Y*. If such f exists, then $C^b(X)$ and $C^b(Y)$ are in bijective correspondence with each other. Therefore, we can identify $C^b(X)$ and $C^b(\Gamma)$ if Γ acts proper-discontinuously on X with compact quotient. The inclusion of an orbit into X is a coarse quasi-isometry. Therefore the universal cover of a semilocally 1-connected compact space is coarse quasi-isometric to its fundamental group with the word metric.

This example shows the great benefit of allowing discontinuous reference maps. In practice one frequently replaces Γ by a uniformly contractible space X containing it, i.e. one for which there is a function $\rho : [0, \infty) \to [0, \infty)$ so that all balls of radius r are null-homotopic in balls of radius $\rho(r)$. For example, we can use $E\Gamma$ if $B\Gamma$ is a finite complex. Then any finite-dimensional object in $C^b(X)$ can be replaced by one where the reference map is continuous. We also note that, for example, proper Lipschitz maps induce natural transformations among these categories.

Having built the category $C^b(X)$, we can now discuss homotopy of maps in $C^b(X)$, homotopy equivalences, Whitehead groups $\mathrm{Wh}^b(M \downarrow X)$, and structure sets $\mathcal{S}^b(M \downarrow X)$.

Definition 8.41. *In the above context, the bounded structure set $\mathcal{S}^b(M \downarrow X)$ is the collection of bounded structures for M, i.e. a bounded homotopy equivalence $M' \to M$, where M' is also a manifold bounded over X, and two such structures are equivalent if they are related by a bounded Cat-isomorphism. We also define the bounded Whitehead groups* $\mathrm{Wh}^b(M \downarrow X)$ *to be the collection of polyhedra P that are bounded homotopy equivalent to M over X up to expansions and collapses of bounded size.*

Example 8.42. *Suppose that W is a noncompact manifold endowed with a metric so that the distance function $d(p, \cdot) : W \to [0, \infty)$ from a basepoint p is a coarse quasi-isometry. Such a distance function can be found iff W is one-ended. Then there are* *maps*

$$\mathrm{Wh}^b(W \downarrow [0,\infty)) \to \mathrm{Wh}^p(W),$$
$$\mathcal{S}^b(W \downarrow [0,\infty)) \to \mathcal{S}^p(W).$$

Now suppose that for $n \in \mathbb{Z}_{\geq 1}$ the maps

$$d^{-1}(n) \to d^{-1}([n, n+1]) \leftarrow d^{-1}(n+1)$$

are all π_1-isomorphisms. Such a condition can be arranged in dimensions at least 5 iff W is π_1-tame. Using the π-π codimension one splitting theorem, one can then see that the maps are isomorphisms.

Remark 8.43. *When $\widetilde{K}_0(\mathbb{Z}[\pi]) = 0$, this example can be viewed as an exercise using the method of proof of Browder and Wall's splitting theorem 1.77, applied repeatedly to the $d^{-1}(n)$. One can consult the work of Siebenmann on the proper h-cobordism [587] for the additional ideas necessary when this condition is eliminated.*

In the following example, we consider a pathological case, and then we discuss how to avoid it.

Example 8.44. *Let M^n be a compact manifold, and let W^n be M^n with a puncture which we will map to $[0,\infty)$ as follows. Let w be a point of W and let Σ_i be a sequence of codimension one spheres surrounding w such that Σ_i and Σ_{i+1} are separated from one another by a sphere. For each i, take a small neighborhood of Σ_i and stretch it to map to $[3^i, 2 \cdot 3^i]$, building the remaining part of the map by the Tietze extension theorem. Then one can easily see that there is a map*

$$\prod \widetilde{L}_n^h(\mathbb{Z}[\pi_1(\Sigma_i)])/\bigoplus \widetilde{L}_n^h(\mathbb{Z}[\pi_1(\Sigma_i)]) \to \mathcal{S}^b(W \downarrow [0,\infty))$$

which is zero in $\mathcal{S}^p(W) \cong \mathcal{S}^h(M)$. Moreover, this map actually splits back by approximate versions of the bounded control results that we discuss below. In any case, we see here that there is often much more information available when we only allow bounded distance movement. This group is typically uncountable.

This pathology is related to, but not identical to, the situation of non-tame ends in the proper theory. To avoid it, we will assume for simplicity that, for our space M, there is a ball of radius 1 centered at every point $x \in X$, at least outside of a compact set K, so that the map $\pi_1(f^{-1}(B_1(x))) \to \pi_1(W \backslash f^{-1}(K))$ is an isomorphism. In this case, one can frequently calculate the *bounded L-groups*, which are the L-groups relevant to this theory.

A special case, essentially due to Chapman [159], is the following:

Theorem 8.45. *If N is compact with dimension n, then*

$$L_{n+k}^{s,b}(N \times \mathbb{R}^k \downarrow \mathbb{R}^k) \cong L_n^{1-k}(\pi_1(N)).$$

Chapman showed that every bounded structure on $N\times\mathbb{R}^k$ is actually a transfer-invariant cover of a manifold homotopy equivalent to $N \times \mathbb{T}^k$. If one thinks about the Shaneson formula for $N \times \mathbb{T}^k$, there are various terms associated to subtori. However, all but the innermost, corresponding to $\mathbb{T}^0$, map non-trivially on an r^k-fold cover that unwraps all directions equally. The change in decoration is the one that occurs in the Shaneson formula, i.e. $\mathcal{S}^s(N\times\mathbb{S}^1) = \mathcal{S}^s(N\times I)\times\mathcal{S}^h(N)$. Similarly, the formula for $\mathcal{S}^h(N\times\mathbb{S}^1) = \mathcal{S}^h(N \times I) \times \mathcal{S}^p(N)$ descends to a projective term in codimension 1. One is led to L-groups whose decorations are parametrized by negative K-groups.

These negative K-groups, which occur for example in an interated Bass-Heller-Swan formula for multiple Laurent series, also arise as the relevant bounded Whitehead groups (Pederson-Weibel [502]) for the relevant bounded h-cobordism theorem

$$\mathrm{Wh}^b(N \times \mathbb{R}^k \downarrow \mathbb{R}^k) \cong \widetilde{K}_{1-k}(\pi_1(N)).$$

Here reduction for K_1 gives the Whitehead group; the reduction for K_0 is the usual reduction, i.e. projective modules modulo stably free modules; and for K_{-1} and below, reduction does nothing. Note that $\mathrm{Wh}^b(M \times \mathbb{R} \downarrow \mathbb{R}) \cong \mathrm{Wh}^p(M \times \mathbb{R})$, as before.

We will soon put these theorems in a somewhat broader context that greatly increases their utility. Let us now apply the above theorem to prove Novikov's theorem about the homeomorphism invariance of rational Pontrjagin classes. This argument is revisionist, not circular.

Theorem 8.46. *Let $M^{4n+k} \in C^b(\mathbb{R}^k)$ and let $h:\ M' \to M$ be a homotopy equivalence in this category with M and M' smooth. Then*

$$f_{M^*}(L_n(M) \cap [M]) = f_{M'^*}(L_n(M') \cap [M'])$$

in $H_k^{lf}(\mathbb{R}^k; \mathbb{Q})$.

This result is clearly a bounded analogue of the Novikov conjecture. See Weinberger [691]. The asserted equality is exactly the fact that the codimension k signature obtained from h is a bounded surgery obstruction. The case for simply connected N appears in the theorem above. Note that smoothness is used to identify this signature with the L-class through the Hirzebruch formula.

The result above implies Novikov's theorem.

Theorem 8.47. *Pontrjagin classes of smooth manifolds are homeomorphism invariant.*

Proof. Suppose that $f:\ A \to B$ is a homeomorphism between smooth manifolds. We replace A and B by their products with large-dimensional spheres. Let h be the induced map. We can rationally represent any homology class by submanifolds with trivial normal bundle. Given any homology class, let V be the corresponding submanifold of B with neighborhood $V \times \mathbb{R}^k$ in B, and let $h^{-1}(V \times \mathbb{R}^k)$ be the inverse image in A. Then $V \times \mathbb{R}^k$ and $h^{-1}(V \times \mathbb{R}^k)$ are two smooth manifolds in $C^b(\mathbb{R}^k)$ and the restriction of h to them is a homotopy equivalence in this category. The theorem then shows that the L-classes of A and B evaluate equally on this homology class, which then implies Novikov's theorem. □

Corollary 8.48. *The Novikov conjecture is true for the fundamental class of a closed nonpositively curved manifold V.*

Proof. If $h: M' \to M$ is a homotopy equivalence and $f: M \to V$ is induced by a map of fundamental groups, then the relevant higher signature associated to $[V]$ can be computed by taking the universal cover of V, and considering the lift of h to the associated cover. However, since this cover of V is a Hadamard manifold, there is a Lipschitz homeomorphism from the universal cover of $V \to \mathbb{R}^v$ given by the inverse of the exponential map. Then the higher signatures of M and M' are equal to the ones whose equality is given by the theorem above. □

Remark 8.49. *With more effort, this proof can be modified to prove the Novikov conjecture for the fundamental groups of such manifolds, even without compactness. See Ferry-Weinberger [245]. The compact case is easier and can be found in Farrell-Hsiang [226].*

Remark 8.50. *There is an interesting example that will lead us to a pair of critical theorems. Suppose that M is a closed manifold with a metric so that small balls are contractible. Let us consider $\mathcal{S}^b(M \times [0,\infty) \downarrow cM)$, where cM is the open cone of M. We ask about maps $W \to M \times [0,\infty)$. The boundary gives us a homotopy equivalence $M' \to M$, but as we approach infinity, we are closer (after rescaling) to the situation of the α-approximation theorem. Therefore we can obtain a homeomorphism near infinity. By using the s-cobordism theorem, we can execute a finite movement on the compact part to obtain a homeomorphism. These observations suggest the following theorem.*

Theorem 8.51. *If M is given as above, then $\mathcal{S}^b(M \times [0,\infty) \downarrow cM) = 0$.*

If we insert this result into the surgery exact sequence, we conclude that $L^b(N \downarrow cM) \cong \widetilde{H}(M; \mathbb{L}_\bullet)$ when N is simply connected. For the non-simply connected case, see Theorem 8.53 below.

This reasoning can be used to prove the following theorems, which are analogues of Quinn's main theorem in [521].

Theorem 8.52. *(Pedersen-Weibel [502]) Suppose M has fundamental group π over the cone cP, where P is a finite polyhedron. Then $\mathrm{Wh}^b(M \downarrow cP)$ is isomorphic to $H_0^{lf}(cP; \mathbb{W}_\bullet(\pi))$, where $\mathbb{W}_\bullet(\pi)$ is a spectrum whose homotopy groups are Whitehead groups in nonnegative dimensions, and are negative K-groups in negative dimensions.*

These groups of course play the expected roles in h-cobordism and boundarization problems.

Theorem 8.53. *(Ferry-Pedersen [243]) Suppose that M has fundamental group π over the cone cP for which P is a finite polyhedron, and suppose that* $\mathrm{Wh}(\pi \times \mathbb{Z}^k) = 0$

for all k (e.g. π trivial or free abelian or $\mathbb{Z}_2$). Then $L_n^b(M \downarrow cP)$ is isomorphic to $H_n^{lf}(cP; \mathbb{L}_\bullet(\pi))$.

Note that these homology groups can be considered as reduced homology groups of P. Indeed, these theorems in the original papers are proved by induction on the cells of P, and the hardest part is to verify a Mayer-Vietoris property. This induction gives a different approach to the case of $cP = \mathbb{R}^k$ used above.

Remark 8.54. *Unfortunately, the K-theory obstructions that obstruct codimension splitting prevent any one decoration from being adequate to express L^{bdd} even for general cones. The usual solution is to work with $L^{-\infty}$, which is the limit of the L-groups where one weakens the K-theoretic hypotheses more and more, or equivalently*

$$\lim_{k\to\infty} L_{n+k}^{bdd}(\,\cdot \times \mathbb{R}^k \downarrow \mathbb{R}^n).$$

Then one can relate the relevant L-groups to the $L^{-\infty}$. This issue, of course, also arises for the Farrell-Jones conjecture.

8.5 CONTROLLED SURGERY

In classical surgery theory, one begins with a Poincaré duality space X and a degree one normal map $f : M \to X$. The problem is to vary (M, f) by a normal cobordism to obtain a homotopy equivalence $f' : M' \to X$. Sometimes it is necessary to have an ε or controlled version of surgery theory. Therefore X comes equipped with a reference map to a metric space K, and the aim is to produce a homotopy equivalence $f' : M' \to X$ which is small measured in K. The existence of such an f' implies that X is a small Poincaré duality space in the sense that cells are close to their dual cells, measured in K. The assumption that X is a small Poincaré duality space must therefore be part of the original data.

We only discuss simply connected controlled topology, since the goal is to introduce the ideas of control, and not the complications of the fundamental group. See Quinn [520–523], Ferry-Pedersen [243], and Ferry [235]. The papers of Hughes [326] and Hughes-Taylor-Williams [329] give an alternate approach. However, the stratified surgery theory in the last section of the chapter will rely on the more general theory.

In this section, we begin by reviewing some ideas from Section 7.1 leading up to the α-approximation theorem.

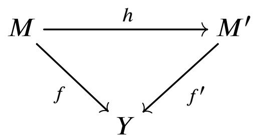
Definition 8.55. *Let $\varepsilon > 0$ and let B be a metric ANR. Suppose that $p_Y : Y \to B$ is a fixed map from a topological space Y to B. If X is a topological space with maps $f, g : X \to Y$, we say that f and g are ε-homotopic with respect to B if there is a homotopy equivalence $F_\varepsilon : X \times I \to Y$ between f and g such that, for all $x \in X$, the set $T_x = \{p_Y \circ F_\varepsilon(x,t) : t \in [0,1]\}$ has diameter less than ε in B. This T_x is defined to* *be the track of x under the homotopy* F_ε.

Definition 8.56. *Let B be a metric ANR and suppose that* $p_Y : Y \to B$ *is a map from the topological space Y to B. Let* $\varepsilon > 0$. *If* $f : X \to Y$ *is a homotopy equivalence, then it is an* ε*-homotopy equivalence over B if there is a homotopy inverse* $g : Y \to X$ *of f such that*

1. $f \circ g$ *and* id_Y *are* ε*-homotopic with respect to B;*
2. $f \circ g \circ f$ *and f are* ε*-homotopic with respect to B.*

Remark 8.57. *We usually take* $p_Y \circ f$ *as a control map.*

Definition 8.58. *Let* $p_Y : Y \to B$ *be a map from a CW complex Y to a metric space B. Fix positive* ε *and* δ. *Let M be a closed Top manifold and* $f : M \to Y$ *be a* δ*-homotopy equivalence over B. The pair* (M, f) *is said to be a* δ*-controlled manifold structure for* $p_Y : Y \to B$. *Two such pairs* (M, f) *and* (M', f') *are* ε*-related, denoted by* $(M, f) \overset{\varepsilon}{\sim} (M', f')$, *if there is a homeomorphism* $h : M \to M'$ *such that the following diagram* ε*-homotopy commutes with respect to B:*



i.e. the maps g and $g' \circ h$ *are* ε*-homotopic over B. We say that two pairs* (M, f) *and* (M', f') *are* ε*-homotopy equivalent if there is a finite sequence* (M_1, f_1), (M_2, f_2), …, (M_r, f_r) *of such pairs such that*

$$(M, f) = (M_1, f_1) \overset{\varepsilon}{\sim} (M_2, f_2) \overset{\varepsilon}{\sim} \cdots \overset{\varepsilon}{\sim} (M_r, f_r) = (M', f').$$

The collection of ε*-homotopy equivalence classes is called the* (ε, δ)*-controlled structure set* $\widetilde{\mathcal{S}}^{Top}_{\varepsilon,\delta}(Y \to B)$. *Our aim in this section is to build a long exact sequence that computes this controlled structure set, which is actually a group in the topological category.*

Definition 8.59. *We define a version of the above for pairs if Y is a Top manifold with boundary. Let* $p_Y : Y \to B$ *be as above. Let* $(M, \partial M)$ *be a Top manifold with boundary and let* $f : M \to Y$ *be a* δ*-homotopy equivalence such that* $f|_{\partial M} : \partial M \to \partial Y$ *is a Top homeomorphism. The object* $(M, \partial M, f)$ *is a* δ*-controlled manifold structure with boundary for* $p_Y : Y \to B$. *Two such objects* $(M, \partial M, f)$ *and* $(M', \partial M', f')$ *are*

ε-related if there is a homeomorphism $h: M \to M'$ such that

$$\begin{array}{ccc} \partial M & \xrightarrow{h|_{\partial M}} & \partial M' \\ & {\scriptstyle f|_{\partial M}} \searrow \quad \swarrow {\scriptstyle f'|_{\partial M'}} & \\ & \partial Y & \end{array}$$

is a diagram that commutes and

$$\begin{array}{ccc} M & \xrightarrow{h} & M' \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle f'} & \\ & Y & \end{array}$$

is a diagram that ε-homotopy commutes. As previously, we can define the notion of ε-homotopy equivalence. The collection of $(M, \partial M, f)$ up to ε-homotopy equivalence is denoted by $\widetilde{S}^{Top}_{\varepsilon,\delta}((Y, \partial Y) \to B)$. The identity $M \to M$ is the control map.

The reader should at this point be reminded of the α-approximation theorem and the definition of UV^1 maps from Section 7.1.

Definition 8.60. *Let $p: K \to B$ be a map of finite polyhedra. If $\delta > 0$, we say that p is $UV^1(\delta)$ if, for every map $\alpha: P^2 \to B$ of a 2-complex P into B with a lift $\alpha_0: P_0 \to K$ defined on a subcomplex P_0, there is a map $\alpha': P \to K$ with $\alpha'|_{P_0} = \alpha_0$ so that $p \circ \alpha'$ is δ-homotopic to α:*

$$\begin{array}{ccc} P_0 & \xrightarrow{\alpha_0} & K \\ \big\downarrow & \overset{\alpha'}{\nearrow} & \big\downarrow {\scriptstyle p} \\ P^2 & \xrightarrow[\alpha]{} & B \end{array}$$

The following theorems and definitions establish the basic classification theorems in the controlled category. Many of the main ideas should be familiar, although the details are quite technical. The following is a controlled version of the simply connected h-cobordism theorem.

Theorem 8.61. *(Thin h-cobordism theorem, Quinn [520, 521]) Let B be a finite polyhedron. For all $\varepsilon > 0$ there is $\delta > 0$ such that, if the following conditions hold:*

1. *$n \geq 4$ and $p: M_0^n \to B$ is a $UV^1(\delta)$ control map from a manifold M_0^n;*
2. *(W, M_0, M_1) is a cobordism with strong deformation retractions $r_t: W \to M_0$ and $s_t: W \to M_1$ such that, for all $x \in W$, the diameters of $p(r_1(r_t(x))$ and $p(r_1(s_1(s_t(x))))$ are less than δ in B,*

then there is a Top homeomorphism $g: W \to M_0 \times I$ such that the maps $W \xrightarrow{r_1} M_0 \xrightarrow{p} B$ and $W \xrightarrow{g} M_0 \times I \xrightarrow{pr_1} M_0 \xrightarrow{p} B$ are ε-homotopic.

Here (W, M_0, M_1) is called a *δ-thin h-cobordism* and W has an *ε-product structure*.

For the purposes of algebraicization, one can consider geometric chain complexes. A free geometric R-module is a free R-module with a basis whose elements are associated to points in B. When we map one free module to another, a basis element is mapped to a linear combination of other basis elements, and we can ask how far away they are, obtaining a notion of the size of a geometric morphism. An ε-chain complex has all morphisms with size less than ε. An example is the cellular chain complex of a polyhedron over B, if one subdivides the simplicial structure so that all simplices have diameter less than ε.

At times one wants the chain complex analogue of this idea. We say that a chain map $f : C \to D$ is an *ε-chain homotopy equivalence* if there exists a chain homotopy inverse $g: D \to C$ and chain homotopies $P: g \circ f \simeq id$ and $Q: f \circ g \simeq id$ such that f, g, P, and Q all have control at most ϵ.

With these notions one can form a controlled K-group on Whitehead groups, as well as homology theories of the control space. See Quinn [521]. In the setting of the thin h-cobordism theorem, the condition on π makes this theory trivial.

Definition 8.62. *Suppose that P is a finite polyhedron and B is a compact metric space with a continuous map $p: P \to B$. We say that P is an unrestricted ε-Poincaré complex of formal dimension n over B if there is a subdivision of P so that*

1. *if Δ is a simplex of P, then $p(\Delta)$ has diameter less than ε in B,*
2. *there is a cycle y in the simplicial chains $C_n(P)$ such that the map given by cap product $y \cap -: C^*(P) \to C_{n-*}(P)$ is an ε-chain homotopy equivalence.*

The definition of a restricted ε-Poincaré complex of formal dimension n is similar except that it is also required that the map $p: P \to B$ be $UV^1(\varepsilon)$.

Definition 8.63. *If $p: P \to B$ is a $UV^1(\delta)$ control map, we say that a map $f: M \to P$ is (δ, k)-connected over B if, whenever*

1. *(L, L_0) is a CW pair with $\dim(L) \leq k$,*
2. *there is a map $\alpha: L_0 \to M$ such that there is $\beta: L \to P$ with $f \circ \alpha = \beta|_{L_0}$,*

then

1. *there is a map $\gamma: L \to M$ with $\gamma|_{L_0} = \alpha$;*
2. *there is a homotopy $h_t: L \to P$ rel L_0 with $h_0 = f \circ \gamma$ and $h_1 = \beta$;*
3. *$\operatorname{diam}(p \circ h(\{x\} \times I)) < \delta$ for each $x \in L$.*

Definition 8.64. *Let P be an unrestricted δ-Poincaré duality space of formal dimension n over a metric space B with a continuous map $p\colon P \to B$. Let ν be a Cat bundle over P. A δ-surgery problem or degree one normal map is a triple (M^n, ϕ, F), where*

1. *M^n is a closed Top manifold;*
2. *$\phi\colon M \to P$ is a map such that $\phi_*[M] = [P]$;*
3. *F is a stable trivialization of $\tau_M \oplus \phi^*\nu$.*

A δ-surgery problem is often denoted by the composition $M \xrightarrow{\phi} P \xrightarrow{p} B$. Note that δ only occurs in the definition of P.

Definition 8.65. *If M and M' are both n-dimensional closed Top manifolds, then two δ-surgery problems (M, ϕ, F) and (M', ϕ', F') are equivalent if there are*

1. *an $(n+1)$-dimensional manifold W with $\partial W = M \coprod M'$;*
2. *a proper map $\Phi\colon W \to P$ extending ϕ and ϕ';*
3. *a stable trivialization of $\tau_W \oplus \Phi^*\nu$ extending F and F'.*

As in the uncontrolled case, surgery is possible below the middle dimension.

Theorem 8.66. *Let $n \geq 6$ and $\varepsilon > 0$. Let $(P^n, \partial P^n)$ be an unrestricted ε-Poincaré duality pair over a finite polyhedron B. Let $\phi\colon (M, \partial M) \to (P, \partial P)$ be an ε-surgery problem. Then ϕ is normally bordant to an ε-surgery problem $\phi'\colon (M', \partial M') \to (P, \partial P)$ such that*

1. *ϕ is $\left(\varepsilon, \left[\frac{n}{2}\right]\right)$-connected over B;*
2. *the restriction $\phi'|_{\partial M'}\colon \partial M' \to \partial P$ is $\left(\varepsilon, \left[\frac{n-1}{2}\right]\right)$-connected over B.*

The following is established by Chapman [160].

Theorem 8.67. *(Controlled Hurewicz-Whitehead) Let $n \geq 1$. There are $k > 0$ and $\delta_0 > 0$ such that, if the following conditions hold:*

1. *$\delta \in (0, \delta_0)$,*
2. *(X, Y) is an n-dimensional polyhedral pair with cells of size at most δ over B,*
3. *$p\colon X \to B$ is a $UV^1(\delta)$ control map such that $p|_Y$ is also $UV^1(\delta)$,*
4. *$C_*(Y) \to C_*(X)$ is a δ-chain homotopy equivalence,*

then $Y \to X$ is a $k\delta$-homotopy equivalence.

In the following theorem, condition (3) implies that $\pi_1(P) \cong \pi_1(\partial P) \cong \pi_1(B)$. These fundamental groups may not be trivial, but implicit in condition (3) is the assumption that the homotopy fibers are simply connected.

Theorem 8.68. *(Controlled π-π theorem) Let $n \geq 6$. Let B be a finite polyhedron with the barycentric metric. There are $k > 0$ and $\varepsilon_0 > 0$ such that, if*

1. $\varepsilon \in (0, \varepsilon_0)$,
2. $(P^n, \partial P^n)$ *is an ε-Poincaré complex over B,*
3. *the maps $(M, \partial M) \xrightarrow{\phi} (P, \partial P) \xrightarrow{p} B$ form an ε-surgery problem such that $p\colon P \to B$ and $p|_{\partial P}\colon \partial P \to B$ are both $UV^1(\varepsilon)$ maps,*

then surgery can be used to obtain a normal cobordism from $\phi\colon (M, \partial M) \to (P, \partial P)$ to a $k\varepsilon$-homotopy equivalence $\phi'\colon (M', \partial M') \to (P, \partial P)$ of pairs.

Definition 8.69. *Define $L_{\eta,B,\delta}(\mathbb{Z}[e])$ to be the abelian semigroup of geometric special quadratic forms of radius at most δ, modulo the equivalence relation generated by $\sim_\delta$.*

Theorem 8.70. *(Squeezing in (ε, δ)-surgery) Let B be a finite complex and $n \geq 5$. Then there are $\varepsilon_0 > 0$ and $T \geq 1$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and all ε-surgery problems $M^n \xrightarrow{f} X \xrightarrow{p} B$ with a UV^1-control map p, there is a well-defined obstruction σ_f in $H_n(B; \mathbb{L}_\bullet^{\langle 1 \rangle})$ that vanishes iff f is normally bordant to a $T\varepsilon$-equivalence over B. The obstruction does not depend on T or ε if ε_0 is sufficiently small.*

Theorem 8.71. *(Controlled Wall realization) Let B be a polyhedron with bounded geometry and let $n \geq 6$. There are $k \in \mathbb{R}_{>0}$ and $\delta_0 > 0$ such that, given*

1. $\delta \in (0, \delta_0)$,
2. *a $UV^1(\delta)$-control map $p\colon V^{n-1} \to B$, and*
3. $\alpha \in L_{\eta,B,\delta}(\mathbb{Z}[e])$,

we can represent the image of α in $L_{\eta,B,k\delta}(\mathbb{Z}[e])$ by a map from a Top manifold $(M, \partial M)$ with boundary to $V \times I$.

Theorem 8.72. *(Stability theorem for L-groups) Let B be a polyhedron and let $q \geq 2\dim(B) + 2$. There is a sequence $\{\delta_i\}$ of positive real numbers monotonically approaching 0 so that the image of the map $j_i\colon L_{q,B,\delta_i}(\mathbb{Z}[e]) \to L_{q,B,\delta_{i-1}}(\mathbb{Z}[e])$ is isomorphic to $H_q(B; \mathbb{L}_\bullet(\mathbb{Z}[e]))$ for all i.*

Remark 8.73. *This stability theorem is essentially the same as the squeezing theorem.*

Theorem 8.74. *(Stability theorem for manifold structures) Let $n \geq 5$. If M^n is a closed n-manifold and $p\colon M \to B$ is a UV^1 control map, then for every $\varepsilon > 0$ there is $\delta > 0$ such that, for all $\mu > 0$, if $\phi\colon N \to M$ is a δ-homotopy equivalence over B, then ϕ is ε-homotopic over B to a μ-homotopy equivalence $\psi\colon N \to M$.*

The reader should consider the work of Hughes [326] and Hughes-Taylor-Williams

[329] on the classification of manifold approximation fibrations.

Theorem 8.75. *Let M be a compact Top manifold of dimension $n \geq 6$ (or $n \geq 5$ if $\partial M = \emptyset$) and let B be a finite polyhedron. If $p : M \to B$ is UV^1, then there are $\varepsilon_0 > 0$ and $T \geq 0$ depending on n and B such that, for all $\varepsilon \in (0, \varepsilon_0)$, there is a surgery exact sequence*

$$\cdots \to H_{n+1}(B; \mathbb{L}_\bullet) \to \mathcal{S}^{Top}_{\varepsilon,T}(M \to B) \to [M : F/Top] \to H_n(B; \mathbb{L}_\bullet)$$

where $\mathbb{L}_\bullet$ is the periodic spectrum of the trivial group and

$$\mathcal{S}^{Top}_{\varepsilon}(M \to B) = \operatorname{im}(\widetilde{\mathcal{S}}^{Top}_{\varepsilon}(M \to B) \to \widetilde{\mathcal{S}}^{Top}_{T\varepsilon}(M \to B)).$$

For all $\varepsilon \in (0, \varepsilon_0)$, we have the stability result $\mathcal{S}^{Top}_{\varepsilon}(M \to B) \cong \mathcal{S}^{Top}_{\varepsilon_0}(M \to B)$.

Proof. We first demonstrate exactness at $[M : F/Top]$. Given a degree one normal map ϕ, we can compute its controlled surgery obstruction $\sigma(\phi)$. If $\sigma(\phi) = 0$, then surgery can be executed on ϕ to produce an ε-homotopy equivalence for all $\varepsilon > 0$. Therefore, it comes from a controlled structure in $\mathcal{S}^{Top}_{\varepsilon,T}(M \to B)$. Conversely, if the normal map ϕ comes from a controlled structure, then $\sigma(\phi) = 0$.

Now we show exactness at $\mathcal{S}^{Top}_{\varepsilon,T}(M \to B)$. If $f : N \to M$ is a controlled structure and $\alpha \in H_{n+1}(B; \mathbb{L}_\bullet)$, then α can be realized as a cobordism (W_α, N, N'), and the controlled equivalence $f' : N' \to M$ is action of α on f. If the image of f' is zero in $[M : F/Top]$, then it is obtained by acting on $id : M \to M$ by some $\alpha' \in H_{n+1}(B; \mathbb{L}_\bullet)$. □

Remark 8.76. *In the case that B is a compact ANR (see Definition 7.2), we may lose the linear dependence in the definition of $\mathcal{S}_{\varepsilon,T}(M \to B)$, but we still have a decent theory. There is $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, there is $\delta > 0$ such that the surgery sequence holds for*

$$\mathcal{S}^{Top}_{\delta,\varepsilon}(M \to B) = \operatorname{im}(\widetilde{\mathcal{S}}^{Top}_{\delta}(M \to B) \to \widetilde{\mathcal{S}}^{Top}_{\varepsilon}(M \to B)).$$

Remark 8.77. *The unreduced notation for the structure set will be used to describe another structure set $\mathcal{S}^{Top}_{\varepsilon}(Y \to B)$ at the end of the section.*

Remark 8.78. *Notice that, if B is a one-point space and Y is a CW complex, then for all $\varepsilon > 0$ the notion of ε-homotopy coincides with the notion of homotopy. Therefore $\widetilde{\mathcal{S}}^{Top}_{\varepsilon}(Y \to pt)$ is simply $\mathcal{S}^{Top}(Y)$. Clearly, for general metric spaces B and Top manifolds Y, the set $\widetilde{\mathcal{S}}^{Top}_{\varepsilon}(Y \to B)$ contains the identity map $id_Y : Y \to Y$ and is therefore nonempty.*

Example 8.79. *We note that the controlled surgery exact sequence for a UV^1 map*

$M \to B$ becomes

$$H_{n+1}(B;\mathbb{L}_\bullet) \to \mathcal{S}^{Top}_{\varepsilon}(M \to B) \to H_n(M;\mathbb{L}^{\langle 1\rangle}_\bullet) \to H_n(B;\mathbb{L}_\bullet)$$

so that the controlled structures are isomorphic to $H_{n+1}(B, M;\mathbb{L}_\bullet)$, aside from the $\mathbb{Z}$ that arises from the difference between periodic and connective L-theory. One might notice that it is similar to the predicted formula for $\mathcal{S}^{Top}(M)$ when the Borel conjecture holds for $\pi_1(M)$. See Chapter 5. The relevant "Borel conjecture" here is the α-approximation theorem, which is itself a rigidity theorem for B over B. Of course, when $M = B$, this homological calculation of the structure set is the α-approximation theorem.

8.6 HOMOLOGY MANIFOLDS

Manifold topology naturally fits into a larger framework of homology manifolds. Homology manifolds arose first as the class of spaces which satisfies Poincaré duality for a local reason. See Swan [632], Bredon [74], and Wilder [710]. In this section we study finite-dimensional absolute neighborhood retracts (ANRs) X for which the local homology isomorphism $H_*(X, X\backslash\{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ holds for every point x in X. Such spaces have many of the properties of manifolds like Poincaré duality. The existence of homology spheres, i.e. topological closed manifolds homologically identical to the standard spheres, gives us a way to produce homology manifolds that are not topological manifolds. A cone on a non-simply connected (manifold) homology sphere will have the property that all deleted neighborhoods of the cone point are non-simply connected, so this space is not a manifold. It is however a trivial calculation to see that the local homology groups are correct.

Another way to obtain many wilder examples is that of decomposition spaces, pioneered by Bing. One starts with a manifold M and describes a collection of subsets that are in a weak sense contractible (technically, cell-like), and identifies each of these subsets to a point. Because every homology sphere bounds a contractible manifold, the first example is a special case of the second. For quite some time it seemed that every homology manifold could be obtained in this fashion. See Kervaire [353] for high dimensions and Freedman [253] for dimension 3.

Definition 8.80. *Let X be a finite-dimensional ANR. We say that X is an (integral) homology n-manifold if, for every $x \in X$, we have*

$$H_i(X, X\backslash\{x\};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

If A is an abelian group, then there is a notion of a homology A-manifold X by replacing all instances of $\mathbb{Z}$ in the above definition with A. In addition, we may define X to be a homology n-manifold with boundary if points $x \in X$ may satisfy the condition that

$H_n(X, X\backslash\{x\}; \mathbb{Z}) = 0$. *The collection of such points is the boundary* ∂X *of* X.

Remark 8.81. *If one dispenses with finite-dimensionality the theory becomes radically different, and, for example, there is no longer uniqueness of resolutions. See Dranishnikov-Ferry-Weinberger [205].*

Remark 8.82. *If a homology n-manifold has boundary* ∂X*, then* ∂X *is a homology* $(n-1)$*-manifold.*

One particular subject of interest in the study of homology manifolds is the extent to which they resemble Top manifolds outside of mere homological considerations. Following Edwards [210], we focus our attention to the *disjoint disks property*. It is the appropriate form of general positioning in this setting because seemingly stronger general position statements can be shown to follow from it.

Definition 8.83. *We say that a topological space* X *has the disjoint disks property (DDP) if, for any* $\varepsilon > 0$ *and maps* $f, g : \mathbb{D}^2 \to X$*, there are maps* $f', g' : \mathbb{D}^2 \to X$ *such that*

1. $d(f, f') < \varepsilon$ *and* $d(g, g') < \varepsilon$,
2. *the intersection* $f'(\mathbb{D}^2) \cap g'(\mathbb{D}^2) = \emptyset$.

Definition 8.84. *Recall that, if* X *and* M *are both finite-dimensional ANRs, we say that a map* $f : M \to X$ *is cell-like (CE) if* f *is a proper surjection and, for all* $x \in X$*, the preimage* $f^{-1}(x)$ *is contractible in any of its neighborhoods. Such a map is a resolution if* M *is a Top manifold. The space* X *is resolvable if there is a resolution* $f : M \to X$ *for some closed Top manifold* M*. If* $(X, \partial X)$ *is a homology manifold with boundary, then we say that* $(X, \partial X)$ *is resolvable if there is a resolution* $f : (M, \partial M) \to (X, \partial X)$ *for some compact manifold* M *with boundary.*

Remark 8.85. *The reader might recall that CE maps are exactly the hereditary homotopy equivalences. Therefore, a map* $\phi : M \to X$ *from a Top manifold* M *to a homology manifold* X *is a resolution of* X *if* $\phi|_{\phi^{-1}(U)} : \phi^{-1}(U) \to U$ *is a homotopy equivalence for all open sets* $U \subseteq X$*. See Lacher [379].*

Edwards proved that, in the presence of the DDP and the ANR conditions, a resolution of a homology manifold, if it exists, must be the limit of Top homeomorphisms. It is an extension of Siebenmann's CE approximation theorem in Chapter 7. In dimensions at least 5, Edwards's theorem characterizes topological manifolds as those ANR homology manifolds with resolution that have the DDP.

Theorem 8.86. *(Edwards [210]) Let* $n \geq 5$ *and let* X^n *be an ANR homology manifold and* $\phi : M \to X$ *a resolution. Then the following are equivalent:*

1. X *satisfies DDP condition;*
2. X *satisfies the ABH property.*

Corollary 8.87. *Suppose that X is an ANR homology manifold of dimension at least 5. Assume that X is resolvable and has the DDP property. Then X is homeomorphic to a Top manifold.*

It is then natural to ask whether or not the resolvability condition in this corollary is actually redundant. If it were redundant, i.e. if every (DDP) ANR homology manifold were resolvable, then one would have the so-called Characterization conjecture.

Conjecture 8.88. *(False) (Characterization conjecture) Every ANR homology manifold of dimension at least 5 which has the DDP property is a Top manifold.*

In fact, there are nonresolvable homology manifolds that fall exactly in the gaps in surgery theory from the difference between $\mathbb{L}_\bullet(e)$ and F/Top.

When the dimension of the singular set of the homology n-manifold is at most $\frac{n}{2} - 1$, Bryant, Cannon, and Lacher [103] prove results that initially supported the validity of the Resolution and Characterization conjectures. Quinn then showed that, even if just a single point has a resolvable neighborhood, then the entire homology manifold is resolvable [523]. However, the surgery theory for homology manifolds that we will describe shows that the Resolution conjecture is false. See Bryant-Ferry-Mio-Weinberger [104]. In this paper they construct homology manifolds in dimensions at least 5 that are not resolvable. In the process, homology manifolds with DDP that are not homotopy equivalent to any closed manifold can also be produced. These statements prove that the characterization conjecture is false, but that nonconnective surgery theory is, up to s-cobordism, exactly correct for these spaces.

Several ingredients are required, including the Quinn obstruction and the method of (ε, δ)-surgery. Consequently one obtains a full covariantly functorial 4-periodic surgery exact sequence in the context of homology manifolds.

We first describe the Quinn index $I(X)$, a local invariant related to the existence of resolvable open sets. It has several different interpretations.

Theorem 8.89. *(Quinn [524]) Let X be a connected finite-dimensional ANR homology n-manifold with $n \geq 4$, not necessarily compact and possibly with boundary. There is an element $I(X) \in 8\mathbb{Z} + 1$ such that*

1. *if U is open in X, then $I(X) = I(U)$;*
2. *if Y is another connected finite-dimensional ANR homology manifold, then*

$$I(X \times Y) = I(X)I(Y);$$

3. *the space X is resolvable iff $I(X) = 1$.*

Proof. Ferry and Pedersen [243] prove that, for any such X, there is a degree one normal map $f : M \to X$ from a Top manifold M to X. By the (ε, δ)-surgery theorem, there are $\varepsilon_0 > 0$ and $T \geq 1$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, there is a surgery obstruction

$$\sigma(f) \in H_n(X; \mathbb{L}_\bullet) \cong [X : F/Top \times \mathbb{Z}]$$

which vanishes iff f is normally bordant to a $T\varepsilon$-equivalence $f_{T\varepsilon} : M_{T\varepsilon} \to X$. If $f : M \to X$ is modified by surgery to $f' : M' \to X$, then the difference $\sigma(f) - \sigma_{f'}$ lies in $[X : F/Top]$. Therefore, to every $f : M \to X$ there is a quantity $i(X)$ in $[X : \mathbb{Z}]$ which is invariant under surgery. Since X is connected and $\mathbb{Z}$ is discrete, this $i(X)$ is simply a constant value in $\mathbb{Z}$. The quantity $I(X) = 1 + i(x)$ is defined to be the *Quinn invariant.*

The quantity $i(X)$ can be interpreted as a difference of signatures, namely $i(x) - i(\mathbb{R}^n)$, so $I(X) \equiv 1 \bmod 8$. Therefore (2) holds. Property (1) follows from the fact that both U and X map to the same component of $F/Top \times \mathbb{Z}$.

We now prove the backward implication for (3), since the forward direction is simpler. Suppose that $I(X) = 1$. The surgery exact sequence for the controlled problem for X over itself is given by

$$H_{n+1}(X; \mathbb{L}_\bullet) \to H_{n+1}(X; \mathbb{L}_\bullet) \to \mathcal{S}_\varepsilon^{Top}(X \to X) \to H_n(X; \mathbb{L}_\bullet) \to H_n(X; \mathbb{L}_\bullet)$$

for sufficiently small $\varepsilon > 0$. Since X has dimension n, we have $H_i(X; \mathbb{Z}) = 0$ for all $i \geq n+1$. The spectra $\mathbb{L}_\bullet$ and $\mathbb{L}_\bullet^{\langle 1 \rangle}$ differ by a factor of $\mathbb{Z}$, so by the Atiyah-Hirzebruch spectral sequence, we have an isomorphism $H_{n+1}(X; \mathbb{L}_\bullet) \cong H_{n+1}(X; \mathbb{L}_\bullet^{\langle 1 \rangle})$ and an injection $H_n(X; \mathbb{L}_\bullet^{\langle 1 \rangle}) \to H_n(X; \mathbb{L}_\bullet)$. Therefore the structure set $\mathcal{S}_\varepsilon(X \to X)$ is trivial as in the manifold case. Now let $\{\varepsilon_i\}$ be a sequence with $\lim_{i\to\infty} \varepsilon_i = 0$ and $\sum_{i=1}^\infty \varepsilon_i < \infty$. There is then a sequence $\{\delta_i\}$ such that, for each i, there is a homeomorphism $h_i : M_{\delta_i} \to M_{\delta_{i+1}}$ such that the diagram

$$\begin{array}{ccc} M_{\delta_i} & \xrightarrow{h_i} & M_{\delta_{i+1}} \\ & \searrow^{f_i} & \downarrow f_{i+1} \\ & & X \end{array}$$

commutes up to ε_i. In other words, we have $d(f_{\delta_{i+1}} \circ h_i, f_{\delta_i}) < \varepsilon_i$ for all i. For each i, define $\phi_i = f_{\delta_i} \circ h_{i-1} \circ h_{i-2} \circ \cdots \circ h_1 : M_{\delta_1} \to X$. Then $d(\phi_{i+1}, \phi_i) = d(f_{\delta_{i+1}} \circ h_i, f_{\delta_i}) < \varepsilon_i$ for all i. The sequence $\{\phi_i\}$ therefore converges to a map $f : M_{\delta_1} \to X$ since $\sum_{i=1}^\infty \varepsilon_i < \infty$. Since ϕ_i is a δ_i-equivalence, it follows that f is a resolution of X. □

Since $I(X)$ is a signature and only depends on any open subset of a connected X, it is often called the *local index.*

To avoid interrupting the discussion later, we include here a few technical lemmas.

Definition 8.90. *We say that a map $p : L \to B$ between finite polyhedra has the abso-*

lute δ-lifting property $\mathrm{AL}^k(\delta)$ *if, whenever* (P, Q) *is a polyhedral pair with* $\dim(P) \le k$, *and* $\alpha_0 : Q \to L$ *is a map, and* $\alpha : P \to B$ *is a map with* $p \circ \alpha_0 = \alpha|_Q$, *there is then a map* $\overline{\alpha} : P \to L$ *extending* α_0 *with* $d(p \circ \overline{\alpha}, \alpha) < \delta$.

Lemma 8.91. *(Ferry [238]) Let* M^n *be a compact connected manifold and let* B *be a finite connected polyhedron. For all* $\varepsilon > 0$ *there is* $\delta > 0$ *such that, for all* $\mu > 0$, *if* $p : M \to B$ *is* $\mathrm{AL}^{k+1}(\delta)$ *with* $k \le \frac{n-3}{2}$, *then* p *is* ε*-homotopic to an* $\mathrm{AL}^{k+1}(\mu)$ *map. It follows that there is* $\delta_0 > 0$ *such that every* $\mathrm{AL}^{k+1}(\delta_0)$*-map is homotopic to a* UV^k *map. If* $U \subseteq B$ *and* $p|_{p^{-1}(U)}$ *is* UV^k *and* $C \subseteq U$ *is compact, then the limiting* UV^k*-map can be chosen to equal* P *on* $p^{-1}(C)$.

Lemma 8.92. *(Ferry [238]) Let* $n \ge 5$. *If* $f : (M^n, \partial M) \to K$ *is a map from a compact manifold to a polyhedron, and if the homotopy fiber of* f *is simply connected, then* f *is homotopic to a* UV^1*-map. If* $f|_{\partial M}$ *is* UV^1, *then* f *is homotopic to a* UV^1*-map relative to the boundary.*

Lemma 8.93. *(Bryant-Ferry-Mio-Weinberger [104]) Let* $n \ge 1$ *and* B *be a finite polyhedron. There are* $\varepsilon_0 > 0$ *and* $T > 0$ *such that, if*

1. $\varepsilon \in (0, \varepsilon_0)$,
2. $(M_1, \partial M_1)$ *and* $(M_2, \partial M_2)$ *are orientable manifolds,*
3. $p_1 : M_1 \to B$ *and* $p_2 : M_2 \to B$ *are* UV^1*-maps,*
4. $h : \partial M_1 \to \partial M_2$ *is an orientation-preserving* ε*-equivalence over* B *(including* $d(p_1, p_2 \circ h) < \varepsilon$*),*

then $M_1 \cup_h M_2$ *is a* $T\varepsilon$*-Poincaré duality space over* B.

Now we sketch the proof of the main theorem of the section by Bryant-Ferry-Mio-Weinberger [104]. In the following we will be using three types of surgery obstructions:

1. the total surgery obstruction $\mathcal{O}(Z) \in \mathcal{S}^{Top}_{n+1}(Z)$,
2. the controlled surgery obstruction $\sigma_c(f) \in H_n(Z; \mathbb{L}_\bullet)$,
3. the uncontrolled surgery obstruction $\sigma(f) \in L_n(\mathbb{Z}[\pi])$.

Definition 8.94. *Let* Z^n *be a (simple) Poincaré complex with* $n \ge 6$. *A homology manifold structure on* Z *is a simple homotopy equivalence* $f : X \to Z$, *where* X *is a closed ANR homology n-manifold with the DDP. We say that two homology manifold structures* $f_1 : X_1 \to Z$ *and* $f_2 : X_2 \to Z$ *on* Z *are equivalent if there is an s-cobordism* $F : (V, X_0, X_1) \to X$ *from a homology* $(n+1)$*-manifold* V *with boundary* $X_0 \coprod X_1$ *restricting to* f_1 *and* f_2 *on* X_1 *and* X_2. *The homology structure set* $\mathcal{S}^H(M)$ *is the set of all equivalence classes of homology manifold structures on* M.

Remark 8.95. *It is possible, of course, to define a relative version of a homology manifold structure. If* Z^n *has a distinguished subset* Y, *we say that a homology structure on*

Z relative to Y is a map $f : (X, \partial X) \to (Z, Y)$ from a homology n-manifold X with boundary ∂X such that $f|_{\partial X} : \partial X \to Y$ is a Top homeomorphism. We use the notation $\mathcal{S}^H(M)$ to indicate the set of all equivalence classes of homology manifold structures on M relative to the boundary.

Theorem 8.96. *(Bryant-Ferry-Mio-Weinberger [104]) Suppose that Z^n is a simple Poincaré complex with $n \geq 6$ and fundamental group π. There is a homology manifold M (simple) homotopic equivalent to Z iff the total surgery obstruction $\mathcal{O}(Z)$ vanishes in $\mathcal{S}^{Top}_{n+1}(Z)$. If this condition holds, then there is a covariantly functorial 4-periodic exact sequence of abelian groups given by*

$$\cdots \to H_n(Z; \mathbb{L}_\bullet) \to L_n(\mathbb{Z}[\pi]) \to \mathcal{S}^H(Z) \to H_{n-1}(Z; \mathbb{L}_\bullet) \xrightarrow{A} L_{n-1}(\mathbb{Z}[\pi]),$$

where $\mathbb{L}_\bullet$ is the periodic surgery spectrum. The sequence works for both decorations h and s.

Here is an outline of the construction of homology manifolds.

We start with a Poincaré complex X and try to find a space Z that is homotopy equivalent to X, and which is a Poincaré complex over itself. Suppose $f : M \to X$ is a degree one normal invariant. We shall build an X_1 which is (1) homotopy equivalent to X, and (2) a very fine Poincaré duality space over X. The key idea is that an ANR X is a homology manifold iff X is a controlled Poincaré complex over itself.

We recall a simple construction of Poincaré complexes. We take two manifolds with boundary and glue them together by a homotopy equivalence of their boundaries. A simple example might be to take a Wall realization of an element $\alpha \in L_{n+1}(\pi_1(N))$, and to glue the ends together to obtain a Poincaré complex with a normal invariant whose surgery obstruction is $-\alpha$. We can do similarly for an element in $L^c_n(M)$. The element acts on a dense codimension one hypersurface that is locally π_1-isomorphic to M, and therefore can be viewed by Lemma 8.92 to be UV^1 over M.

When we glue the pieces together, we obtain a Poincaré complex which is largely a manifold controlled over M, but not homotopy equivalent to M. Using an element of the controlled L-group $L^c_n(M)$ which vanishes in $L_n(\pi_1(M)) = L_n(\mathbb{Z}[e])$, we can execute surgery on this map using manifold surgery to obtain an uncontrolled homotopy equivalence to M.

We can consider $M \to X_1$ and repeat the construction to build an X_2 that is controlled Poincaré over X_1 and controlled homotopy equivalence to X_1 over X (to the scale that we have chosen for the Poincaré duality of X_1). This process produces X_2, but there is no slack in this construction as $L^c(X_1) \cong L^c(X)$. The rest of the construction is to produce $X_3 \to X_2$ and then continue. Ultimately we take the limit of the X_i to produce an X_∞ that is controlled Poincaré over itself.

The details of this argument are somewhat complicated, and we refer to Bryant-Ferry-Mio-Weinberger [104] for them.

The following is a somewhat more detailed sketch.

Proof. From Theorem 4.54 we have the algebraic surgery exact sequence

$$\cdots \to H_n(Z;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi]) \xrightarrow{\alpha} S_n^{Top}(Z) \xrightarrow{g} H_{n-1}(Z;\mathbb{L}_\bullet) \xrightarrow{A} L_{n-1}(\mathbb{Z}[\pi]).$$

A Poincaré duality space Z has total obstruction $\mathcal{O}(Z) \in S_n^{Top}(Z)$ so that $g(\mathcal{O}(Z)) \in H_n(Z;\mathbb{L}_\bullet)$ is the obstruction to lifting the Spivak fibration to Top. If $g(\mathcal{O}(Z)) = 0$, then $\mathcal{O}(Z) = \alpha(\sigma(f))$, where $f : M \to Z$ is any degree one normal map.

Let Z be a Poincaré duality space with $\mathcal{O}(Z) = 0$. Then there is a degree one normal map $f : M \to Z$ and $\sigma(f) \in \operatorname{im} A$ by exactness. Let $\zeta \in H_n(Z;\mathbb{L}_\bullet)$ with $A(\zeta) = \sigma(f)$. Now let $\delta_i \geq 0$ be a sequence such that $\sum \delta_i < \infty$.

Without loss of generality, we can assume that $f : M \to Z$ is connected up to the middle dimension. Lemma 8.91 implies that f can be assumed to be a UV^1-map. Let (ε_0, T_0) be stability constants for Z. Take a triangulation of M with mesh less than $\mu_0 < \varepsilon_0$. Let C_0 be the regular neighborhood of the 2-skeleton of M. Let $D_0 = \overline{M - C_0}$ and let $N_0 = C_0 \cap D_0$. Let $q_0 : M \to M$ be chosen arbitrarily close to the identity so that $q_0|_{C_0}$, $q_0|_{D_0}$ and $q_0|_{N_0}$ are all UV^1. Consider the sequence

$$\cdots \to H_n(Z;\mathbb{L}_\bullet) \to S_\mu(N_0 \to Z) \to [N_0 : F/Top].$$

For $\mu = \mu_0$, there is a cobordism

$$(F_\sigma, id, f_\sigma) : (V, N_0, N_0') \to (N_0 \times I, N_0 \times \{0\}, N_0 \times \{1\})$$

realizing σ, where $f_\sigma : N_0' \to N_0$ is a μ_0-equivalence over Z. Lemma 8.92 implies that F_σ is a UV^1-map. Here V may not be a product since $\sigma(F_\sigma) = \sigma(f)$ may not be zero. Let X_0' be obtained by cutting M at N_0 and glue in C_{f_σ} and $-V$. We want a degree one UV^1-map $X_0' \to Z$. Let $g_\sigma : N_0 \to N_0'$ be a UV^1 μ_0-controlled homotopy inverse to f_σ and let $G_\sigma' : V \xrightarrow{F_\sigma} N_0 \times I \xrightarrow{g_\sigma \times id} N_0' \times I$. Then Lemma 8.91 implies that there is

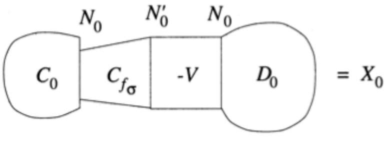


Figure 8.1: Constructing X_0 by glueing

$G_\sigma \simeq G_\sigma'$ such that (1) $G_\sigma|_{N_0'} = id_{N_0'}$, (2) $G_\sigma|_{N_0} = g_\sigma$, and (3) G_σ is UV^1. Form the space X_0'' by inserting C_{f_σ} and C_{g_σ} between C_0 and D_0. Then we have maps

$$X_0' \xrightarrow{G_\sigma} X_0'' \xrightarrow{c} M \to Z$$

where c is a collapse map.

This composition $p_0' : X_0' \to Z$ is a degree one UV^1-map with respect to which X_0' is a $T\mu_0$-Poincaré duality space, i.e. $T\mu_0 < \eta_0$.

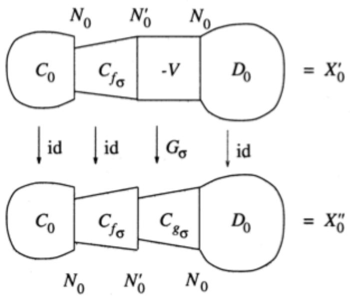


Figure 8.2: Constructing the map $X_0' \to X_0''$

The usual surgery obstruction of p_0' is $-\sigma + \sigma = 0$, so we can execute surgery on X_0' to achieve an η_0-controlled Poincaré space X with respect to a UV^1 homotopy equivalence $p_0 : X_0 \to Z$. Now one can iterate the construction, being careful to ensure convergence. A relative version of existence gives the whole surgery exact sequence. □

Corollary 8.97. *Let M^n be a simply connected closed Top manifold of dimension $n \geq 6$. There is a nonresolvable ANR homology manifold X that is homotopy equivalent to M. This X can be constructed to have the DDP. In fact, if $\gamma \in 8\mathbb{Z} + 1$, then a homology n-manifold X that is homotopy equivalent to M can be constructed with $I(X) = \gamma$.*

Corollary 8.98. *There is a homology manifold that is not homotopy equivalent to a Top manifold. This homology manifold is an ANR and can be constructed to have the DDP.*

Proof. One constructs a Poincaré complex from a torus by doing the Wall realization on the boundary or a regular neighborhood of the 2-skeleton using an element of $L_n(\mathbb{Z}[\mathbb{Z}^n])$ that is not in the image of $[\mathbb{T}^n : F/Top]$. This Poincaré complex is homotopy equivalent to an ANR homology manifold, but not to a manifold. □

Remark 8.99. *In the case when Z is actually a Top manifold with boundary, we know that ∂X is a Top manifold, since $f|_{\partial X}$ is a Top homeomorphism onto $Y = \partial Z$. If a collar $\partial X \times I$ is added to X, then we obtain a homology manifold X' with manifold points. Then $I(X) = I(X') = 1$, and so X is a manifold. By the manifold s-cobordism theorem, we have $\mathcal{S}^H(Z) = \mathcal{S}^{Top}(Z)$, so in this case $\mathcal{S}^H(Z)$ consists entirely of manifold structures.*

Remark 8.100. *The assembly map is an infinite loop map. As a result, when X is a closed homology n-manifold with $n \geq 6$, the homology structure set $\mathcal{S}^H(X)$ is actually an abelian group because it is the fiber of the assembly map.*

Remark 8.101. *The following is an important point that we want to emphasize. Suppose that M is a Top manifold with fundamental group π. We have discussed the fibration $\mathcal{S}^H(M) \to H_*(M; \mathbb{L}_\bullet) \to L_*(\mathbb{Z}[\pi])$. However, the sequence*

$$\mathcal{S}^{Top}(M) \to H_*(M; \mathbb{L}_\bullet) \to L_*(\mathbb{Z}[\pi])$$

is not exact, as was first noted by Nicas. The above is correct with homology manifolds instead of topological manifolds. For topological manifolds we have instead the exact sequence

$$\mathcal{S}^{Top}(M) \to H_*(M; \mathbb{L}_\bullet^{\langle 1 \rangle}) \to L_*(\mathbb{Z}[\pi]),$$

where $\mathbb{L}_\bullet^{\langle 1 \rangle}$ is a connective form of the L-spectrum.

Remark 8.102. *If $\partial M \neq \emptyset$, it does follow that $\mathcal{S}^{Top}(M \operatorname{rel} \partial M) = \mathcal{S}^H(M \operatorname{rel} \partial M)$ and subsequently $\mathcal{S}^H(M) \cong \mathcal{S}^H(M \times \mathbb{D}^4 \operatorname{rel} \partial) \cong \mathcal{S}^{Top}(M \times \mathbb{D}^4 \operatorname{rel} \partial)$. These groups do not equal $\mathcal{S}^{Top}(M)$, a fact that we illustrate in the next example. The isomorphism means that all elements of homology structure set have representatives that are topological spaces, the homology manifolds, but that these representatives may not be closed Top manifolds.*

Example 8.103. *Let $n \geq 6$. Then the homology structure set $\mathcal{S}^H(\mathbb{S}^n)$ of the n-sphere is isomorphic to $\mathbb{Z}$, detected by the realizability of the Quinn index.*

Corollary 8.104. *Let M^n be a closed manifold with $n \geq 6$. Then the Siebenmann periodicity map $c : \mathcal{S}^H(M) \to \mathcal{S}^H(M \times \mathbb{D}^4)_{\text{rel}}$ is an isomorphism.*

Remark 8.105. *This correspondence fails when H is replaced with Top and therefore the Top surgery exact sequence contains the term $H_*(-; \mathbb{L}_\bullet)$. On the other hand, the extension of the Poincaré conjecture fails when we move to the homology manifold category. However, the right-hand side consists entirely of Top structure representatives. In other words, we do not know an s-cobordism theorem for DDP homology manifolds.*

Remark 8.106. *At the writing of this text, it is not known if two DDP homology manifolds that are homotopy spheres with the same Quinn index are Top homeomorphic.*

As in the topological setting, we can pose a version of the Borel conjecture. In this case, we define a closed homology manifold X to be *homology-manifold rigid* if $\mathcal{S}^H(X)$ is trivial. We have already seen that the n-sphere $\mathbb{S}^n$ is not homology-manifold rigid when $n \geq 6$, showing that the Poincaré conjecture cannot be extended to the setting of homology manifold setting. However, we can reasonably conjecture the following.

Conjecture 8.107. *(Extended Borel conjecture) Let X be a closed homology manifold which is an Eilenberg-Maclane space $K(\pi, 1)$. Then $\mathcal{S}^H(X)$ is trivial.*

Proposition 8.108. *Let X^n be a closed Top manifold with $n \geq 6$ which is an Eilenberg-Maclane space $K(\pi, 1)$. If the Novikov conjecture holds for π, then there is an isomorphism $\mathcal{S}^H(X) \cong \mathcal{S}^{Top}(K(\pi, 1))$.*

Theorem 8.109. *If the Borel conjecture is true for π and for all $\pi \times \mathbb{Z}^k$, then the Extended Borel conjecture holds for π.*

At the writing of this book, there is no known closed homology manifold which is aspherical but which is not homotopy equivalent to a closed Top manifold. However, they presumably exist.

8.7 STRATIFIED SURGERY

Many spaces that arise naturally are not manifolds or even homology manifolds, but are constructed of pieces which are themselves manifolds. Stratified spaces can be considered as manifolds with singularities which themselves may be singular. The differences between strata are manifolds or, more generally, ANR homology manifolds, and the pieces should fit together in some prescribed fashion. We list some examples of stratified spaces that appear naturally throughout mathematics.

1. Any manifold $(M, \partial M)$ with boundary is a stratified space with two strata. The bottom stratum is the boundary ∂M, and the top stratum is the whole manifold M. The pure strata, which are by definition the manifold parts in a stratum minus lower ones, are the boundary and the interior.
2. A polyhedron P can be stratified so that any PL homeomorphism $P \to P$ preserves the strata. Here the simplices are aggregated according the number of times that their link desuspends. An example of Anderson [12], which is a variation of Milnor's counterexample to the Hauptvermutung, shows that there is no topologically intrinsic stratification.
3. Another interesting source of stratified spaces comes from embeddings $f : N \to M$. We artificially create a stratum in the ambient manifold M, namely the submanifold N. We can then consider embedding theory as a proper subset of the theory of stratified spaces. Similarly, immersions $g : N \to M$ give rise to at least two interesting stratified spaces: the image $g(N)$ of the immersed manifold and the ambient manifold M with the image $g(N)$ as a union of strata. The singular set consists of double points, and the space is stratified by triple points, quadruple points, and so on.
4. Algebraic varieties are stratified spaces because of their triangulability. See Hironaka [310]. But one may prefer algebraically natural stratifications to the PL intrinsic stratification.
5. Suppose that a compact group or even a finite group G acts on a manifold M.

If $H \leq G$ is a subgroup, then there are fixed sets M^H. In the smooth case, the fixed sets M^H are actually manifolds. In the topological cases, one should assume this property as well as local flatness. In all cases, the quotient space becomes stratified using the orbit types of the action as a stratification. Some authors will use a more refined stratification in which different components of fixed sets of subgroups are assigned different indices.

6. A last source of stratified space is the compactification of moduli spaces or other varieties, or even Riemannian manifolds, under appropriate geometric assumptions.

These lists overlap with each other, and each suggests its own line of questioning, vocabulary of invariants, and methods of attack. Each theory has its historical successes and outstanding problems. These subjects can be viewed within one context, so that some of the ideas that naturally would arise in one area or another can be applied to all of them. In each case, one could imagine the examples that should be included. These choices are reflected in the choice of category of stratified spaces.

One fundamental problem in extending the classification of manifolds to stratified spaces is the lack a good tangent bundle theory. Since the local structure changes from point to point, it is clear that sheaf-theoretic ideas should play a role. Moreover, there is a trivial local-global inseparability built into the theory. By taking the cone on a space, we transfer a global problem on one stratified space to a local one on another. In actuality, however, the process reverses: induction on the number of strata can be used, and global information about spaces with fewer strata is used to solve local problems that occur with more strata. By analogy with the surgery exact sequence for manifolds, one would desire a sequence

$$\cdots \to L_{*+1}(X \times I) \to \mathcal{S}^{Top}(X) \to H_0(X; \mathbb{L}_\bullet(\mathrm{loc})) \to L_*(X)$$

where $L_*(X)$ is an inductively defined surgery theory constructed from the surgery theory of the strata, and the homology term is a cosheaf homology adapted to the local structure. The main result is that this statement is almost true in certain categories of stratified spaces. There is a slight deviation that can sometimes be ignored if one is lucky with K-theory and decorations, and the deviation is always 2-torsion. Here the L-term depends on the space and not just on its fundamental group, orientation and dimension.

The material in this section is largely based on Browder-Quinn [94], Quinn [525] and Weinberger [691].

8.7.1 Browder-Quinn theory: the transverse stratified classification

Many classes of stratified spaces have been defined and studied: Whitney stratified spaces, PL stratified spaces, PL weakly stratified spaces, manifold stratified spaces, stratified Poincaré complexes, stratified homotopy type. We first present some general definitions.

Definition 8.110. *Let S be a finite partially ordered indexing set. Then a space X is a stratified space (or S-stratified space) if it has the following properties:*

1. *for each $s \in S$, there is a closed subset X_s;*
2. *for each $s, s' \in S$, if $s \leq s'$, then $X_s \subseteq X_{s'}$ is a cofibration.*

The X_s is called a stratum of X and the difference of the form $X^s \equiv X_s - \bigcup_{t<s} X_t$ is called a pure stratum of X.

Definition 8.111. *Suppose that X and Y are both S-stratified spaces. A stratified map $f : X \to Y$ is a continuous function such that $f(X^s) \subseteq Y^s$ for all $s \in S$.*

Definition 8.112. *A strongly stratified space with two strata is given by a filtration $X_0 \subseteq X_1$ such that*

1. *X_0 and X_1 are compact topological spaces;*
2. *X_0 and $X_1 \backslash X_0$ are manifolds, possibly disconnected with components of different dimension;*
3. *there is a closed neighborhood N_i of every component $X_0^{(i)}$ of X_0 in X_1 such that $F_i \to \partial N_i \xrightarrow{p_i} X_0^{(i)}$ is a block bundle or fiber bundle with some manifold fiber F_i and N_i homeomorphic to the mapping cylinder of p_i.*

We consider maps that are stratified and transverse to each pure stratum. These maps are called *transverse* by Browder-Quinn [94] and *normally smooth* by Fulton-MacPherson [258] and Goresky-MacPherson [265, 266]. The key point here is that there is a theory of neighborhoods of a pure stratum X_s and we can pull back the neighborhood under a map from a manifold to X_s.

Wall's book shows that the key ingredient for any surgery theory is the π-π theorem, which gives the triviality of surgery obstructions in the relative situation. For strongly stratified spaces, one proves the π-π theorem inductively. For the bottom stratum, it holds by the result for manifolds. Then we use the bundle structure for a neighborhood and transversality to give a solution in a neighborhood at the bottom stratum. On removing this neighborhood one has a relative problem that is still stratified π-π. These considerations from Wall also define L-spaces and spectra. We can therefore define Browder-Quinn groups for a stratified space X as a cobordism theory. See Chapter 9 of Wall [672]. As before, its homotopy groups are the Browder-Quinn obstruction groups to transverse stratified surgery. The proof of the following is done by induction on the strata of X.

Proposition 8.113. *(Transverse isovariant π-π theorem) Let (X, Y) be a strongly stratified pair with $Y = \partial X$, and suppose that each pure stratum of X touches exactly one* *stratum of Y for which the inclusion is a 1-equivalence (stratified π-π condition). If all strata of Y are of dimension at least 5, then any normal invariant $(V, W) \to (X, Y)$ is stratified normally cobordant to a stratified (simple) homotopy equivalence. If there are several strata, one extends this definition inductively.*

Definition 8.114. *Let X be a strongly stratified space with closed pure strata $\overline{X^i}$. An h-cobordism with boundary X is a stratified space Z with boundary $X \coprod X'$, where the inclusions of X and X' into Z are stratified homotopy equivalences, and the neighborhood data for the strata of Z are the pullbacks with respect to the retractions of the data for X and X'.*

Definition 8.115. *For any space Y, let* $\mathrm{Wh}(Y)$ *denote the Whitehead group* $\mathrm{Wh}(\pi_1(Y))$. *If X is a stratified space with closed pure strata $\overline{X^i}$, then* $\mathrm{Wh}^{BQ}(X) = \bigoplus_i \mathrm{Wh}(\overline{X^i})$ *is called the Browder-Quinn Whitehead group.*

Theorem 8.116. *(Stratified h-cobordism theorem) Let X be a PL stratified space. Then there is a bijective correspondence between h-cobordisms with a boundary component PL homeomorphic to X that are products on strata of dimension ≤ 4 and the Browder-Quinn Whitehead group of X relative to strata of dimension ≤ 4.*

Proof. We can prove the statement inductively with the classical h-cobordism theorem as the base case, and the strong stratifications to lift product structures to the next stratum. The stratified h-cobordism theorem holds in the general stratified case because in the PL case inclusions of boundary components are automatically transverse, and because PL homeomorphisms, by definition, preserve all block structures. □

We recall that, in the Browder-Quinn setting, we are given a map $(Y_1, Y_0) \to (X_1, X_0)$ such that (i) $Y_0 \xrightarrow{\sim} X_0$ is covered by a bundle map $\partial Y_1 \to \partial X_1$, (ii) $Y_1 \xrightarrow{\sim} X_1$ restricts to $\partial Y_1 \to \partial X_1$. This condition can be made to examine surgery problems in any of the three categories.

Definition 8.117. *Define $\mathcal{S}^{BQ}(X)$ to be the collection of strongly stratified spaces Y with a transverse stratified simple homotopy equivalence $Y \to X$, up to Cat strongly stratified simple isomorphism.*

Interestingly, the L-theory is independent of the *type* of local structure with pullback.

The appropriate L-groups for the stratified setting are the *Browder-Quinn L-groups*.

Definition 8.118. *If X_0 is the minimal stratum of a stratified space X, let* $\mathrm{cl}(X \backslash X_0)$ *be the closed complement of X_0. Its boundary* $\partial\mathrm{cl}(X \backslash X_0)$ *is a stratified block bundle over X. Then the Browder-Quinn L-group $L^{BQ}(X)$ are defined to be the homotopy groups of the homotopy fiber of the composition:*

$$L_*(X) \xrightarrow{tr} L_*^{BQ}(\partial \mathrm{cl}(X \backslash X_0)) \xrightarrow{i} BL_*^{BQ}(\mathrm{cl}(X \backslash X_0) \text{ rel } \partial),$$

where tr is the transfer map and i is induced by inclusion.

Remark 8.119. *In this notation we are tacitly using X to keep track of dimensions, so that, for a closed manifold, the L-group would simply be $L_{\dim X}(X)$. Here B denotes a delooping, which is possible because all L-spectra are 4-periodic. Because the boundary of a regular neighborhood of X_0 is one dimension lower than X, the delooping is required. With this notation, transfers do not change indices.*

The main result is the following.

Theorem 8.120. *Let X be a strongly stratified space. The Browder-Quinn groups $L_*^{BQ}(X)$ fit into a long exact sequence*

$$\cdots \to [\Sigma X : F/Cat] \to L_*^{BQ}(X \times I)_{\text{rel}} \to \mathcal{S}^{BQ}(X) \to [X : F/Cat] \to L_*^{BQ}(X).$$

Here $[X : F/Cat]$ can be identified with the transverse isovariant normal invariant set $\mathcal{N}^{BQ}(X)$.

The proof requires a number of steps. First, we must identify normal invariants with $[X : F/Cat]$. One can use the strong stratification and transversality. By the usual π-π theorem, the L-groups can be defined by Wall's cobordism method. These L-groups fit into the exact sequence relating X, $X \backslash X_0$, and X_0, and therefore coincide with the definition above.

8.7.2 PL theory

We will explain the classification of PL stratified spaces in this section with the topological theory lurking in the background. As usual, the mathematics is somewhat simpler than in the topological case, but the results are somewhat less perfect, as we shall see.

Definition 8.121. *A filtered space X is a PL stratified space if all the X_s are polyhedral and, for any two points x and y in a component of some pure stratum, there is a PL isotopy $f_t : X \to X$ such that $f_0 = id_X$ and $f_1(x) = y$. We say that X is a PL weakly stratified space if the PL isotopy condition is replaced with a Top isotopy condition.*

We can also introduce similar concepts with respect to homotopies.

Definition 8.122. *Let X be a filtered space with two strata. We define the homotopy link or holink of $X_1 \subseteq X$ to be the collection of all paths $\alpha : [0,1] \to X$ such that $\alpha(1) \in X_1$ and $\alpha(t) \in X \backslash X_1$ for all $t < 1$. We then say that X is homotopy stratified if the map $\beta : \text{holink}(X_1 \subseteq X) \to X_1$ given by evaluation at the endpoint is a fibration. A manifold stratified space is a homotopically stratified space for which all pure strata are manifolds. See Quinn [525] for a full discussion and the case of multiple strata.*

In the PL case, one can use genuine links which are themselves stratified spaces with fewer strata.

Definition 8.123. *Assume that X and Y are homotopy stratified spaces and let $f : X \to Y$ be a stratified map. We say that f is homotopy transverse if there is a diagram*

$$\begin{array}{ccc} \text{holink}(X_1 \subseteq X) & \xrightarrow{f_*} & \text{holink}(Y_1 \subseteq Y) \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & Y_1 \end{array}$$

where the top map f_* is a fiberwise homotopy equivalence (fibration). There is no condition on the part away from the strata. A stratified map $f : X \to Y$ is a stratified homotopy equivalence if it is a homotopy equivalence on every stratum, with no transversality conditions. Such a map will pull back the holink fibrations.

The significance of this notion is underscored in the following simple result of supreme ideological importance. It suggests the applicability of Browder-Quinn surgery in the non-transverse setting.

Theorem 8.124. *A stratified homotopy equivalence is homotopy transverse.*

Proof. The statement is true almost by definition, since stratified maps take the system of deleted neighborhoods in any stratum touching a point to the corresponding neighborhood system in the target space. By using a map and its homotopy inverse, one can establish an equivalence between the holink fibrations in the domain and range. □

Remark 8.125. *In other words, the transversality condition arising in the Browder-Quinn theory does not quite hold in the geometric setting of the genuine neighborhood, but it automatically holds at the fibration level.*

Corresponding to the different types of spaces, there are different types of transversality conditions that one can impose on the map. Suppose first that one has a PL stratified space. Then around each stratum, one has the structure of a stratified block bundle whose fiber is the (geometric) stratified link that is stratified homotopy equivalent to the stratified local holink. The proof is similar to that for the existence of block bundle structures for submanifolds of a PL manifold. See Rourke-Sanderson [557] or Stone [620] for a complete proof.

Definition 8.126. *Define a map $f : X \to Y$ to be stratified transverse*[1] *if f is a PL stratified map that defines a PL homeomorphism between the boundary of a regular neighborhood of each pure stratum in X and the pullback of the corresponding stratified block bundle over the target stratum in Y.*

[1] Sometimes the adjectives are used in the reverse order.

Unfortunately, in the PL case, this notion will not suffice if we are interested in ideas that enable an isomorphism classification. One of two routes can be taken.

1. One can dispense with PL homogeneity as a hypothesis in PL stratified topology. Doing so, one can use L^h-spectra throughout the theory. One will have to recall that there will be very large numbers of PL structures on such spaces whenever Whitehead groups are non-trivial. In addition, classification will be actually up to concordance rather than up to PL isomorphism.
2. If one can insist on PL homogeneity as a condition for PL stratified spaces, then we do not automatically have a Browder-Quinn type transversality condition because simple homotopy transversality is required rather than the automatic homotopy transversality. In this case, besides the usual assumption of simple homotopy equivalence needed by classification theorems, an additional Whitehead group condition is required at each holink. This annoyance is however manageable.

Theorem 8.127. *(PL stratified classification) Let X be a PL stratified space with no four-dimensional strata and no neighboring strata whose dimensions differ by fewer than 6. Then there is a fibration for computing $\mathcal{S}^{PL}(X)$, the collection of simple homotopy transverse simple homotopy equivalences $Y \to X$ up to PL homeomorphism. This fibration is given by*

$$\mathcal{S}^{PL}(X) \to H_0(X; \mathbb{L}_\bullet^{BQ}(\text{loc})) \to L^{BQ}(X) \oplus (H_{i-4}(X_i; \mathbb{Z}_2) \oplus \mathbb{Z}).$$

Remark 8.128. *In order to understand the neighborhood of the bottom stratum, it suffices to consider a problem in blocked stratified surgery with one fewer stratum. This process is inductively analyzable, since surgery theory handles block bundles just as easily as single spaces. However, in understanding links, one must in principle avoid the four-dimensional ones, given the difficulties in that dimension. Therefore one would require a codimension 5 hypothesis for the surgery, in addition to the usual dimension 5 conditions.*

Remark 8.129. *The codimension 5 restrictions are a nuisance and can often be removed at the cost of complicating the sequence some more. They mask our ignorance of low-dimensional topology and the failure of low-dimensional h-cobordism theorems. An example is the case of manifolds with boundary, which certainly violates the codimension condition, but in fact is not a problem. One can also certainly manage these bad situations if one works relatively to them.*

Remark 8.130. *The $\bigoplus H_{i-4}(X_i; \mathbb{Z}_2)$ term is a collection of Kirby-Siebenmann obstructions of pure strata. A homology calculation shows that these classes can actually be placed in the homology group of the closed stratum, because of the codimension assumption. The copies of $\mathbb{Z}$ reflect the difference between F/Top and $\mathbb{L}_0$. The map from $H_0(X; \mathbb{L}_\bullet^{BQ})$ into the i-th copy of $\mathbb{Z}$ can be computed as the result of applying a restriction of the cosheaf to the i-th pure stratum, and there restricting further to any small* *copy of* $\mathbb{R}^i$. *In the topological case, these copies of* $\mathbb{Z}$ *can be removed at the cost of allowing homology manifold stratified spaces, as mentioned above. We will refer to the homology term as the stratified normal invariant set. Certain simple homotopy transverse maps with bundle data give rise to elements of this group. Then it is possible to define characteristic classes for arbitrary stratified spaces, first introducing coefficients into the theory.*

8.7.3 The stratified Top category

The topological theory has some very desirable features, but it also has some complications.

In this section we would like to explain how to calculate the structure set of a manifold stratified set, i.e. the (homology) manifold stratified spaces that are simple homotopy equivalent to a given X, up to Top homeomorphism or s-cobordism. Unfortunately, the surgery theory is not quite so simple because of the constant shifts of decorations that occur in the formula for $L_*(\mathbb{Z} \times G)$, for example. We need two fibrations, one for a so-called stable calculation and one for destabilizing.

We refer to reader to Quinn [525] for the precise definitions, which are somewhat more subtle in Top than the PL case.

Except for our ignorance of the homogeneity properties of ANR homology manifolds, Top homogeneity is better than in the PL case. Quinn [525] shows that Top homogeneity holds if we assume that the pure strata are manifolds and that the topological stratification has homotopy homogeneity, so that the holink neighborhood maps down as a fibration. Furthermore, according to Quinn [525], the Whitehead groups decompose according to the strata, i.e. $\mathrm{Wh}^{Top}(X) \cong \bigoplus \mathrm{Wh}^{Top}(X_i \,\mathrm{rel}\, X_{i-1})$.

However, the Whitehead groups $\mathrm{Wh}^{Top}(X_i \,\mathrm{rel}\, X_{i-1})$ are not the same as the PL answer. We have seen polyhedra that are homeomorphic, but are not PL homeomorphic. The source of these examples is that the boundaries of the open strata are not topologically well-defined. Indeed we see this situation in point singularities, where the proper Whitehead group appears on the right side. In general, there is an issue of a controlled h-cobordism class, giving rise to a sequence

$$\begin{aligned} H_0(X; \mathrm{Wh}(\pi_1(\mathrm{loc}))) \to \mathrm{Wh}(\pi_1(X_i \backslash X_{i-1})) \to \mathrm{Wh}^{Top}(X_i \,\mathrm{rel}\, X_{i-1}) \\ \to H_0(X; \widetilde{K}_0(\pi_1(\mathrm{loc}))) \to \widetilde{K}_0(\pi_1(X_i \backslash X_{i-1})) \end{aligned}$$

which is the natural combination of Siebenmann's proper theory with the calculation of controlled Whitehead groups.

Then we run into the trouble mentioned above that controlled surgery, unlike controlled K-theory, is not generally a homology theory. The structure sets are not fibers of assembly maps. Only $L^{-\infty}$ has this feature. As a result, it is necessary to stabilize first and then destabilize to obtain objects that directly relate to the spectra that we have studied. More specially, we cross with tori and then remove them. Equivalently, we replace the

L-group by $L^{-\infty}$ and then move up the sequence of Rothenberg sequences along the terms $H^*(\mathbb{Z}_2; K_{1-i}(\pi_1(W), \pi_1^\infty(W))$, where the relative K-groups are bounded proper Whitehead groups of $W \times \mathbb{R}^i$, to obtain the correct answer for the stratified structure set. Using the end theorems of Quinn [522], we can deploy a similar method to deal with general stratified spaces.

Finally, we note an issue that already arises in the PL case: the Browder-Quinn L-groups L^{BQ} do not generally decompose in any way into pieces that are relative to lower strata. As a result, the answer is necessarily more holistic, and we will presently give a few examples. We leave a more detailed discussion and many more examples to Weinberger's book [691].

Theorem 8.131. *(Stable Top stratified classification) Let X be a manifold stratified space with no four-dimensional strata. Then there is a fibration for computing the collection $\mathcal{S}^{Top}(X)$ of topologically simple homotopy equivalences $Y \to X$ up to homeomorphism:*

$$\mathcal{S}^{-\infty}(X) \to H_0(X; \mathbb{L}_\bullet^{-\infty}(\text{loc})) \to L^{-\infty}(X) \oplus \bigoplus \mathbb{Z}.$$

Here the L-spectrum belongs to BQ-theory. If one works with homology manifold stratified spaces up to s-cobordism, the sequence is a fibration without the $\bigoplus \mathbb{Z}$ term.

Remark 8.132. *It is a beautiful fact that* $\mathrm{Wh}(\mathbb{S}^1 \times cM \text{ rel } \mathbb{S}^1 \times M)$ *is a sum of Nil groups. This phenomenon is related to the asphericity of the circle and the Farrell-Jones conjecture, and reflects the situation of Farrell fibering for an approximate fibration over the circle rather than merely a fiber bundle.*

We will move on to other theories of stratified spaces and their classification.

8.7.4 Examples

A. Manifolds with boundary

If we study manifolds with boundary, then we already know the nature of the link of each simplex on the boundary. The holink is a point, which has the same geometry as predicted by surgery theory. In other words, surgery predicts that its structure set should be contractible and therefore so should spaces of point block bundles. It can be checked by inspection, and as a result we do not need the caveats of the general PL theory.

Now the Whitehead groups of the links are trivial, so simple homotopy transversality is just homotopy transversality. In other words, we want stratified homotopy equivalence, which is equivalent to just a homotopy equivalence of pairs.

1. The relevant Whitehead group is merely the sum of the Whitehead groups of the strata. Note however that the involution is unusual; it does not preserve the sum decomposition, but must be modified by the map on Whitehead groups induced by the inclusion.

2. The L-group $L^{BQ}(M, \partial M)$ is the usual relative L-group $L_m(\pi_1(M), \pi_1(\partial M))$.
3. The cosheaf of L-spectra is interesting here. There are two kinds of points, interior and boundary points. For an interior point, one has the usual $\mathbb{L}_\bullet(\mathbb{R}^n)$, which is a shifted $\mathbb{L}_\bullet$. At boundary points, one has the cosheaf $\mathbb{L}_\bullet(\mathbb{R}^n_+, \mathbb{R}^{n-1})$ corresponding to the closed upper half-plane. The π-π theorem asserts that the latter is contractible. Therefore the cosheaf homology is simply relative $\mathbb{L}_\bullet$-homology. It will play a role when we study other kinds of structure sets besides $\mathcal{S}^{BQ}$.

B. Isolated singularities

Suppose that X is a space with an isolated singularity. In the PL case, the space X is given the structure of a PL manifold M with boundary ∂M with the cone $c(\partial M)$ on the boundary attached. In other words, the space X can be considered as $M \cup c(\partial M)$ for some manifold M with boundary, and the identification $\mathcal{S}^{PL}(X) = \mathcal{S}^{PL}(M \text{ rel } \partial M)$ is forced by transversality.

1. Simple homotopy transversality implies that any equivalence will restrict to simple equivalences on the boundary components, and Wh^{PL} is simply the Whitehead group of the closed pure stratum.
2. For simplicity, let us only describe the rel singularity theory. In the Browder-Quinn theory, the neighborhood of the singularity must be $c(\partial M)$, and the L-group is the usual L-group $L_*(\pi_1(M))$ of the complement. In the more flexible PL theory, however, the L-group would be the L-group of a pair, because the boundary is not fixed by a mere simple homotopy transversality.
3. Classically, the singular point does not contribute at all and we have the relative $\mathbb{L}_\bullet$-homology of the complement. From the stratified point of view the cosheaf homology has different stalks at the manifold points and at the singularity: at the manifold points we have the usual $\mathbb{L}_\bullet$, but at the singular point we have $\mathbb{L}_\bullet(G)$, where G is the fundamental groupoid of the boundary, giving precisely the difference in the surgery terms between the classic and the stratified sequence.

These ideas are interrelated by the following diagram:

$$\begin{array}{ccccc}
 & & \mathbb{H}(*;\, \mathbb{L}_\bullet(\pi_1(M))) & \longrightarrow & \mathbb{L}(\pi_1(M)) \\
 & & \uparrow & & \uparrow \\
\mathbb{S}(W,\, M) & \longrightarrow & \mathbb{H}(W,\, M;\, \mathbb{L}_\bullet) & \longrightarrow & \mathbb{L}(\pi_1(W),\, \pi_1(M)) \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{S}(W \cup cM, \text{rel } *) & \longrightarrow & \mathbb{H}(W \cup cM; \mathbb{L}_\bullet^{BQ}(\text{rel } *)) & \longrightarrow & \mathbb{L}(\pi_1(W))
\end{array}$$

C. Submanifolds and the Browder splitting theorem

Suppose that we consider a stratified space X consisting of a manifold W^n with a locally flat submanifold M^m. There are two forgetful maps that can be combined to construct a map $L^{BQ}(X) \to L^s_n(W) \oplus L^s_m(M)$. If $n-m \geq 3$, there is an isomorphism $\pi_1(W\setminus M) \to \pi_1(W)$. Therefore we have split the defining exact sequence for $L^{BQ}(X)$, with the result that $L^{BQ}(X) \to L^s_n(W) \oplus L^s_m(M)$ is an isomorphism. By comparing surgery exact sequences, we then have

$$L^s_{m+1}(M) \to \mathcal{S}^{BQ}(X) \to \mathcal{S}^s(W) \to L^s_m(M).$$

We can interpret this sequence in the following way.

Theorem 8.133. *(Browder splitting) A homotopy equivalence $f : W' \to W$ can be split along a submanifold M^m of codimension at least 3 iff the surgery obstruction of f restricted to the transverse inverse image $f^{-1}(M)$ vanishes in $L_m(\pi_1(M))$.*

We will later return to $\mathcal{S}^{str}(X)$ without the strong transversality conditions of the Browder-Quinn theory.

8.7.5 Some final applications

A. Supernormal spaces

Definition 8.134. *A manifold stratified space is X supernormal if all the local holinks are simply connected in each stratum (i.e. $X^i \subseteq X^i \cup X^j$).*

Definition 8.135. *Let X be a stratified space with singular set Σ. Then $\mathcal{S}^{str}(X \text{ rel } \Sigma)$ of (X, Σ) is the collection of stratified homotopy equivalences $f : (Y, \Sigma) \to (X, \Sigma)$, where Y is a manifold stratified space and $f|_\Sigma = id$.*

Remark 8.136. *Supernormality implies that the complement $X\setminus\Sigma$ is simply connected near Σ. In other words, for each $x \in \Sigma$ and a path γ in $X\setminus\Sigma$ near x, there is a disk $\mathbb{D}^2$ in $X\setminus\Sigma$ near x such that $\gamma = \partial\mathbb{D}^2$. Here Σ is not necessarily even a manifold. This condition is often called 1-LCC, i.e. 1-locally-coconnected.*

Theorem 8.137. *(Cappell-Weinberger [133]) Suppose that X is an n-dimensional supernormal space with $n \geq 5$ and singular set Σ. Let $G = \pi_1(X)$. Then there is an isomorphism*

$$\mathbb{S}(X \text{ rel } \Sigma) \cong \text{Fib}(\mathbb{H}_n(X; \mathbb{L}_\bullet) \to \mathbb{L}_n(\mathbb{Z}[G])).$$

In other words, the structure set of X relative to Σ is given by the same homological description as for manifolds.

Remark 8.138. *For many applications related to group actions, the 1-LCC condition unfortunately does not hold, giving difficulties in both K-theory and L-theory. Indeed, the non-1-LCC situation is critical to understanding the Farrell-Jones conjecture for* *double quotient spaces $\Gamma\backslash G/K$ where Γ is a lattice with torsion.*

B. Spheres and complex projective spaces as links

Definition 8.139. *Let X and Y be CW complexes. A Poincaré embedding of X into Y consists of a spherical fibration $\xi \downarrow X$, a Poincaré pair (Z, E), homotopy equivalences between E and the total space of ξ, and Y with the union of the mapping cylinder of ξ and Z along E.*

When the links are spheres, then the entire complex is a manifold, and the singular set is just a locally flat submanifold. Our first result is that one can reduce the question of the existence of PL or Top locally flat embedding to homotopy theory (cf. Theorem 4.65). We used this theorem in our discussion of Siebenmann periodicity.

Theorem 8.140. *(Browder, Casson, Haefliger, Sullivan, Wall) Let M^m and N^n be Top manifolds with $m - n \geq 3$. Then M embeds in N locally flatly iff M Poincaré embeds in N. More precisely, every Poincaré embedding comes from a geometric embedding.*

Let X consist of a manifold W with a submanifold M. Here we obtain $\mathcal{S}^{Top}(X \text{ rel } M) \cong \mathcal{S}^{Top}(W)$. This isomorphism means that M embeds in any manifold homotopy equivalent to W. We can improve this calculation by returning to the Browder-Quinn L-group calculation that yielded the Browder splitting theorem. It now applies both globally and at the cosheaf level to give an isomorphism $\mathcal{S}^{Cat}(X) \cong \mathcal{S}^{Cat}(W) \times \mathcal{S}^{Cat}(M)$. In other words, any manifold homotopy equivalent to M embeds in any manifold homotopy equivalent to W in a unique way compatible with a given Poincaré embedding. By eliminating the transversality condition, we have also eliminated the obstruction in Browder's theorem. This theorem was used in our approach to Siebenmann periodicity. Of course, the decomposition into strata does not hold in general. We typically have $\mathcal{S}^{Top}(X) \ncong \mathcal{S}^{Top}(X \text{ rel } \Sigma) \times \mathcal{S}^{Top}(\Sigma)$, for example in the case of manifolds with boundary.

We note here that the holink of a manifold with boundary is a point, i.e. a manifold with signature 1. In the case of submanifolds of codimension at least 3, the holink is a sphere of dimension at least 3, which kills surgery obstructions. These observations suggest the following theorem, which is a special case of a general theory of *multiaxial group action*. See Davis-Hsiang [197] and Cappell-Weinberger-Yan [135].

Definition 8.141. *If the group G acts on homology manifolds X and Y, then a map $f : X \to Y$ is called isovariant if it preserves isotropy subgroups, i.e $G_x = G_{f(x)}$ for all $x \in X$. The set $\mathcal{S}^{iso}(Y)$ consists of isovariant stratified homotopy equivalences $f : X \to Y$. We note that $\mathcal{S}^{iso}(Y)$ can be thought of as $\mathcal{S}^{str}(Y/G)$.*

Theorem 8.142. *(Cappell-Weinberger-Yan [135]) Suppose $\mathbb{S}^1$ acts locally smoothly and semifreely on M, then the isovariant structure set satisfies*

$$\mathcal{S}^{iso}(M) \cong \mathcal{S}^{iso}(\overline{M\backslash F_2} \text{ rel } F_0) \times \mathcal{S}^s(F_0),$$

where F_0 *(respectively* F_2*) is the union of the components of the fixed sets whose codimensions are 0 (respectively 2) mod 4.*

Remark 8.143. *Actually the above theorem is only exactly correct if one allows homology manifolds to be strata, not just manifolds. This issue arises when there is a fixed set of codimension 6, say an action on* $M \times \mathbb{R}^6$*. The manifold structure set of the complement is* $\mathcal{S}(M \times \mathbb{CP}^2)$ *when issues of decoration are ignored. The normal invariant set can be computed as* $\mathcal{N}(M) \times \mathcal{N}(M \times \mathbb{D}^2) \times \mathcal{N}(M \times \mathbb{D}^4)$*. When the Siebenmann periodicity map* $\mathcal{N}(M) \to \mathcal{N}(M \times \mathbb{D}^4)$ *is not surjective, the existence of a component in the cokernel will require that the structure set of the original* G*-manifold contain a homology manifold that is homotopy equivalent to* M *but itself is not a Top manifold.*

Remark 8.144. *We require the closure of* $M - F_2$ *because, since the holinks are simply connected, there is a canonical completion of this complement, which we will denote by* $\mathrm{cl}\,((M - F_2)/\mathbb{S}^1)$*. It has an equivariant mapping cylinder structure. As a result the relevant classification involves* $L^s_{m-1}(\pi_1(M), \pi_1(F_2))$ *and not the proper* L*-group.*

Remark 8.145. *A semifree circle action always has fixed sets of even codimension, and therefore its fixed sets are unions of components of various codimensions. In the quotient, the links of the components with codimension* $2d$ *are homotopy complex projective spaces* $\mathbb{CP}^{d-1}$*. These links have signature 0 for* d *even and signature 1 for* d *odd.*

Appendix A

Some background in algebraic topology

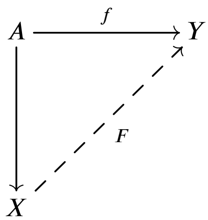
A.1 OBSTRUCTION THEORY

Many problems in geometric topology can be reduced to lifting or extension problems. An early example is the Smale-Hirsch theory of immersions, in which smooth immersions of one manifold in another are reduced to understanding a space of sections of some bundle, i.e. to solving a lifting problem. Indeed, the manifold M^n immerses in $\mathbb{R}^{n+k}$ iff the stable normal bundle has a lift to BO_k. Many more examples of *h-principles* appear in Gromov's book "Partial Differential Relations" [272], Spring [607], and Eliashberg-Mishachev [215], which provide ways to change certain kinds of smooth analytic problems into algebraic topological ones.

More relevant to our concerns are the results of classical smoothing theory (Hirsch-Mazur [316]) and its extension to the triangulation theory (Kirby-Siebenmann [360]) that reduce the problems of smoothing PL manifolds, or triangulating topological manifolds, to lifting problems. Surgery can be considered similarly; it is a situation in which the h-principle or lifting criterion for a solution is obstructed. The additional obstruction in this case lies in the Wall L-groups.

In order for this kind of reduction to be useful, one needs some technology for addressing lifting and extension problems. Obstruction theory is useful for obtaining information about both extension problems and lifting problems. One can even combine them, and study relative lifting problems.

The extension problem is as follows. Given a CW pair (X, A) and a map $f : A \to Y$ we wish to find an extension $F : X \to Y$ that extends the map $A \to Y$.



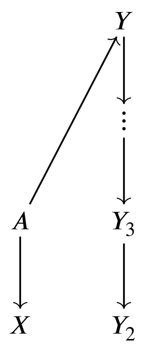
Obstruction theory is a classical approach to solving the extension problem $X \hookleftarrow A \to Y$ cell by cell, extending the given map on each skeleton of X before moving to the next. For simplicity we will assume that the targets are simply connected. Without this assumption, the cohomology groups below would require local coefficient systems. The following theorem captures the principal idea of obstruction theory.

Theorem A.1. *(Main theorem of obstruction theory) Let (X, A) be a relative CW complex. Assume that Y is a simply connected space and that $g : A \to Y$ is a continuous map. Then g is always extendable to the union $X^{(1)} \cup A$ of the 1-skeleton $X^{(1)}$ of X and A. If $n \geq 1$ and $g : A \to Y$ has been extended to $g^{(n)} : X^{(n)} \cup A \to Y$, then we have the following:*

1. *There is a cellular cocycle $\vartheta^{n+1}(g^{(n)}) \in C^{n+1}(X, A; \pi_n(Y))$ which vanishes iff $g^{(n)}$ extends to a map $g^{(n+1)} : X^{(n+1)} \cup A \to Y$.*
2. *The cohomology class $[\vartheta^{n+1}(g^{(n)})] \in H^{n+1}(X, A; \pi_n(Y))$ vanishes iff the restriction*
$$g^{(n)}|_{X^{(n-1)}\cup A} : X^{(n-1)} \cup A \to Y$$
extends to a map $g^{(n+1)} : X^{(n+1)} \cup A \to Y$. Note that in general the map $g^{(n+1)}$ redefines the values of $g^{(n)}$ on $X^{(n)} \cup A$.

Notice that the $(n + 1)$-st obstruction $[\vartheta^{n+1}(g^{(n)})]$ is defined when and only when the lower obstructions are zero, and depends on the choices of extension chosen.

The cohomology group containing the obstruction can be understood dually if we consider a Postnikov system $\{X_n\}$ of X.



Inductively we will try to lift the map $X \to Y_i$ to Y_{i+1}. Let us concentrate on that part of the Postnikov system.

$$\begin{array}{ccccc} A & \longrightarrow & Y_{i+1} & & \\ \big\downarrow & & \big\downarrow & & \\ X & & Y_i & \longrightarrow & K(\pi_{i+1}(Y);\ i+2) \end{array}$$

The k-invariant $Y_i \to K(\pi_{i+1}(Y), i+2)$ classifies the fibration $Y_{i+1} \to Y_i$ with fiber $K(\pi_{i+1}(Y), i+1)$. Since we have a lift on A, the restriction to A can be assumed to lie in the basepoint of $K(\pi_{i+1}(Y), i+2)$. A lift to Y_{i+1} is then the same as the triviality of the relative cohomology class $H^{i+2}(X, A; \pi_{i+1}(Y))$.

Corollary A.2. *If the cohomology classes*

$$\omega_n = [\vartheta^{n+1}(g^{(n)})] \in H^{n+1}(X, A; \pi_n(Y))$$

vanish for all n, then the map $g : A \to Y$ can be extended to a map $g' : X \to Y$. In particular, suppose that Y is a connected abelian CW complex; i.e. $\pi_1(Y)$ is abelian and acts trivially on higher homotopy. If (X, A) is a CW pair such that $H^{n+1}(X, A; \pi_n(Y)) = 0$ for all n, then every map $A \to Y$ can be extended to a map $X \to Y$.

Note that, in some cases, the groups $H^{n+1}(X, A; \pi_n(Y))$ will all vanish. For example, when the dimension of X is r and Y is $(r-1)$-connected, then these cohomology groups are trivial for all n, making the extension problem a trivial verification.

Corollary A.3. *Any map $f : X \to Y$ from an n-dimensional CW complex to an n-connected space is null-homotopic.*

Remark A.4. *The problem of finding a homotopy between two maps is a special case of the extension problem. Suppose that $f_1, f_2 : X \to Y$ are maps of topological spaces and $A \subseteq X$ is a subspace such that $f_1|_A, f_2|_A : A \to Y$ are homotopic via a homotopy $H : A \times I \to Y$. Define the map $K : (X \times \{0,1\}) \cup (A \times I) \to Y$ given by*

$$K = \begin{cases} f_1 & \text{on } X \times \{0\}, \\ H & \text{on } A \times I, \\ f_2 & \text{on } X \times \{1\}. \end{cases}$$

The homotopy can be extended iff certain obstructions $\omega_n = \omega_n(f_1, f_2)$ vanish in

$$H^{n+1}(X \times I, (X \times \{0,1\}) \cup (A \times I); \pi_n(Y)) \cong H^n(X, A; \pi_n(Y))$$

for all nonnegative integers n. Notice that, if A is empty, then the homotopy problem above gives obstructions $\omega_n(f_1, f_2)$ in $H^n(X; \pi_n(Y))$ to finding a homotopy between f_1 and f_2. For $A = X \times \mathbb{S}^k \subseteq X \times \mathbb{D}^{k+1} \simeq X$, the obstructions lie in $H^{n+k}(X; \pi_n(Y))$.

More precisely we can write the following theorem.

Theorem A.5. *Let (X, A) be a relative CW complex and let Y be an n-simple space. Suppose that $f_1, f_2 : X \to Y$ are maps which agree on A and $F^{(n)} : X^{(n-1)} \times I \to Y$ is a homotopy from $f_1|_{X^{(n-1)}}$ to $f_2|_{X^{(n-1)}}$ relative to A. Then there is a cohomology class*

$$[\vartheta^{n+1}(F^{(n)})] \in H^{n+1}(X \times I, (X \times \{0,1\}) \cup (A \times I); \pi_n(Y)) \cong H^n(X, A; \pi_n(Y))$$

which vanishes iff the restriction $F^{(n)}|_{X^{(n-2)} \times I} : X^{(n-2)} \times I \to Y$ extends to a homotopy $F^{(n+1)} : X^{(n)} \times I \to Y$ of $f_1|_{X^{(n)}}$ to $f_2|_{X^{(n)}}$ relative to $A \times I$.

Remark A.6. *Obstruction theory is the geometry behind the Atiyah-Hirzebruch spectral sequence for computing generalized cohomology theories. We will often use it to obtain crude estimates on the sizes of algebraic topological invariants. Note that the k-invariants describe in principle how the choices of extensions on lower skeleta influence higher obstructions. When X is an infinite loop space, then they give rise to the differentials in the Atiyah-Hirzebruch spectral sequence.*

Remark A.7. *Suppose that $p : E \to B$ is a fibration with fiber F, and let $f : X \to B$ be a map which we wish to lift to E. Also suppose that the fibration can be "classified" by a map to Z where $\Omega Z \simeq F$. For any CW complex X, there is then an exact sequence*

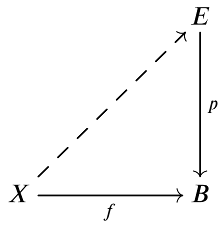
$$[X : F] \to [X : E] \xrightarrow{p_*} [X : B] \xrightarrow{q_*} [X : Z].$$

The map $f : X \to B$ can be lifted to a map $g : X \to E$ iff $q_*[f] = q \circ f$ is null-homotopic. Hence the lifting question can be recast into a null-homotopy problem. The obstructions to finding a homotopy from $q \circ f : X \to Z$ to a constant map lie in the cohomology groups $H^k(X; \pi_k(Z))$. However, we know that $\pi_k(Z) = \pi_{k-1}(\Omega Z) = \pi_{k-1}(F)$. Therefore, the obstructions lie in the groups $H^k(X; \pi_{k-1}(F))$ as k ranges over the positive integers. The classifying condition is satisfied in many spaces like the classifying spaces F/Cat that are relevant to surgery theory. Obstruction theory, however, can handle lifting problems even if they are not classified.

Remark A.8. *When the target is not simply connected, or the fibration is not principal, the same ideas apply, but the details are a little different. In this case, we obtain not a cochain in $C^n(X; \pi_n(F))$, but instead a cochain in cohomology with local coefficients. Every $\alpha \in \pi_1(B)$ can be assigned to a homotopy class h_α of self-homotopy equivalences of F, namely the "monodromy" over this loop. This h_α induces an automorphism of $[\mathbb{S}^n : F]$ by sending f to $h_\alpha \circ f$. The fibration then determines a representation $\rho' : \pi_1(B) \to \mathrm{Aut}\,(\pi_n(F))$. This local coefficient system over B can be pulled back via $f : X \to B$ to a local coefficient system over X creating the composite map $\rho : \pi_1(X) \xrightarrow{f_*} \pi_1(B) \xrightarrow{\rho'} \mathrm{Aut}\,(\pi_n(F))$. In this context, we can construct an obstruction cocycle as in the following theorem.*

The dual point of view is that, when X is not simply connected, the fibrations in the Postnikov system are not principal fibrations and are classified by $BAut(K(\pi, i+1))$, which encodes the monodromy. In this case, obstruction theory gives a series of obstructions in cohomology with twisted or local coefficients.

A special case of the lifting problem is the *cross section problem*, which asks whether a map $p: E \to B$ permits a map $s: B \to E$ such that $p \circ s = id_B$. In other words, the question asks whether we can solve the diagram



in the case when $X = B$ and $f = id_B$.

As a simple yet important example, suppose that $p: E \to B$ is a real oriented n-plane vector bundle. Let E_0 be the complement in E of the zero section, and consider the bundle $\mathbb{R}^n - \{0\} \to E_0 \to B$. A cross section exists if all obstructions vanish in the groups $H^{i+1}(B; \pi_i(\mathbb{R}^n - \{0\})) \cong H^{i+1}(B; \pi_i(\mathbb{S}^{n-1}))$. If B is a CW complex of dimension n, then the only possible nonzero cohomology is in dimension $i = n - 1$, whereupon we have a possible obstruction $e(p)$, the *Euler class*, in the group $H^n(B; \mathbb{Z})$. Since it is the only obstruction, and certainly the one with the lowest degree, the Euler class is well-defined. In fact, if $p: TB \to B$ is the tangent bundle of a closed, oriented n-manifold B, then there is a pairing $\langle e(p), [B] \rangle = \chi(B)$. Therefore, there is a nowhere zero section of the bundle $p: TB \to B$, i.e. a nonzero vector field on B, iff the Euler characteristic $\chi(B)$ vanishes. See Milnor-Stasheff [461, §12].

A.2 PRINCIPAL BUNDLES AND CHARACTERISTIC CLASSES

The idea of a classifying space is a generalization of the important role that Grassmannians play in the theory of vector bundles, or that Eilenberg-MacLane spaces play in cohomology theory. They are a useful organizing tool for certain kinds of homotopy functors and are indispensable throughout topology. In this section we assume that all spaces are countable CW complexes with a given basepoint, and all maps are basepoint-preserving.

Definition A.9. *A contravariant homotopy functor*

$$F : Spaces \to Pointed{-}Sets$$

is representable if there is a space B such that $F(X) = [X : B]$ by a natural isomorphism. We say that B is a classifying space for F.

When such a functor F is representable, then properties of F are reflected in properties of B. Information on the structure of B can give a better understanding of the functor F. Oftentimes, constructions performed on spaces B are most easily done by constructions on F.

A definitive theorem describing representable functors is the *Brown representation theorem.*

Theorem A.10. *(Brown [96]) A homotopy functor*

$$F : Spaces \to Pointed{-}sets$$

is representable iff (1) F takes wedges to products, and (2) for any CW complex W covered by two subcomplexes U and V, and for any elements $u \in F(U)$ and $v \in F(V)$, if u and v restrict to the same element of $F(U \cap V)$, then there is an element $w \in F(W)$ restricting to u and v, respectively.

For example, we can apply this idea to principal G-bundles over a space X. We give a definition.

Definition A.11. *If G is a topological group and X is a topological space, a principal G-bundle over X is a fiber bundle $E \to X$ equipped with a continuous right action $E \times G \to E$ such that G preserves the fibers of E and acts freely and transitively on them. The collection of principal G-bundles of X, up to an appropriate equivalence, is denoted by* $\mathrm{Prin}_G(X)$.

Remark A.12. *These bundles can clearly be glued and pulled back. The classical argument for vector bundles (see e.g. Atiyah [20]) shows that principal G-bundles are a homotopy functor with the trivial bundle as the distinguished point. If G is a discrete group, a principal G-bundle with total space can be identified with a regular covering map with G as group of deck transformations.*

The Brown representation theorem then gives the following.

Proposition A.13. *Let X be a topological space. For any topological group G, there is a classifying space BG for which the collection* $\mathrm{Prin}_G(X)$ *of principal G-bundles over X is in bijective correspondence with* $[X : BG]$.

Remark A.14. *For $G = O_n$, SO_n, U_n and other topological groups, the spaces BG are classical. Note that principal G-bundles are a certain type of vector bundle. From a vector bundle, one considers the automorphisms of the bundle with an appropriate metric. Conversely, from the bundle E one can consider $E \times_G V$ where V is the n-dimensional vector space with G acting on it by its defining representation.*

If ΣX denotes the suspension of X, then we have $\mathrm{Prin}_G(\Sigma X) = [X : G]$ because the G-bundle is trivial over each cone of X. One only needs to specify the bundle by using the clutching isomorphism over each point of X. We therefore have the following.

Proposition A.15. *Let G be as above, and let Ω denote the loopspace functor. Then $\Omega(BG) \cong G$.*

For general groups G, there is a general construction of BG given by Milnor [445]. Consider the n-fold join $G^{(n)} = G * G * \cdots * G$ of G and define $EG = \lim_{n\to\infty} G^{(n)}$. The n-th stage $G^{(n)}$ is $(n-1)$-connected, and EG is contractible and is endowed with a G-action obtained by multiplying simultaneously on the right in each of the factors. Then BG can be defined by $BG = EG/G$. Indeed, the map $EG \to BG$ will be a principal bundle, and given a map $X \to BG$, one can form the pullback of this bundle on X. The association $G \to BG$ is a functor from the category of topological groups to the homotopy category of CW complexes.

Remark A.16. *This description gives rise the standard resolutions that are used for studying cohomology of groups, but we will not discuss them.*

Vector bundles give a very concrete description of classifying spaces when we consider a real n-plane bundle as a principal O_n-bundle. Because the Grassmannians are explicit smooth manifolds, calculations are very possible. In this case, the space BO_n can be identified with the *infinite Grassmann manifold* $\mathrm{Gr}_n = \mathrm{Gr}_n(\mathbb{R}^\infty)$, the set of all n-dimensional linear subspaces of $\mathbb{R}^\infty$, topologized as the direct limit of the sequence $\mathrm{Gr}_n(\mathbb{R}^n) \subset \mathrm{Gr}_n(\mathbb{R}^{n+1}) \subset \cdots$.

Definition A.17. *The canonical real n-plane bundle γ^n over Gr_n is constructed as follows by considering the subset of all pairs (V, v) in $\mathrm{Gr}_n \times \mathbb{R}^\infty$, where V is an n-plane in $\mathbb{R}^\infty$ and $v \in V$.*

Then any real n-plane bundle ξ over a paracompact base space admits a bundle map $\xi \to \gamma^n$. We use γ^n to describe the correspondence between $\mathrm{Prin}_{O_n}(X)$ and $[X : BO_n]$ in the following. For every map $X \to BO_n$, there is a pullback n-plane bundle $f^*\gamma^n$ over X. For every n-plane bundle η over X, there is a map $\eta \to \gamma^n$ whose restriction is a map $f : X \to BO_n$. In fact, there is also an identification

$$BO_n = \lim_{n\to\infty} O_{m+n}/(O_m \times O_n).$$

The same kinds of constructions can be performed for n-dimensional complex vector bundles classified by $BU_n = \lim_{m\to\infty} U_{m+n}/(U_m \times U_n)$. There is also a version for real oriented n-dimensional real vector bundles: $BSO_n = \lim_{m\to\infty} SO_{m+n}/(SO_m \times SO_n)$.

Remark A.18. *It is a well-known stability theorem that the quotients U_{m+n}/U_m are highly connected. One can equivalently say that high-dimensional vector bundles on low-dimensional spaces are isomorphic to the Whitney sum of unique low-dimensional bundles and an appropriate low-dimensional trivial bundle.*

We will now give a brief discussion of characteristic classes. For a real vector bundle E over a space X, there are canonically defined *Stiefel-Whitney classes* $w_k \in H^k(M;\mathbb{Z})$ that describe the obstructions to constructing everywhere independent sets of sections of the bundle. For a complex vector bundle E over a base X, there are *Chern classes* $c_i \in H^{2i}(X;\mathbb{Z})$. The *Pontrjagin classes* $p_k \in H^{4k}(M;\mathbb{Z})$ of a real vector bundle E' are the even Chern classes of the complexification $E = E' \otimes \mathbb{C}$. Lastly the *Euler class* $e(E) \in H^r(X;\mathbb{Z})$ of an oriented, real vector bundle E of rank r over X generalizes the notion of the classic Euler characteristic.

We mention here the *splitting principle*, which states that general formulas for characteristic classes can be verified for sums of line bundles, and then the formulas will be true in general.

Theorem A.19. *(Splitting principle) Let ξ be a vector bundle of rank n over a paracompact space X. There exists a space Y, called the flag bundle associated to E, and a map $p\colon Y \to X$ such that*

1. *the induced cohomology homomorphism $p^*: H^*(X) \to H^*(Y)$ is injective, and*
2. *the pullback bundle $p^*\xi$ breaks up as a direct sum $\ell_1 \oplus \ell_2 \oplus \cdots \oplus \ell_n$ of line bundles over Y.*

The theorem above holds for complex vector bundles with integer coefficients or for real vector bundles with $\mathbb{Z}_2$-coefficients. Under the splitting principle, characteristic classes for complex vector bundles correspond to symmetric polynomials in the first Chern class c_1 of complex line bundles.

These characteristic classes appear as generators in the cohomology ring of various classifying spaces.

Theorem A.20. *Let $w_1, w_2, \ldots, w_n$ be the Stiefel-Whitney classes of the canonical bundle γ^n over BO_n. Let the Chern classes c_i and the Pontrjagin classes p_i be similarly defined. Let e be the Euler class. Then we have the following identifications:*

1. $H^*(BO_n;\mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \ldots, w_n]$ *where* $\deg w_i = i$,
2. $H^*(BU_n;\mathbb{Z}) = \mathbb{Z}[c_1, c_2, \ldots, c_n]$ *where* $\deg c_i = 2i$,
3. $H^*(BSO_{2n};\mathbb{Z}[1/2]) = \mathbb{Z}[1/2][p_1, p_2, \ldots, p_n, e]$ *where* $\deg p_i = 4i$ *and* $\deg e = 2n$.

Remark A.21. *Compact Lie groups G have maximal tori. In the above cases, the natural map $B\mathbb{T} \to BG$ is associated to the map of functors that associates to n line bundles the Whitney sum, which is an n-dimensional line bundle.*

Starting with the obvious map $O_n \to O_{n+1}$, we can use functoriality to construct a map $BO_n \to BO_{n+1}$ for all $n \geq 1$. The limit of this process is a space BO. For a space X, the space BO classifies all vector bundles over X with the following equivalence: two n-dimensional real vector bundles are *equivalent* if they are isomorphic after adding a trivial bundle. In other words, the bundles ξ and η are equivalent if there is a k-

dimensional trivial bundle ε^k over X such that $\xi \oplus \varepsilon^k$ and $\eta \oplus \varepsilon^k$ are isomorphic. This classification is called a *stable theory*. A stable theory of complex bundles similarly produces a classifying space BU.

Both BO and BU have remarkable a property called *Bott periodicity* which is given by

$$\mathbb{Z} \times BU = \Omega^2(BU) \text{ and } \mathbb{Z} \times BO = \Omega^8(BO).$$

See Bott [72], Atiyah [21], Milnor [453], and Lawson-Michelsohn [390].

Theorem A.22. *The homotopy groups $\pi_*(BU)$ are 2-periodic and the homotopy groups $\pi_*(BO)$ are 8-periodic:*

$$\pi_n(BU) = \begin{cases} \mathbb{Z} & \text{if} \quad n \equiv 0 \mod 2, \\ 0 & \text{if} \quad n \equiv 1 \mod 2, \end{cases}$$

$$\pi_n(BO) = \begin{cases} \mathbb{Z} & \text{if} \quad n \equiv 0, 4 \mod 8, \\ \mathbb{Z}_2 & \text{if} \quad n \equiv 1, 2 \mod 8, \\ 0 & \text{if} \quad n \equiv 3, 5, 6, 7 \mod 8. \end{cases}$$

Remark A.23. *The periodicity above shows that the classifying spaces are actually infinite loop spaces, i.e. that the stable versions $K^*(X)$ and $KO^*(X)$ are cohomology theories. In fact, the groups $K^*(X)$ and $KO^*(X)$ have ring structures. Addition is given by Whitney sum, and multiplication is given by tensor products. Rationally, here is a Chern character ring isomorphism $ch\colon K^*(X) \to H^{even}(X;\mathbb{Q})$ given by $ch(\psi) = \Sigma e^{c_1(\xi)}$, where exponentiation is performed by the formal power series in the cohomology class, and $c_1(\xi)$ is the first Chern class of line bundles which, using the splitting principle, add up to ψ. Similarly there is a rational ring isomorphism: $ch\colon KO(X) \to \oplus H^{4i}(X;\mathbb{Q})$.*

There are classifying spaces BPL_n, $BSPL_n$, $BTop_n$, and $BSTop_n$ appropriate for the PL and Top categories along with their stable analogues BPL, $BSPL$, $BTop$, and $BSTop$. According to a theorem of Boardman-Vogt [61] and May [433], they are infinite loop spaces as well (see Adams [5]). The standard reference for their structure and those of related spaces is Madsen-Milgram [420]. We will generically refer to all these classifying spaces as $BCat$ or $BSCat$.

Besides fiber bundles of various sorts, we can also classify fibrations. For surgery theory, we are particularly interested in *spherical fibrations* on a space X, i.e. maps $E \to X$ whose homotopy fiber is homotopy equivalent to $\mathbb{S}^{k-1}$. If F_k denotes the set of homotopy self-equivalences of $\mathbb{S}^{k-1}$ with the compact open topology, then BF_k is the classifying space for $(k-1)$-dimensional spherical fibrations. The oriented version is denoted by BSF_k.

Theorem A.24. *(Stasheff [614]) If F_k is given as above, then $\Omega(BF_k) = \Omega^{k-1}(\mathbb{S}^{k-1})$.*

See Hatcher [293], Spanier [604], and May [434] for more on fibrations.

Note that equatorial inclusion $\mathbb{S}^n \subseteq \mathbb{S}^{n+1}$ induces inclusions $F_n \to F_{n+1}$ for all $n \geq 1$ that give rise to maps $BF_n \to BF_{n+1}$. Their limit is an infinite loop space BF whose homotopy groups $\pi_n(BF)$ are essentially the stable homotopy groups π_n^S of spheres. The cohomology theory that arises from the homotopy classes of maps $X \to BF$ is denoted by $KSph_*(X)$.

Theorem A.25. *Let* $\varinjlim \pi_{n+k-2}(\mathbb{S}^{k-1}) = \pi_{n-1}^S$ *be the stable homotopy groups of spheres. The homotopy groups of* BF *are then given by*

$$\pi_n(BF) = \begin{cases} \pi_{n-1}^S & \text{if } n \geq 2, \\ \mathbb{Z}_2 & \text{if } n = 1. \end{cases}$$

The low-dimensional homotopy groups are therefore given as follows:

n	1	2	3	4	5	6	7	8	9	10
$\pi_n(BO)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\pi_n(BF)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$

There is an obvious map $O_n \to F_n$ that identifies an $n \times n$ orthogonal matrix with a homeomorphism (and therefore homotopy equivalence) on the $(n-1)$-sphere. Functoriality gives a map $BO_n \to BF_n$, which is geometrically realized by passing from a real n-plane vector bundle to its underlying $(n-1)$-spherical fibration obtained by removing the zero section. The passage from stable real vector bundles to stable spherical fibrations defines a map $J : BO \to BF$ inducing the stable J-homomorphism $J_* : \pi_*(BO) \to \pi_*(BF)$ on homotopy groups. If one thinks of π_n^S as bordism of framed n-manifolds, the J-homomorphism describes framed spheres.

There is also a natural map $BCat \to BF$ when Cat is PL or Top, and the work of Boardman-Vogt and May mentioned above shows that it is actually an infinite loop map, so that the fibers of these maps, which we will denote by F/Cat, are infinite loop spaces.

Remark A.26. *Although the tangent bundle of a manifold is not homotopy invariant in* $KCat^*(M)$ *in any of the geometric categories, Atiyah [19] showed that the image in* $KSph^*(X)$ *is homotopy invariant. Adams conjectured, and Quillen and Sullivan proved, that the map* $BO \to BF$ *factors through a summand of* BF*, giving very precise information about the part of* $KO^*(X)$ *that survives this forgetful map. We explain some of these ideas in Chapter 4.*

For completeness, we give the definitions for Top- and PL-bundles. The smooth theory developed in this section can be reiterated using these definitions for the other two manifold categories.

Definition A.27. *Let* B *be a topological space. A Top* $\mathbb{R}^n$*-bundle over* B *is an* $\mathbb{R}^n$*-bundle* $p\colon E \to B$ *equipped with a fixed section* $s\colon B \to E$*. If* $\xi = \{p\colon E \to B\}$

and $\eta = \{q\colon Y \to X\}$ are two Top $\mathbb{R}^n$-bundles, then a Top morphism $\phi\colon \xi \to \eta$ is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[f]{} & X \end{array}$$

for which the restriction $p^{-1}(b) \to q^{-1}(f(b))$ of g to every fiber $p^{-1}(b)$ of ξ is a Top homeomorphism. In addition, we require that ϕ take sections of $E \to B$ to sections of $Y \to X$.

Definition A.28. *Let B be a topological space. A PL (piecewise linear) $\mathbb{R}^n$-bundle over B is a Top $\mathbb{R}^n$-bundle $p\colon E \to B$ such that E and B are polyhedra equipped with PL maps $p\colon E \to B$ and $s\colon B \to E$. We also require that, for every simplex $\Delta \subseteq B$, there is a PL homeomorphism $h\colon p^{-1}(\Delta) \to \Delta \times \mathbb{R}^n$ with $h(s(\Delta)) = \Delta \times \{0\}$. If $\xi = \{p\colon E \to B\}$ and $\eta = \{q\colon Y \to X\}$ are two PL $\mathbb{R}^n$-bundles, then a PL morphism $\phi\colon \xi \to \eta$ is a commutative diagram Top for which the restriction $p^{-1}(b) \to q^{-1}(f(b))$ of g to every fiber $p^{-1}(b)$ of ξ is a PL homeomorphism.*

We end this section by briefly discussing the fiber F/Cat of the map $BCat \to BF$. We use the general principle that, if H is a subgroup of G, then the fiber G/H of the map $BH \to BG$ classifies H-principal bundles that are trivialized with respect to G. For example, the quotient SO/U classifies complex bundles that are given real trivializations.

We give a concrete definition of this idea. The following notation is nonstandard.

Definition A.29. *Let n be a positive integer and let X be a finite complex. A proper homotopy equivalence $\theta\colon E \to X \times \mathbb{R}^n$ is an $(F, Cat)_n$ bundle over X if there is a Cat $\mathbb{R}^n$-bundle $\pi\colon E \to X$ for which*

$$\begin{array}{ccc} E & \xrightarrow{\theta} & X \times \mathbb{R}^n \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p_1} \\ X & \xrightarrow[id]{} & X \end{array}$$

is a homotopy commutative diagram. Two $(F, Cat)_n$-bundles $\theta_1\colon E_1 \to X \times \mathbb{R}^n$ and $\theta_2\colon E_2 \to X \times \mathbb{R}^n$ over X are equivalent if there is a Cat bundle equivalence $b\colon E_1 \to$

E_2 such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\theta_1} & \\ {\scriptstyle b}\downarrow & & X \times \mathbb{R}^n \\ E_2 & \xrightarrow{\theta_2} & \end{array}$$

is properly homotopy commutative.

Proposition A.30. *For all $n \geq 1$, there is classifying space $(F/Cat)_n$ such that, if X is a finite complex as above, there is a bijective correspondence between $(F, Cat)_n$-bundles over X and the set $[X : (F/Cat)_n]$. The correspondence $\theta \to \theta \times id_{\mathbb{R}}$ defines stabilization maps $(F/Cat)_n \to (F/Cat)_{n+1}$ and the stable limits are denoted by F/Cat.*

Proposition A.31. *The space F/Cat classifies stable Cat-bundles that are given stable fiber homotopy equivalences to the trivial bundle.*

It can be shown that the two different definitions of F/Cat produced in this section are homotopy equivalent. Using a Whitney sum operation both of these spaces become homotopy associative and homotopy commutative H-spaces.

In Chapter 1, we give a completely different interpretation of $[M : F/Cat]$ for a Cat manifold M that makes these "homogeneous" classifying spaces critical to surgery theory. Indeed, despite the simplicity of BO in the study of smooth manifolds, the theories in PL and Top are actually easier to understand because F/PL and F/Top have a simpler structure than F/O.

The results in this subsection hold similarly for the topological and piecewise linear categories. The study of manifolds in the various categories will rely on an understanding of the following diagram, which gives the relationship between these classifying spaces. We assert nothing about the "exactness" of any row or column.

$$\begin{array}{ccccccccccc}
PL/O &&&&&&&&&& \\
\downarrow &&&&&&&&&& \\
Top/O & \longrightarrow & Top/PL &&&&&&&& \\
\downarrow && \downarrow &&&&&&&& \\
F/O & \longrightarrow & F/PL & \longrightarrow & F/Top &&&&&& \\
\downarrow && \downarrow && \downarrow &&&&&& \\
BO & \longrightarrow & BPL & \longrightarrow & BTop & \longrightarrow & BF &&&& \\
&&&&&& \downarrow &&&& \\
&&&&&& B(F/O) & \longrightarrow & B(F/PL) & \longrightarrow & B(F/Top)
\end{array}$$

A.3 GENERALIZED HOMOLOGY THEORIES

Definition A.32. *A generalized cohomology theory h^* is a contravariant functor from the category of pairs of topological spaces to the category of graded abelian groups which satisfy all the Eilenberg-Steenrod axioms except for the dimension axiom. In other words, they must satisfy the following.*

1. *If $f, g : (X, A) \to (Y, B)$ are homotopic, then the induced maps $f^*, g^* : h^*(Y, B) \to h^*(X, A)$ are the same.*
2. *If (X, A) is a pair and U is a open set of X whose closure is in the interior of A, then the inclusion map $i : (X - U, A - U) \to (X, A)$ induces an isomorphism in homology.*
3. *If $X = \coprod X_\alpha$ is the disjoint union of a family of spaces X_α, then $h^*(X) = \bigoplus_\alpha h^*(X_\alpha)$.*
4. *For any pair (X, A), there is a family of boundary maps $\delta^* : h^*(A) \to h^{*+1}(X, A)$ such that the sequence*

$$\cdots \to h^n(X, A) \to h^n(X) \to h^n(A) \to h^{n+1}(X, A) \to \cdots$$

is exact. Here the first map is induced by the inclusion map $(X, \emptyset) \to (X, A)$.

Remark A.33. *To define reduced cohomology, we let $i : pt \to X$ be the inclusion of a point and $\pi : X \to pt$ be the collapsing map. Then $\pi \circ i = id$ and so $i^* \circ \pi^* = id$ on $h^*(pt)$. Let $\widetilde{h}^*(X) = \ker i^*$ be the reduced cohomology of X. Then $h^*(X) = \widetilde{h}^*(X) \oplus h^*(pt)$ as $h^*(pt)$-modules.*

Generalized cohomology theories are constructed using loopspaces. If ΩK denotes the space of loops on a CW complex X; i.e. ΩK is the *loopspace of* K, then basepoint-preserving maps $\Sigma X \to K$ are the same as the basepoint-preserving maps $X \to \Omega K$. This fact gives the basic adjoint relationship $[\Sigma X : K] = [X : \Omega K]$. Taking $X = \mathbb{S}^n$ in this relation, we see that $\pi_{n+1}(K) = \pi_n(\Omega K)$ for all $n \geq 0$. When we pass from a space to its loopspace, the homotopy groups shift down a dimension.

Remark A.34. *The association $X \mapsto \Omega X$ is functorial. A basepoint preserving map $f : X \to Y$ induces a map $\Omega f : \Omega X \to \Omega Y$ by composition with f. A homotopy $f \simeq g$ induces a homotopy $\Omega f \simeq \Omega g$. Therefore $X \simeq Y$ implies $\Omega X \simeq \Omega Y$.*

The following important condition gives us a way to construct generalized cohomology theories.

Definition A.35. *An Ω-spectrum is a sequence of CW complexes $K_1, K_2, \ldots$ together with weak homotopy equivalences $K_n \to \Omega K_{n+1}$.*

Theorem A.36. *If $\{K_n\}$ is an Ω-spectrum, then the functors $X \mapsto h^n(X) = [X : K_n]$ define a reduced cohomology theory on the category of basepointed CW complexes and basepoint-preserving spaces.*

Example A.37. *Let G be an abelian group. Because $\Omega K(G, n+1)$ is an Eilenberg-Maclane space of type $K(G, n)$, there is a homotopy equivalence given by $\varepsilon_n : K(G, n) \to \Omega K(G, n+1)$. We use the notation $K(G)$ to denote the Eilenberg-Maclane spectrum for the group G whose n-th space is $K(G, n)$ together with the maps ε_n.*

Brown's representability theory claims that the converse of Theorem A.36 is true; i.e. every reduced cohomology theory on CW complexes arises from an Ω-spectrum in this way. Note that there is essentially no difference between cohomology theories on basepointed CW complexes and cohomology theories on non-basepointed CW complexes.

Every cohomology theory h^* has an associated homology theory, denoted by h_*. Generalized homology theories are the obvious analogue of generalized cohomology theories; they are a collection of covariant functors on the homotopy category of pairs, linked by a natural boundary map, satisfying all the Eilenberg-Steenrod axiom besides the dimension axiom. The simplest way to define homology uses Spanier-Whitehead duality. For a finite complex X, the homology $h_*(X)$ can be defined as the cohomology of the dual $h^*(DX)$. Homology axioms are satisfied because D is a contravariant functor on the stable homotopy category, and because $D^2 = id$.

Remark A.38. *There are other approaches to homology, such as using the notion of smash product of a space with a spectrum. One defines $h_*(X) = \pi_*(X_+ \wedge \mathbb{E}_\bullet)$, where X_+ just means X with a disjoint basepoint added and $\mathbb{E}_\bullet$ is the representing spectrum for h. (We refer to Adams [5] for details.)*

Remark A.39. *One can compute homology with a spectral sequence. If $F \to E \to B$ is a Serre fibration, there is an Atiyah-Hirzebruch spectral sequence $E_2^{p,q} = H^p(B; h^q(F))$.*

If B is a finite CW complex, then the sequence converges and its limit term is associated to $h^*(E)$. If F is a point, then the spectral sequence $H^p(X; h^q(pt))$ converges to $h^n(X)$, allowing $h^*(X)$ sometimes to be computed in terms of $H^*(X)$ and $h^*(pt)$. Similarly, the spectral sequence $E^2_{p,q} = H_p(B; h_q(F))$ converges to $h_{p+q}(E)$.

We quickly describe how bordism gives rise to a generalized homology theory. Thom's remarkable paper [642] defined transversality and used it as the key geometric tool to enable the reduction of cobordism to homotopy theory. Conner-Floyd [172] took a decisive next step and made bordism into a functor. If X is a space, then a *Cat manifold in* X is a pair (M, f) consisting of a closed Cat manifold M and a map $f: M \to X$. Bordism of Cat pairs (M, f) is an equivalence relation and the resulting equivalence classes form an abelian group $\Omega_*^{Cat}(X)$. Since Cat structures on M and N induce a Cat structure on $M \times N$, then $\Omega_*^{Cat}(pt)$ becomes an associative, commutative, unitary ring and $\Omega_*^{Cat}(X)$ becomes a $\Omega_*^{Cat}(pt)$-algebra, where $M \cdot (N, f)$ is the class $f \circ p: M \times N \to N \to X$. A pair $Y \subseteq X$ gives rise to relative group $\Omega_*^{Cat}(X, Y)$ with a long exact sequence

$$\cdots \to \Omega_n^{Cat}(Y) \to \Omega_n^{Cat}(X) \to \Omega_n^{Cat}(X, Y) \to \Omega_{n-1}^{Cat}(Y) \to \cdots$$

One can observe that $\Omega_*^{Cat}(\cdot)$ is an additive homotopy functor.

Whenever transversality is available, one obtains a generalized homology theory $\Omega_i^{Cat}(\cdot)$. See Williamson [711] for PL transversality, which is quite elementary once one knows about block bundles, and Kirby-Siebenmann [361] for topological transversality, which is very deep.

In short, bordism $\Omega_*^{Cat}(\cdot)$ is represented by a spectrum, which we will call $MCat$. In other words, there is an Ω-spectrum $MCat_k$ with maps $f_k: MCat_k \to \Omega MCat_{k+1}$ such that

$$\Omega_n^{Cat}(pt) = \lim_{k\to\infty} [\, \mathbb{S}^{n+k}: \; MCat_k] = \pi_n^S(MCat),$$

$$\Omega_n^{Cat}(X) = \lim_{k\to\infty} [\, \mathbb{S}^{n+k}: \; X_+ \wedge MCat_k] = \pi_n^S(X_+ \wedge MCat).$$

Here π_n^S is used to indicate a stability phenomenon as in the stable homotopy groups of spheres.

Thom worked in the Diff category. His cobordism theorem gives a concrete model for the Thom spectrum MO as the spectrum of Thom spaces. However, the formal aspects work whenever one has a transversality theorem.

Theorem A.40. *(Thom cobordism) Let γ^k_{Cat} be the universal space for Cat k-bundles. The complement of the zero section gives a spherical fibration whose Thom space $T(\gamma^k_{Cat})$ is the mapping cone of the universal Cat sphere bundle γ^k_{Cat}. If Cat satisfies transversality of large codimension, then $MCat_k = T(\gamma^k_{Cat})$ for all k, giving a geometric interpretation of the spectrum guaranteed by Brown.*

Theorem A.41. *(Thom [642]) Let X be a topological space with subset A and denote by*

$\Omega_n^O(X)$ the bordism ring of unoriented Diff n-manifolds. Then there is an isomorphism given by

$$\Omega_n^O(X) \cong \bigoplus_i H_{n-i}(X, A;\, \Omega_i^O(*)).$$

In other words, bordism can be calculated by the usual homology groups with coefficients in the bordism of a point. The relationship comes from a natural isomorphism of functors that commute with appropriate boundary maps. In the language of spectra, there are positive integers n_i for which $MO \simeq \bigvee_{i=0}^{\infty} \Sigma^{n_i} K(\mathbb{Z}_2)$.

Remark A.42. *We can define a map $\alpha_n : MO_n(X) \to H_n(X; \mathbb{Z}_2)$ for each n by setting $\alpha_n[M, f] = f_*[M] \in H_i(X; \mathbb{Z}_2)$, where $f_* : H_i(M; \mathbb{Z}_2) \to H_i(X; \mathbb{Z}_2)$ is induced from $f : M \to X$ and $[M]$ is the fundamental class of M.*

Remark A.43. *In Appendix A.4, we discuss the spectrum MSO representing the bordism of oriented manifolds.*

The above formula of Thom is most valuable if one understands the bordism groups $\Omega_i(pt)$ and $\Omega_i^{SO}(pt)$. It is worth noting that, for every homology theory E_*, and any space X, we have

$$E_k(X) \otimes \mathbb{Q} \cong \bigoplus H_{k-n}(X; E_n(pt) \otimes \mathbb{Q}).$$

Because the left-hand side commutes with direct limits, so one can verify it for finite complexes, and Serre's version of the Hurewicz homomorphism theorem guarantees that the n-th suspension of an n-complex is rationally a wedge of spheres. All the maps arising in the Hurewicz theorem can be wedged together, and together they give a rational isomorphism. In other words, the Atiyah-Hirzebruch spectral sequence always collapses rationally at the E^2-stage. For general reasons, there is no isomorphism of this sort for $\mathbb{E} = MSO$ when $\mathbb{Q}$ is replaced with $\mathbb{Z}_{(2)}$.

Thom proves that MO and $MSO \otimes \mathbb{Q}$ are both polynomial algebras. The first is a polynomial algebra $\mathbb{F}_2[x_2, x_4, x_5, \ldots]$ with one generator in every dimension not of the form $2^k - 1$. The even-dimensional generators can be taken to be $\mathbb{RP}^{2i}$, while the odd-dimensional generators were constructed by Dold. We also have a polynomial algebra

$$MSO \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \mathbb{CP}^6, \ldots],$$

with complex projective spaces as generators. Thom's method shows that the Stiefel-Whitney numbers for MO and Pontrjagin numbers for MSO detect the cobordism classes rationally. In other words, the homology class of the cycle $t_*([M])$ in the homology of the Grassmannian detects the bordism class. Here $t_* : M \to BSO$ is the classifying space of the tangent bundle.

Using the above rational calculation, and the simple fact that signature is a multiplicative, oriented bordism invariant, i.e. $\mathrm{sig}(M \times N) = \mathrm{sig}(M)\,\mathrm{sig}(N)$, and $\mathrm{sig}(M) = 1$, Hirzebruch proved the following theorem.

Theorem A.44. *(The Hirzebruch signature theorem) There are universal polynomials $L_k(p_1, p_2, \dots, p_k)$ in the Pontrjagin classes such that*

1. $L(M \times N) = L(M)L(N)$ *(exterior product in the cohomology of the product),*
2. *the coefficient of p_k in L_k is nonzero,*
3. $\operatorname{sig}(M^{4k}) = \langle L_k(M), [M] \rangle$,

where we write $L = 1 + L_1 + L_2 + \cdots$ for the total graded cohomology class.

The first few polynomials are

$$L_0 = 1$$

$$L_1 = \frac{p_1}{3}$$

$$L_2 = \frac{7p_2 - p_1^2}{45}$$

$$L_3 = \frac{62p_3 - 13p_1p_2 + 2p_1^3}{3^3 \cdot 5 \cdot 7}$$

Note that (3) implies that an r-fold cover N of a closed manifold M satisfies $\operatorname{sig}(N) = r \cdot \operatorname{sig}(M)$. We see in Chapter 2 that it does not hold for Poincaré complexes.

The Hirzebruch signature theorem was also used by Milnor to show that the boundary of the E_8-manifold is an exotic 7-sphere; it is the Wall realization of the generator of $L_8(\mathbb{Z}[e])$ acting on $\mathcal{S}^{Diff}(\mathbb{S}^7)$. The normal data implies that $p_1 = 0$, so Hirzebruch's formula, together with the fact that the p_i are all integral cohomology classes, implies that 7 divides $\operatorname{sig}(M)$. We could close off M by adding a ball to the other side. As 8 is not divisible by 7, the boundary is indeed exotic.

A.4 LOCALIZATION

Definition A.45. *Let R be a commutative ring and let S be a multiplicatively closed subset of R with no zero divisors. The localization of R at S is the set $S^{-1}R$ of equivalence classes of formal fractions $\frac{r}{s}$, where $r \in R$ and $s \in S$, with addition and multiplication defined in the usual way.*

Notation A.46. *Let R be a commutative ring. All primes are assumed to be positive integers in $\mathbb{Z}$.*

1. *Let p be a prime and $S \subseteq \mathbb{Z}$ be given by $S = \{1, p, p^2, p^3, \dots\}$. Then $S^{-1}\mathbb{Z}$ is denoted by $\mathbb{Z}[1/p]$, the smallest subring of $\mathbb{Q}$ containing $\mathbb{Z}$ and $1/p$. This ring is often called the localization of $\mathbb{Z}$ away from p. More generally, if S is a multiplicatively closed subset of $\mathbb{Z}$, then denote by $\mathbb{Z}[1/s]$ the localization $S^{-1}\mathbb{Z}$,*

called the localization of $\mathbb{Z}$ away from S.

2. *If I is a prime ideal of R, then $R\backslash I$ is multiplicatively closed, and the notation R_I will be shorthand for the localization $(R\backslash I)^{-1}R$, and is called the localization of R at I. In the case when $R = \mathbb{Z}$ and $I = (p)$ is the ideal generated by the prime p, we have $\mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at p, consisting of all rational numbers whose denominator is not a multiple of p.*
3. *In particular, if P is a set of primes in $\mathbb{Z}$, let (P) be the ideal generated by the primes in P. Then $\mathbb{Z}_{(P)}$ is called the localization of $\mathbb{Z}$ at P, consisting of all rational numbers whose denominators are not divisible by any element in P. When P is empty, then (P) is simply the zero ideal (0). Therefore $\mathbb{Z}_{(0)} = \mathbb{Q}$.*

Remark A.47. *Localization is an important idea in algebra. In topology it is also possible to localize simply connected spaces and H-spaces. See Hilton-Mislin-Roitberg [308] for additional information, including the extension to nilpotent spaces. See also Bousfield-Kan [73] for a functorial version that we will occasionally find useful.*

Definition A.48. *Consider topological spaces X_i and a sequence of maps*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n.$$

Let M_i be the mapping cylinder of f_i given by

$$((X_i \times I) \amalg X_{i+1})/((x \times \{1\} \sim f_i(x)).$$

The mapping cylinders f_{i+1} and f_i can be attached together by glueing $X_{i+1} \subseteq M_i$ to $X_{i+1} \times \{0\} \subseteq M_{i+1}$. Combining them for all i, we obtain one single space called the mapping telescope of the sequence of maps. The mapping telescope can certainly be formed with an infinite sequence of maps.

We can concretely construct the *localization of a simply connected space X at a prime p.* A sphere of dimension at least 1 has self-maps of arbitrary degree. Consider such maps of a sphere of all degrees coprime to p, and execute the infinite mapping telescope construction above. The result is defined is the *sphere localized at p.* By taking the cone on this localized sphere, we obtain a *disk localized at p.* For any simply connected CW complex X, we can replace every cell with the localized version to obtain a localization $X_{(p)}$ of the space at p. When X is not simply connected, the construction is actually not well-defined.

Remark A.49. *Localization is even more transparent when X is an H-space. Let X be an H-space with unit. Let S be a multiplicatively closed set of positive integers. If $n \in \mathbb{Z}_{\geq 1}$, then let $e_n : X \to X$ be $e_n(x) = x^n$ for all $x \in X$. The induced map $e_{n*} : \pi_k(X) \to \pi_k(X)$ is then given by multiplication by n. Let $\{n_i\}$ be a sequence of integers in S that is cofinal in S under multiplication; i.e. for all $s \in S$ there is i such* *that $s|n_i$. Set the localization $X[{}^1/s]$ of X to be the mapping telescope of the sequence*

$$X \xrightarrow{e_{n_1}} X \xrightarrow{e_{n_2}} X \xrightarrow{e_{n_3}} \cdots$$

One achieves the same result as the construction above.

It is not hard to see that in the homotopy category this process is independent of the choice of the sequence $\{n_i\}$. Note also that the process of localization will not preserve any Cat manifold structure that might exist on the original space.

Remark A.50. *Localization $X \mapsto X[{}^1/s]$ is functorial and commutes with spaces and induced maps of homotopy and homology. One can construct it in general by localizing a Postnikov tower for any nilpotent space X. See Hilton-Mislin-Roitberg [308] for such Postnikov systems.*

The following result follows from Serre's thesis.

Theorem A.51. *Let X be a simply connected CW complex and let S be a multiplicatively closed set of positive integers. The homotopy and homology groups of $X[{}^1/s]$ satisfy the following properties for all n:*

$$\pi_n(X[{}^1/s]) = (\pi_n(X))[{}^1/s],$$

$$H_n(X[{}^1/s]) = (H_n(X))[{}^1/s].$$

Notation A.52. *Suppose that X is a simply connected or nilpotent CW complex. Let $S = \{2^n : n \in \mathbb{Z}_{\geq 1}\}$. The resulting space $X[{}^1/s]$ is often denoted by $X[{}^1/2]$ and called the odd homotopy type of X. Let $T = \{m \in \mathbb{Z}_{\geq 1} : (m, 2) = 1\}$. The resulting space $X[{}^1/T]$ is denoted by $X_{(2)}$ and called the 2-local homotopy type of X. Similar constructions give localizations $X_{(p)}$ and $X[{}^1/p]$ for any prime p, and also $X_{(P)}$ and $X[{}^1/P]$ for any finite set P of primes. If P is empty, we can localize appropriately to obtain a space denoted by $X_{\mathbb{Q}}$ or $X_{(0)}$, the rationalization of X.*

Remark A.53. *If X is a simply connected CW complex and P is a collection of primes, then there is a pullback square*

$$\begin{array}{ccc} X & \longrightarrow & X_{(P)} \\ \big\downarrow & & \big\downarrow \\ X[1/P] & \longrightarrow & X_{(0)} \end{array}$$

relating the localizations of X at P, away from P, and at 0. In other words, the homotopy fibers of both vertical arrows are the same, and this condition is equivalent to the statement that the horizontal arrows have the same homotopy fibers as each other.

Theorem A.54. *(Thom [642], Milnor [449], Wall [661]) The spectrum $MSO_{(2)}$ has* *the homotopy type of a wedge of Eilenberg-Maclane spaces. Indeed, we have*

$$MSO_{(2)} \simeq \bigvee_i \Sigma^{r_i} K(\mathbb{Z}_2) \vee \bigvee_j \Sigma^{4s_j} K(\mathbb{Z}_{(2)})$$

for some integers r_i and s_j.

Remark A.55. *In other words, for all X we have*

$$MSO_n(X)_{(2)} \cong \bigoplus_i H_i(X; MSO_{n-i}(pt))_{(2)}.$$

Remark A.56. *As before with MO, we can define a map $\alpha_{MSO_{(2)}} : (MSO_{(2)})_i(X) \to H_i(X; \mathbb{Z}_{(2)})$. For each i, set $\alpha_{MSO_{(2)}}[M, f] = f_*[M] \in H_i(X; \mathbb{Z}_{(2)})$, where the map $f_* : H_i(M; \mathbb{Z}_{(2)}) \to H_i(X; \mathbb{Z}_{(2)})$ is induced from $f : M \to X$ and $[M]$ is the fundamental class of M.*

We quickly discuss coefficients into homology theories.

Definition A.57. *Let G be an abelian group and $n \geq 1$. We say that a CW complex X is a Moore space if $H_n(X; \mathbb{Z}) \cong G$ and $\widetilde{H}_i(X; \mathbb{Z}) = 0$ for all $i \neq n$. We will often write $X = M_n(G)$.*

Remark A.58. *A Moore space is a homology analogue of an Eilenberg-Maclane space $K(G, n)$. The sphere $\mathbb{S}^n$ is a Moore space of $\mathbb{Z}$.*

If E is a homology theory, then $E_i(X; G)$ is by definition $E_{i+2}(X \wedge M_2(G))$, which fits into the obvious exact sequence. It is not hard to describe this construction in terms of the spectra defining E.

For example, if E is an infinite loop space representing E^*, then we can use the loop structure to take the degree p self-map of E to itself and take its homotopy fiber. It will obviously deloop using a delooping of E, and defines the cohomology theory $E^{*+1}(\,\cdot\,; \mathbb{Z}_p)$.

Homology with coefficients is useful, but more drastic, in concentrating the problems at a prime p. It loses rational information, but it can give vital information. After all, a map of finite simply connected complexes $f : X \to Y$ is a $\mathbb{Z}_{(p)}$-equivalence iff it is an isomorphism on $\mathbb{Z}_p$-homology. As part of Sullivan's work on the Adams conjecture [626], he introduced the notion of p-adic completion of a space that is even more drastic than localization at p; an arithmetic square explains how it misses the rational information.

A.5 WHITEHEAD TORSION

Definition A.59. *If X is a CW complex, then an elementary expansion of X is a complex Y obtained by glueing $\mathbb{D}^n$ to X along $\mathbb{D}^{n-1}_- \to X$, where $\partial\mathbb{D}^n = \mathbb{S}^{n-1} = \mathbb{D}^{n-1}_+ \cup \mathbb{D}^{n-1}_-$. Here the union is taken over the sphere $\mathbb{S}^{n-2}$. We also say that X is an elementary collapse of Y. A map $f : X \to Y$ of finite CW complexes is a simple homotopy equivalence if it is homotopic to a finite sequence of elementary expansions and collapses.*

Definition A.60. *Let R be a unital ring with identity. Define the abelian group $K_1(R) = \lim_{n\to\infty} \mathrm{GL}_n(R)^{ab}$, where the superscript ab indicates the abelianization of the group in question. The direct limit is based on the sequence of maps $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$ given by $A \mapsto \mathrm{diag}(A, 1)$.*

Let $\mathbb{Z}[G]$ denote the integral group ring of the group G. For each $\gamma \in G$, define $(\pm\gamma)$ to be 1×1 invertible matrices in $\mathrm{GL}_1(\mathbb{Z}[G])$ and therefore in $K_1(\mathbb{Z}[G])$. Then $\mathrm{Wh}(G) = K_1(\mathbb{Z}[G])/(\pm\gamma)$ is called the Whitehead group of G.

If R is a unital ring and $f_* : A_* \to B_*$ is a chain homotopy equivalence of finitely based free R-chain complexes, let (C_*, c_*) be the mapping cone on f_*. Let $d_* : C_* \to C_{*+1}$ be any chain contraction of C_*, i.e. $c_{n+1} \circ d_n + d_{n-1} \circ c_n = id_{C_n}$. If we split C_* into $C_{odd} = \bigoplus_{n\,odd} C_n$ and $C_{even} = \bigoplus_{n\,even} C_n$, there is then an isomorphism $(c_* + d_*)_{odd} : C_{odd} \to C_{even}$. Although it is not well-defined, it is well-defined up to upper triangular matrices. Then define the *torsion* $\tau(f_*) \in \widetilde{K}_1(R)$ to be the class $[A]$, where A is the matrix of $(c_* + d_*)_{odd}$ with respect to the given bases.

The Whitehead group $\mathrm{Wh}(G)$ has been the subject of much research. In particular we have the following results.

1. The Whitehead group of the trivial group is trivial.
2. The Whitehead group of the abelian group $\mathbb{Z}^n$ is trivial (Bass-Heller-Swan [47]).
3. Higman [303] proved that $\mathrm{Wh}(\mathbb{Z}_5) \cong \mathbb{Z}$. An example of a non-trivial unit in the group ring arises from the identity

$$(1 - t - t^4)(1 - t^2 - t^3) = 1$$

   where t is a generator of the cyclic group of order 5. This example is closely related to the existence of units of infinite order in the ring of integers of the cyclotomic field generated by fifth roots of unity.
4. More generally, if p is an odd prime, we have $\mathrm{Wh}(\mathbb{Z}_p) \cong \mathbb{Z}^r$, where $r = \left[\frac{p-3}{2}\right]$.

It is conjectured that $\mathrm{Wh}(G)$ vanishes for any torsion-free group G. An excellent monograph on the calculation of Whitehead groups for finite groups is Oliver [491].

If $f : X \to Y$ is a homotopy equivalence of finite CW complexes with fundamental group G, then there is a particular element in $\mathrm{Wh}(G)$ called the *Whitehead torsion*, which we will now define. This torsion gives the first algebraic invariant of finite complexes that extends beyond the homotopy type.

Definition A.61. *Denote by* $p_* : \widetilde{K}_1(\mathbb{Z}[G]) \to \mathrm{Wh}(G)$ *the natural projection map. If* $h : X \to Y$ *is a homotopy equivalence of connected finite CW complexes with fundamental group G, let* $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ *be the lift of f to the universal covers. If* $\widetilde{f}_* : C_*(\widetilde{X}) \to C_*(\widetilde{Y})$ *is the induced* $\mathbb{Z}[G]$*-chain homotopy equivalence, then we define the Whitehead torsion* $\tau(f)$ *to be the element* $p_*(\tau(\widetilde{f}_*))$ *in* $\mathrm{Wh}(G)$.

Theorem A.62. *(J. Whitehead [707]) A homotopy equivalence* $f : X \to Y$ *has vanishing torsion* $\tau(f)$ *in* $\mathrm{Wh}(\pi_1(Y))$ *iff it is homotopic to a finite sequence of collapses and expansions.*

The Whitehead torsion $\tau(f)$ of a homotopy equivalence $f : X \to Y$ has a number of important properties. We list a few here.

1. Let $f, g : X \to Y$ be homotopy equivalences of finite connected CW complexes. If f and g are homotopic, then $\tau(f) = \tau(g)$.
2. If $f : X \to Y$ is a PL homeomorphism, then $\tau(f) = 0$.
3. If $f : X \to Y$ is a Top homeomorphism of finite connected CW complexes, then $\tau(f) = 0$. This difficult result was proved by Chapman [156].
4. If $f : X \to Y$ and $g : Y \to Z$ are homotopy equivalences of finite connected CW complexes, then $\tau(g \circ f) = g_*\tau(f) + \tau(g)$. Here g_* is the map on Whitehtead groups induced by g.
5. If $f : (X, A) \to (Y, B)$ and $g : (U, A) \to (V, B)$ are homotopy equivalences of pairs and $f|_A = g|_A = h : A \to B$, then $\tau(f \cup_h g) = \tau(f) + \tau(g) - \tau(h)$. Note that this result readily implies an excision statement.
6. If X, Y, Z are all finite complexes, then the Whitehead torsion of a map $f \times id_Z : X \times Z \to Y \times Z$ satisfies $\tau(f \times id_Z) = \chi(Z)\tau(f)$, where $\chi(Z)$ is the Euler characteristic of Z.

A.5.1 The *h*-cobordism theorem

The h-cobordism theorem is one of the most important results in topology. Most homeomorphisms or diffeomorphisms between high-dimensional manifolds are constructed using it. The original simply connected version was proved by Smale as part of his solution of the Poincaré conjecture. Barden, Mazur, and Stallings gave the general non-simply connected version. See Kervaire [352].

Definition A.63. *An* $(n+1)$*-dimensional cobordism* W *between n-dimensional manifolds* M *and* N *is an* $(n+1)$*-dimensional h-cobordism if the inclusion maps* $M \hookrightarrow W$ *and* $N \hookrightarrow W$ *are homotopy equivalences. The h stands for homotopy equivalence.*

Theorem A.64. *(h-cobordism theorem, see Milnor [455]) Let* $n \geq 5$ *and suppose that*

W is a compact $(n+1)$-dimensional h-cobordism between M and N in the category Cat = Diff, PL, or Top. If W, M, and N are all simply connected, then W is Cat isomorphic to the cylinder $M \times [0,1]$. The isomorphism can be chosen to be the identity on $M \times \{0\}$.

Remark A.65. *In lower dimensions, we have the following.*

1. *For $n = 4$, the h-cobordism theorem is true in Top, proved by Freedman [252] using a four-dimensional Whitney trick. But it is false in PL and Diff, as shown by Donaldson [202].*
2. *For $n = 3$, the h-cobordism theorem for smooth manifolds has not been proved and, because of the three-dimensional Poincaré conjecture, is equivalent to the hard open question of the existence of non-standard smooth structures on the 4-sphere.*
3. *For $n = 2$, the h-cobordism theorem is equivalent to the Poincaré conjecture, which was proved by Perelman using Hamilton's Ricci flow.*

If we drop the assumption that M and N are simply connected, then h-cobordisms need not be cylinders. Instead, the obstruction is exactly the Whitehead torsion $\tau(f)$ of the inclusion $f : M \hookrightarrow W$.

The following result is also called the s-cobordism theorem by some authors, where the s stands for simple homotopy equivalence.

Theorem A.66. *(h-cobordism theorem, Mazur, Stallings, Barden) Let $n \geq 5$. Suppose that (W, M, N) is a compact $(n+1)$-dimensional h-cobordism in the category Cat = Diff, PL, or Top with fundamental group G. Then W is Cat isomorphic to the cylinder $M \times [0,1]$ iff the inclusion $i : M \to W$ is a simple homotopy equivalence; i.e. its Whitehead torsion $\tau(i)$ vanishes in* $\mathrm{Wh}(G)$.

Notation A.67. *If (W, M, N) is an h-cobordism, then we can write $\tau(W, M)$ for $\tau(i)$, where $i : M \hookrightarrow W$ is the inclusion map.*

We wish to emphasize the two aspects of Theorem A.66 in the following two theorems, because both halves are remarkable. When the Whitehead group vanishes, it turns homotopy data into geometry, and when the Whitehead group is non-trivial, it gives a way of constructing interesting new manifolds. For example, using the Wall realization theorem, one can easily construct, for all odd primes at least 5, nonlinear free $\mathbb{Z}_p$-actions on any odd-dimensional sphere of dimension at least 4, for example an action that produces a linear lens space.

Theorem A.68. *(Realization theorem) Let $n \geq 5$. Suppose M is a closed Cat n-manifold with fundamental G. If τ_0 is an element in* $\mathrm{Wh}(G)$, *then there is an $(n+1)$-dimensional h-cobordism (W, M, N) such that $\tau(W, M) = \tau_0$.*

Theorem A.69. *(Classification theorem) Let $n \geq 5$. If (W, M, N) and $(W'; M, N')$ are $(n+1)$-dimensional h-cobordisms with $\tau(W, M) = \tau(W'; M)$. Then (W, M, N) is Cat homeomorphic to (W', M, N').*

If (W, M, N) and (W', M', N) are two cobordisms with a common boundary part N, oriented oppositely from each other, we can define

$$(W, M, N) \circ (W', M', N) = (W \cup_N W', M, M')$$

where $W \cup_N W'$ is the union of W and W' glued along N. If (W, M, N) has both a right and left inverse, then we say that it is an *invertible cobordism.* Of course, by realizing $-\tau(W, M)$, one notes, using the h-cobordism theorem, that all h-cobordisms are invertible. The Whitehead torsion τ is additive in the sense that $\tau((W, M) \circ (W', M')) = \tau(W, M) + \tau(W', M')$ in $\mathrm{Wh}(G)$.

Theorem A.70. *Let (W, M, N) be an $(n+1)$-dimensional cobordism of n-dimensional PL manifolds M and N of dimension at least 5. Then $W \backslash N$ is PL isomorphic to $M \times [0, 1)$.*

Proof. Consider

$$W \cup -W \cup W \cup -W \cup W \cup \cdots,$$

where $-W$ is the inverse of the h-cobordism. On the one hand, using the h-cobordism theorem, we have

$$\begin{aligned}(W \cup -W) \cup (W \cup -W) \cup \cdots \\ &\cong (M \times [0, 1]) \cup (M \times [1, 2]) \cup (M \times [2, 3]) \cup \cdots \\ &\cong M \times [0, 1).\end{aligned}$$

On the other, it is also

$$\begin{aligned}W \cup (-W \cup W) \cup (-W \cup W) \cup \cdots \\ &= W \cup (N \times [0, 1]) \cup (N \times [1, 2]) \cup \cdots \\ &= W \cup (N \times [0, 1), \\ &\cong W \backslash N,\end{aligned}$$

as required. ☐

Corollary A.71. *If (W, M, N) is an h-cobordism of dimension ≥ 5, then the following two polyhedra are Top homeomorphic:*

1. *the closed cone on M,*
2. *the union of W and a cone on N.*

The corollary follows from the theorem because the first is the one-point compactification of $M \times [0, 1)$ and the second is the one-point compactification of $W \backslash N$.

A.5.2 The Hauptvermutung conjectures

Definition A.72. *We say that a subset $K \subseteq \mathbb{R}^n$ is a polyhedron if, for every point $x \in K$, there exist finitely many linear simplices $\sigma_1, \dots, \sigma_r \subseteq K$ such that $\bigcup_{i=1}^r \sigma_i$ contains a neighborhood of x. A polyhedron is simply the space described by a simplicial complex.*

Definition A.73. *An abstract simplicial complex K determines a polyhedral topological space $|K|$. If X is a topological space, then a triangulation (K, f) of X is a pair for which K is a simplicial complex and $f : |K| \to X$ is a Top homeomorphism. We say that the topological space X is triangulable if it admits a triangulation (K, f).*

In trying to distinguish the Top category and the PL category, one may ask first whether every Top continuous map of polyhedra is homotopic to a PL map. The simplicial approximation theorem answers this question in the affirmative. See Hatcher [293].

Theorem A.74. *(Simplicial approximation theorem) Suppose that K and L are simplicial complexes and $f : |K| \to |L|$ is a continuous map. Then there is a simplicial subdivision K' of K and a simplicial PL map $g : K' \to L$ such that f is homotopic to the topological realization $g' : |K'| \to |L'|$.*

The simplicial approximation theorem, however, does not prove that every Top homeomorphism of polyhedra is homotopic to a PL homeomorphism.

Conjecture A.75. *(Polyhedral Hauptvermutung) Every Top homeomorphism of polyhedra is homotopic to a PL homeomorphism. In other words, every homeomorphism $f : |K| \to |L|$ between polyhedra is homotopic to the topological realization of a simplicial isomorphism $f' : K' \to L$, where K' and L' are simplicial subdivisions of K and L.*

Remark A.76. *Poincaré had asked this question in order to establish the topological invariance of simplicial homology. Of course, once one has the idea of homotopy invariance and the simplicial approximation theorem, the Hauptvermutung is not necessary for this purpose. However, it is nonetheless a fundamental problem.*

Milnor [451] obtained the first counterexamples to the Polyhedral Hauptvermutung in 1961, using Reidemeister torsion and some results by Mazur [436]. He showed that, when $n \geq 3$, there is no PL homeomorphism between

$$(L(7,1) \times \Delta^n) \cup \operatorname{cone}(L(7,1) \times \mathbb{S}^{n-1})$$

and

$$(L(7,2) \times \Delta^n) \cup \operatorname{cone}(L(7,2) \times \mathbb{S}^{n-1}),$$

where Δ^n is the n-simplex, even though these spaces are Top homeomorphic. Stallings [610] modified the construction and showed how to use any non-trivial h-cobordism to produce a similar counterexample.

Let M be a manifold with fundamental group $\mathbb{Z}_p$, where p is an odd prime at least 5, and let W be an h-cobordism with non-trivial torsion, constructed using the Wall realization theorem. Consider the two polyhedra in Corollary A.71 which are Top homeomorphic but not combinatorially equivalent. We claim they are not PL homeomorphic. If they were, then after some subdivisions of these complexes, this isomorphism would take the cone points to one another, and the stars to each other. It will produce a PL isomorphism of W with $M \times [0, 1]$, contradicting the non-triviality of the torsion of the pair (W, M).

Remark A.77. *It is remarkable that the counterexamples to the Hauptvermutung used Whitehead torsion, since the theorem of Chapman states that the torsion of a homeomorphism is trivial. The secret, of course, is that we are not taking the torsion of a homeomorphism, but of its restriction to a particular PL subset of the polyhedra that would be PL invariant, i.e. the complement of the open star.*

All of these constructions are non-manifold in nature, arising as homeomorphisms of one-point compactifications of open PL manifolds, leaving open the following.

Conjecture A.78. *(Manifold Hauptvermutung) Every homeomorphism $f : |K| \to |L|$ of the polyhedra of compact m-dimensional manifolds is isotopic to a homeomorphism.*

This conjecture is false because of the work of Cannon [110] and Edwards [210] on the double suspension theorem. See also Daverman [185]. If X^n is a PL manifold non-simply connected homology sphere, then the double suspension $\Sigma^2 X$ is a Top manifold and indeed homeomorphic to $\mathbb{S}^{n+2}$. However, it is not a PL manifold because it has 1-simplices with non-simply connected links.

Conjecture A.79. *(PL Manifold Hauptvermutung) If two PL manifolds are Top homeomorphic, then they are PL homeomorphic.*

Remark A.80. *The PL manifold Hauptvermutung is also false; there are PL homotopy tori that are PL distinct, but they are all homeomorphic to the standard torus. The result is a consequence of a beautiful low-dimensional topological result, Rokhlin's theorem.*

As a consequence, Kirby and Siebenmann also identified the obstruction to PL triangulation in $H^4(M; \mathbb{Z}_2)$.

Theorem A.81. *(Kirby-Siebenmann [360]) There is an obstruction $ks(M)$ which is zero in $H^4(M; \mathbb{Z}_2)$ when M has a compatible PL structure.*

We explain this theorem in Chapter 3.

One can ask about the existence of non-PL triangulations of Top manifolds, i.e. whether every compact Top manifold can be triangulated by a locally finite simplicial complex. Casson's work on the Casson invariant shows that Freedman's E_8-manifold in dimension 4 is not triangulable in any sense (see Akbulut-McCarthy [8]). This result also also follows from the three-dimensional Poincaré conjecture.

Remark A.82. *Let Θ_3 be the abelian group under connected sum of homology cobordism classes of oriented PL homology 3-spheres. In dimensions at least 5, Galewski-Stern [260] and Matumoto [432] showed that all topological manifolds of dimension at least 5 have triangulations as polyhedra iff there is a homology 3-sphere Y with Rokhlin invariant 1 which has order 2 in Θ_3. Manolescu [425] proved that such Y does not exist. In particular, he showed that any manifold for which this mod 2 invariant is non-trivial is actually of infinite order in Θ_3, and consequently there are completely nontriangulable manifolds in all dimensions at least 5.*

Appendix B

Geometric preliminaries

B.1 THE SMOOTH CATEGORY

Differential topology has an extremely appealing foundation. The fundamental notions are (1) bundle theory, with explicit classification possible using Grassmanians, (2) transversality, due to Thom, which depends essentially on Sard's theorem, (3) handlebody theory, which is the result of a geometric form of the Morse lemma and is used in the proof of the h-cobordism theorem, and (4) immersion theory, in the sense of Smale-Hirsch. There are many good books for each of these topics, and several that cover most of these topics.

Surgery itself as a process implicitly arises in this list. When one passes through a Morse singularity for a height function on a cobordism, the level sets differ from one another by surgeries. So finding cobordisms is the same as finding sequences of surgeries.

B.1.1 Neighborhoods and immersions

The tubular neighborhood theorem, whose proof uses the existence of Riemannian metrics, the exponential map, and the existence and uniqueness of solutions of ordinary differential equations, proves the existence of a normal bundle structure in a small neighborhood of an embedded or immersed smooth manifold into one of higher dimension.

Theorem B.1. *(Tubular neighborhood, see Hirsch [315]) Let N be a Diff n-manifold and M a Diff m-manifold with $n < m$. An embedding (immersion) $f : N \to M$ extends to a codimension 0 embedding (immersion) $E(\nu_f) \to M$ of the total space of the normal $(m-n)$-dimensional Diff bundle $\nu_f : N \to BO_{m-n}$.*

For classifying spaces, see Appendix A.2.

For a surgery problem we first need to identify a degree one normal map from a manifold and then execute surgery on it in the effort to modify it to a homotopy equivalence. Surgery on Diff manifolds requires an embedded product of a sphere and a disk in M. By the tubular neighborhood theorem, we require a sphere embedded in N with a trivial normal bundle. Such spheres can be constructed by the (strong) Whitney embedding theorem and a bundle calculation, which we explained above in the discussion of the h-cobordism theorem.

Theorem B.2. *(Whitney immersion) Let M^m and N^n be Diff manifolds.*

1. *If $m \leq \frac{n}{2}$ then every map $f : M^m \to N^n$ is homotopic to an immersion $g : M \to N$. In fact, the immersion g can be chosen arbitrarily close to f.*
2. *If $m \leq \frac{n-1}{2}$ then any two homotopic immersions are regular homotopic; i.e. the homotopy is a one-parameter family of immersions.*

Theorem B.3. *(Whitney embedding) Let M^m and N^n be smooth manifolds.*

1. *If $m \leq \frac{n-1}{2}$ then every map $f : M \to N$ is arbitrarily close to an embedding $g : N \to M$.*
2. *If $m \leq \frac{n}{2} - 1$ then any two homotopic embeddings f and g are isotopic; i.e. the homotopy is a one-parameter family of homeomorphisms.*

The proofs of these results use transversality.

Theorem B.4. *(Smale-Hirsch [311]) Suppose that $m < n$ or that M is open with $m = n$. For all maps $f : M^m \to N^n$, there is a homotopy equivalence between $\mathrm{Imm}_f(M, N)$ and the space of fiberwise embeddings $TM \subseteq f^*(TN)$.*

B.1.2 Transversality and handlebody decompositions

Definition B.5. *Let Z be a complex, and suppose that ξ^k is a Diff bundle with base space X embedded in Z. We say that a map $f : N^n \to Z$ from a Diff manifold N^n to Z is transverse to (X, ξ^k) if the preimage $V^{n-k} \equiv f^{-1}(X)$ is a Diff $(n-k)$-submanifold of N^n with Diff normal bundle ν^k, and if there is an embedding of ν^k into a tubular neighborhood of V^{n-k} in N^n such that $f|_{\nu^k} : \nu^k \to \xi^k$ is a Diff bundle map.*

Remark B.6. *Transversality in the smooth category is typically defined using the notion of tangency. Because of the orthogonal relation between the tangent and normal bundles, it is often possible to define concepts related to one in terms of the other. This use of normality to describe transversality is useful when we discuss maps into non-manifolds.*

Remark B.7. *To amplify, sometimes we will map $f : N \to Z$ and will want to approximate f by a map "transverse to $X \subseteq Z$" even if X and Z are not manifolds. It is possible to make approximations if X is a closed subset with a neighborhood that is a vector bundle over X.*

Definition B.8. *Let $\eta : X \to BO_k$ be a k-dimensional Diff bundle over a space X.*

1. *Denote by $D(\eta)$ the disk bundle $\{v \in E(\eta) : ||v|| \leq 1\}$ of η and the sphere*

bundle $S(\eta) = \{v \in E(\eta) : ||v|| = 1\}$. The one-point compactification of $E(\eta)$ is constructed as $T(\eta) = D(\eta)/S(\eta)$, the Thom space of η.

2. *Let N be an n-manifold and let Z be a CW complex. Suppose that a Diff bundle η with base space X is embedded in Z. Consider a map $g : N \to Z$ and the composition $g : N \to Z \to T(\eta)$. Here $Z \to T(\eta)$ is the collapse map that sends every point of $Z \backslash E(\eta)$ to the compactification point of $T(\eta)$. We say that $g : N^n \to T(\eta)$ is transverse at the zero section $X \hookrightarrow T(\eta)$ if $g^{-1}(X)$ is an $(n-k)$-dimensional submanifold M of N and its normal k-dimensional Diff bundle ν_M in N is precisely the pullback bundle $(g|_M)^*\eta : M \xrightarrow{g|_M} X \xrightarrow{\eta} BO_k$. Note that, if such transversality is possible, then there is a pullback bundle map $(f, b) : (M, \nu_M) \to (X, \eta)$.*

The theorem of Sard on regular values can be used to show that the smooth category Diff satisfies transversality; i.e. Z satisfies Diff transversality of every codimension and every CW complex Z.

Theorem B.9. *(Sard-Thom Transversality) Let $\eta : X \to BO_k$ be a k-dimensional Diff bundle over X and let N be a Diff n-manifold. Every continuous map $N^n \to T(\eta)$ is homotopic to a map $g : N \to T(\eta)$ which is transverse at the zero section with a pullback bundle map $(f, b) : (M, \nu_M) \to (X, \eta)$ with $f = g|_M$, where $M \equiv f^{-1}(X)$ is a Diff submanifold of N.*

Recall that an h-cobordism (W, M, M') satisfies the condition that the inclusion maps $M \hookrightarrow W$ and $M' \hookrightarrow W$ are homotopy equivalences. An s-cobordism (W, M, M') requires that the inclusions be simple homotopy equivalences. Smale's proof of the h-cobordism theorem uses the handlebody decomposition of Diff cobordisms guaranteed by Morse theory, which itself is implied by Sard's Theorem. Morse theory studies differentiable manifolds M by considering differentiable functions $f : M \to \mathbb{R}$ whose critical points are all nondegenerate.

A CW complex X is obtained from $\emptyset$ by attaching cells of increasing dimension i; i.e. one can write

$$X = \bigcup_{i=0}^{\infty} (\mathbb{D}^i \cup \mathbb{D}^i \cup \cdots \cup \mathbb{D}^i).$$

Morse theory, in Smale's formulation, proves that an m-manifold M can be obtained from $\emptyset$ by successively attaching handles of increasing index i, giving M the structure of a CW complex, i.e. we can write

$$M = \bigcup_{i=0}^{m} (\mathbb{D}^i \times \mathbb{D}^{m-i} \cup \cdots \cup \mathbb{D}^i \times \mathbb{D}^{m-i}),$$

where the glueings are performed, respecting the boundaries of each $\mathbb{D}^i \times \mathbb{D}^{m-i}$. Note that the cell structure of a CW complex is part of the definition, whereas a handle decomposition of a manifold must be proved.

The basic connection between Morse theory, handles and surgery is the following. If $a < b \in \mathbb{R}$ are regular values of a Morse function $f : M^m \to \mathbb{R}$, then

$$(M[a,b], N_a, N_b) \equiv f^{-1}([a,b], \{a\}, \{b\})$$

is an m-cobordism of $(m-1)$-manifolds. If every $t \in [a,b]$ is a regular value, then each $N_t = f^{-1}(t)$ is diffeomorphic to N_a, with a diffeomorphism

$$(M[a,b], N_a, N_b) \cong N_a \times (I, \{0\}, \{1\}).$$

If $[a,b]$ has regular values except for one critical value of index i, then $(M[a,b], N_a, N_b)$ is the trace of an $(i-1)$-surgery on N_a, with

$$M[a,b] \cong (N_a \times I) \cup (\mathbb{D}^i \times \mathbb{D}^{m-i})$$

obtained from $N_a \times I$ by attaching an i-handle. Therefore M is obtained from $\emptyset$ by attaching an i-handle for each critical value of f with index i, and M has the structure of a finite CW complex with one i-cell for each i-handle.

Definition B.10. *A function $f : M \to \mathbb{R}$ is Morse if its critical points are all nondegenerate. The index of a critical point of a Morse function is the number of negative eigenvalues in the Hessian.*

Theorem B.11. *(Morse): Every m-dimensional manifold M admits a Morse function $f : M \to \mathbb{R}$.*

Proof. (Milnor [453]) There is an embedding $M^m \hookrightarrow \mathbb{S}^{m+k}$ for k large by the Whitney embedding theorem. Let $a \in \mathbb{S}^{m+k} \backslash M$ and define $f_a : M \to \mathbb{R}$ by $f(x) = ||x-a||$. Then f_a is Morse for all a except for a set of measure zero. □

Definition B.12. *In the following all manifolds are in the Diff category.*

1. *Given an $(m+1)$-manifold W with boundary and an embedding $\mathbb{S}^{i-1} \times \mathbb{D}^{m-i+1} \hookrightarrow \partial W$ (where $0 \le i \le m+1$), define the $(m+1)$-manifold with boundary $(W', \partial W')$ obtained from W by attaching an i-handle to be*

$$W' = W \bigcup_{\mathbb{S}^{i-1} \times \mathbb{D}^{m-i+1}} (\mathbb{D}^i \times \mathbb{D}^{m-i+1}).$$

2. *An elementary $(m+1)$-dimensional cobordism of index i is given by the cobordism (W, M, M') obtained from $M \times I$ by attaching an i-handle at $\mathbb{S}^{i-1} \times \mathbb{D}^{m-i+1} \hookrightarrow M \times \{1\}$ with $W = (M \times I) \cup (\mathbb{D}^i \times \mathbb{D}^{m-i+1})$.*
3. *The dual of an elementary $(m+1)$-dimensional cobordism (W, M, M') of index i is the elementary $(m+1)$-dimensional cobordism (W, M', M) of index $(m-i+1)$ obtained by reversing the ends, and regarding the i-handle attached to $M \times I$ as an $(m-i+1)$-handle attached to $M' \times I$.*

4. *A cobordism* (W, M, M') *is trivial if it is diffeomorphic to* $M \times (I, \{0\}, \{1\})$.

Lemma B.13. *For any* $i \in \{0, \dots, m+1\}$*, the Morse function* $f : \mathbb{D}^{m+1} \to \mathbb{R}$ *given by*

$$f(x_1, \dots, x_{m+1}) = -\sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{m+1} x_j^2$$

has a unique interior critical point $0 \in \mathbb{D}^{m+1}$*, which is of index i. For* $\varepsilon \in (0, 1)$*, the* $(m+1)$*-dimensional manifolds with boundary defined* $W_{-\varepsilon} = f^{-1}(-\infty, -\varepsilon]$ *and* $W_\varepsilon = f^{-1}(-\infty, \varepsilon]$ *are related in the following way. The manifold* W_ε *is obtained from* $W_{-\varepsilon}$ *by attaching an i-handle:* $W_\varepsilon = W_{-\varepsilon} \cup \mathbb{D}^i \times \mathbb{D}^{m-i+1}$.

Proposition B.14. *Let* $f : W^{m+1} \to I$ *be a Morse function on an* $(m+1)$*-dimensional manifold cobordism* (W, M, M') *with* $f^{-1}(0) = M$ *and* $f^{-1}(1) = M'$ *with all critical points of* f *in the interior of* W.

1. *If* f *has no critical points then* (W, M, M') *is a trivial cobordism, with a diffeomorphism given by* $(W, M, M') \cong M \times (I, \{0\}, \{1\})$*, which is the identity on* M.
2. *If* f *has a single critical point of index i, then* W *is obtained from* $M \times I$ *by attaching an i-handle using an embedding* $\mathbb{S}^{i-1} \times \mathbb{D}^{m-i+1} \hookrightarrow M \times \{1\}$*, and* (W, M, M') *is an elementary cobordism of index i with a diffeomorphism*

$$(W, M, M') \cong (M \times I \cup \mathbb{D}^i \times \mathbb{D}^{m-i+1}, M \times \{0\}, M').$$

Proof. We give a proof for (ii). In a neighborhood of the unique critical point $p \in W$, we have

$$f(p + (x_1, \dots, x_{m+1})) = f(p) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{m+1} x_j^2$$

with respect to a coordinate chart $W \hookrightarrow \mathbb{R}^{m+1}$ such that $0 \in \mathbb{R}^{m+1}$ corresponds to $p \in W$ and $c = f(p) \in \mathbb{R}$ is the critical value. For any $\varepsilon > 0$, there are diffeomorphisms $f^{-1}(-\infty, c - \varepsilon] \cong M \times I$ and $f^{-1}[c + \varepsilon, \infty) \cong M' \times I$. By Lemma B.13, there is a diffeomorphism

$$f^{-1}[c - \varepsilon, c + \varepsilon] \cong (M \times I) \cup (\mathbb{D}^i \times \mathbb{D}^{m-i+1}).$$

□

Theorem B.15. *(Handle decomposition, Smale)*

1. *Every cobordism* $(W^{m+1}; M, M')$ *has a handle decomposition as the union of a finite sequence*

$$(W, M, M') = (W_1, M_0, M_1) \cup (W_2, M_1, M_2) \cup \cdots \cup (W_k, M_{k-1}, M_k),$$

where each (W_j, M_{j-1}, M_j) *is an elementary cobordism with index* i_j*, such that*

$0 \le i_1 \le \cdots \le i_k \le m+1$. *Additionally, we have* $M_0 = M$ *and* $M_k = M'$.

2. *Two closed m-manifolds* M *and* M' *are cobordant iff* M' *can be obtained from* M *by a sequence of surgeries.*

Proof. The proofs go as follows.

1. Any cobordism admits a Morse function $f : (W, M, M') \to I$ where $M = f^{-1}(0)$ and $M' = f^{-1}(1)$, such that all the critical points are in the interior of I. Since W is compact, there are only finitely many critical points p_j, where $1 \le j \le k$. Let the critical values by $c_j = f(p_j) \in \mathbb{R}$, and let i_j be the index of p_j. Choose f such that $0 < c_1 < c_2 \cdots < c_k < 1$ and $0 \le i_1 \le i_2 \le \cdots \le i_k \le m+1$. Let $r_j \in I$ be regular values such that

$$0 = r_0 < c_1 < r_1 < c_2 < \cdots < r_{k-1} < c_k < r_k = 1.$$

Then each $(W_j; M_{j-1}, M_j) = f^{-1}([r_{j-1}, r_j], \{r_{j-1}\}, \{r_j\})$ is an elementary cobordism of index i_j.

2. The trace of a surgery is an elementary cobordism, so surgery-equivalent manifolds are cobordant. Conversely, every elementary cobordism is the trace of a surgery, and by (a) every cobordism is a union of elementary cobordisms. □

Remark B.16. *If* (W, M, M') *has a Morse function* $f : W \to I$ *with critical points of index* $0 \le i_0 \le i_1 \le \cdots \le i_k \le m+1$, *then* W *has a handle decomposition*

$$W = (M \times I) \cup (h^{i_0} \cup \cdots \cup h^{i_k})$$

with $h^i = \mathbb{D}^i \times \mathbb{D}^{m-i+1}$ *a handle of index i.*

Corollary B.17. *Every closed m-manifold* M^m *can be obtained from* $\emptyset$ *by attaching handles. A Morse function* $f : M \to \mathbb{R}$ *with critical points of index* $0 \le i_0 \le i_1 \le \cdots \le i_k \le m+1$ *determines a handle decomposition* $M = h^{i_0} \cup \cdots \cup h^{i_k}$ *so that* M *is a finite m-dimensional CW complex with one i-cell for each critical point of index i.*

B.2 THE PL CATEGORY

The piecewise linear category is much less familiar, but it is considerably more elementary. The required ideas for piecewise linear (PL) surgery are essentially the same as for the smooth category, but with more combinatorial definitions and frequently for simple reasons. For example, there is no need for Sard's theorem: the existence of "regular values" relies merely on linear algebra. The most important change is in the bundle theory: one must replace the idea of bundle with that of a block bundle (see Rourke-Sanderson [557]). These block bundles have a more subtle theory of pullbacks, and a much less visualizable classifying space. Tubular neighborhoods are replaced by regular neighborhoods: they are combinatorially defined, and can be defined even for non-manifolds. The analogue of Smale-Hirsch theory is due to Haefliger-Poenaru [280]. This appendix reviews some of the basic information about the PL category. We will see that PL behaves geometrically just like Diff, although the Poincaré conjecture is true in dimensions at least 5 because of coning, i.e. the Alexander trick of radially extending homeomorphisms of the sphere to the ball, which makes F/PL so fantastically computable. See Chapter 3.4. We recommend the books by Rourke-Sanderson [563] and Hudson [325].

In terms of classifying spaces, other beautiful features arise because of F_c/PL_c stability, which is a reflection of the unknotting of spheres in higher-dimensional spheres, where the codimension at least 3. This stability implies the existence of many PL embeddings in much lower codimensions than is possible in Diff. The main hurdle is the computation of bordism theory, which is much more complicated than smooth bordism. See Madsen-Milgram [420].

The relevant theorems in the PL category that make the PL surgery exact sequence possible were largely established by Rourke and Sanderson in a sequence of papers on block bundles and their properties (see [557], [558], and [559]).

PL topology begins with the notion of a *Euclidean polyhedron* P, i.e. a compact subset L of $\mathbb{R}^n$ such that $N = aL$ is a cone neighborhood of a in P. We assume inductively that L locally has this structure, so inductively any such P amounts to a union of simplices. Here N is the *star* of a in P and L is a *link* of a, the symbol aL signifies the union of all the line segments connecting a with points of L. A map $f : P \to Q$ between polyhedra is *piecewise linear* if, for all $a \in P$, there is a link L such that

$$f(\lambda a + (1 - \lambda)x) = \lambda f(a) + (1 - \lambda)f(x)$$

for all $x \in L$ and $\lambda \in [0, 1]$. If X is a topological space, then we can use "coordinate charts" $f : P \to X$ from a polyhedron with the usual compatibilities for a *piecewise linear structure on X*. With this structure, we say that X is a *piecewise linear space*. We will often use the abbreviation PL to mean piecewise linear.

If K is a simplicial complex, we often express its underlying polyhedron as $|K|$. One can speak about a *simplicial map* $f : K \to L$ as a continuous map $f : |K| \to |L|$ which maps vertices of K to vertices of L and simplices of K linearly onto simplices of L. A simplicial map is piecewise linear and is determined by its values on the vertices of K.

It should be clear how to define a piecewise linear manifold. A *piecewise linear m-ball* is a polyhedron which is PL homeomorphic to an m-simplex. A *piecewise linear m-sphere* is a polyhedron which is PL homeomorphic to the boundary of an $(m + 1)$-simplex. A *PL manifold of dimension m* is a Euclidean polyhedron M in which every point has a (closed) neighborhood which is a piecewise linear m-ball.

There are polyhedra which are not PL manifolds but are topological manifolds. In such examples there are points where links are not PL spheres.

To understand the notion of a regular PL neighborhood, we need to define the notion of a *collapse*. Suppose that $P_0 \subseteq P$ are Euclidean polyhedra such that $B = \mathrm{cl}_P(P - P_0)$ is a PL ball for which $B \cap P_0$ is a face. Then we say that *P collapses to P_0 by an elementary collapse*, denoted by $P \searrow_e P_0$. We say that *P collapses to the subpolyhedron P_0* and write $P \searrow P_0$ if there is a finite sequence $P = P_r \searrow_e P_{r-1} \searrow_e \cdots \searrow_e P_0$. If X is a polyhedron contained in the PL m-manifold M, then $N \subseteq M$ is called a *regular neighborhood of X in M* if (1) N is closed neighborhood of X in M, (2) N is a piecewise linear m-manifold, and (3) $N \searrow X$.

Regular neighborhoods always exist. See Rourke-Sanderson [560] and Hudson [324]. In fact, the regular neighborhood theorem states that, if N_1 and N_2 are regular neighborhoods of X in Y, then there is a PL homeomorphism $h : Y \to Y$ with $h(N_1) = N_2$ and h is the identity on X and outside of some compact subset of Y.

One can define $N(X)$ to be the union of all the simplices of Y that intersect X, perhaps after barycentric subdivision.

Definition B.18. *Let K be a locally finite simplicial complex. A q-block bundle $\xi^q \downarrow K$ consists of a total PL space $E(\xi)$ containing K such that*

1. *for each n-simplex σ_i in K, there is a PL $(n+q)$-ball $\beta_i \subseteq E(\xi)$, called the block over σ_i, such that (β_i, σ_i) is an unknotted ball pair, i.e. PL homeomorphic to the standard pair $(\mathbb{D}^{n+q}, \mathbb{D}^n)$;*
2. *the total space $E(\xi)$ is the union of the balls β_i;*
3. *for all i, j, k, if $\sigma_i \cap \sigma_j = \sigma_k$ (or $\emptyset$), then $\beta_i \cap \beta_j = \beta_k$ (or $\emptyset$).*

As usual there is a notion of *isomorphic q-block bundles* over K, and the collection of the associated isomorphism classes is called $I_q(K)$. This $I_q(K)$ is invariant under subdivision of K. As expected there is a classifying space for such bundles.

Theorem B.19. *There is a locally finite simplicial complex BPL_q such that $I_q(X)$ is in bijective correspondence with $[X : BPL_q]$. In addition, there is a q-block bundle $\gamma^q \downarrow BPL_q$ that provides a natural equivalence $T : [\,* : BPL_q] \to I_q(*)$ of functors given by $T[f] = f^*\gamma^q$.*

Theorem B.20. *(Existence of normal block bundles) Let $M^n \subseteq Q^{n+q}$ be a proper inclusion of a compact PL submanifold M in a regular neighborhood Q, and suppose that $(M, \partial M) = (|K|, |L|)$, where L is a subcomplex of K. Then there is a q-block bundle $\xi \downarrow K$ with a PL homeomorphism $E(\xi) \to Q$. If U is a regular neighborhood of ∂M in ∂Q and $\eta^q \downarrow L$ is a q-block bundle with $E(\eta) = U$, we can choose ξ such that $\xi|_L = \eta$.*

Using block bundles, we can define transversality and prove a transversality theorem.

Definition B.21. *Let $N, M \subseteq Q$ be proper submanifolds of Q and let ξ be a blocked regular neighborhood of M in Q. We say that N and M are transverse with respect to ξ, notated $N \perp_\xi M$, if there is a subdivision ξ' of ξ such that $E(\xi' \downarrow N \cap M) = E(\xi) \cap N$.*

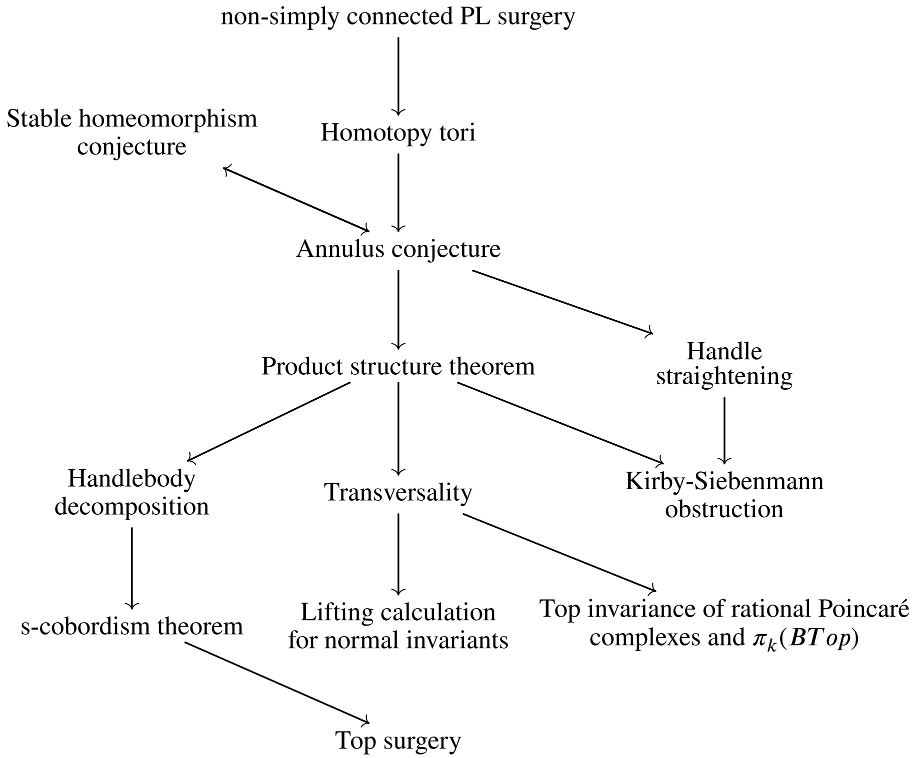
Theorem B.22. *(Transversality, Williamson [711]) Let $\xi, N, M,$ and Q be as above. Then there is an ε-isotopy of Q producing a submanifold N' that is transverse with respect to ξ. If ∂N and ∂M are transverse with respect to $\xi|_{\partial M}$, then the isotopy can be chosen relative to ∂Q.*

Remark B.23. *Note that map transversality can be obtained from this theorem for $f : N \to Q$ by making* Graph(f) *transverse to $N \times M$ in $N \times Q$.*

B.3 THE TOPOLOGICAL CATEGORY

Topological manifolds have a beautiful theory for a number of reasons. The classifying space F/Top has a periodicity that F/PL lacks. It is the category of manifolds in which rigidity is strongest. Surgery theory has a much more natural description in the Top category, as we saw in Chapter 4. On the other hand, it is more difficult to lay the foundations of the theory.

The work of Kirby-Siebenmann [361], along with missing cases finished by Quinn [522]), showed that the Top category can be endowed with a handlebody theory, an h-cobordism theorem, transversality, and other necessary ingredients. The proof for Top surgery actually depends on the PL surgery theory (or controlled topology) through the classification of homotopy tori. The mechanism for learning about topological manifolds from facts about tori is Kirby's torus trick. We recommend reading Kirby-Siebenmann [361] for the detailed story, or just taking the previous paragraph as a provisional summary for a first attempt at understanding the shape of the subject.



When one arrives at stratified spaces or locally smooth topological group actions, analogous statements about handlebody structures and transversality may actually not hold. These topological theorems, as beautiful as they are, require some deep calculational inputs.

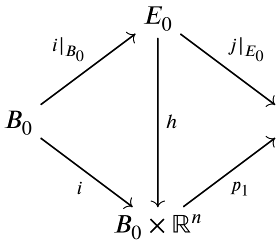
This short appendix is a brief overview of some of the structure of the Kirby-Siebenmann theory. We will not review the analogous and simpler theory of smoothing PL manifolds, due to Hirsch and Mazur, partly because it is not necessary for establishing the basic theorems of PL.

Top surgery requires a certain amount of difficult machinery. In this section we will discuss the dependencies in the following diagram, whose elaboration is a large part of Kirby and Siebenmann's book [361].

B.3.1 Top bundle theory

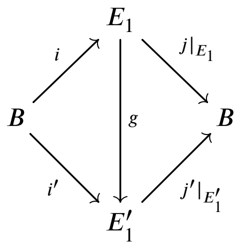
We begin with a discussion of Milnor's Top microbundles, which are somewhat less intuitive than bundles. Microbundles are required in discussing smoothings, immersions, and embeddings.

Definition B.24. *Let B and E be locally finite simplicial complexes. Then ξ is a Top microbundle of dimension n if there is a composition $B \xrightarrow{i} E \xrightarrow{j} B$ of maps that equals the identity map i_B such that, for each $b \in E$, there are neighborhoods B_0 of b and E_0 of $i(b)$ with a Top homeomorphism $h: E_0 \to B_0 \times \mathbb{R}^n$ such that the diagram*



commutes. Here the map i is the obvious inclusion and p_1 is the obvious projection.

Definition B.25. *Suppose we have two Top microbundles $\xi : B \xrightarrow{i} E \xrightarrow{j} B$ and $\xi : B \xrightarrow{i'} E' \xrightarrow{j'} B$ over the same space B. Then they are Top isomorphic (i.e. $\xi \approx \xi'$) if there are neighborhoods E_1 of $i(B)$ and E_1' of $i'(B)$ with a Top homeomorphism $g : E_1 \to E_1'$ such that the diagram*



commutes. We also say that ξ and ξ' are Top micro-identical.

Example B.26. *There are a few canonical examples of Top microbundle structures.*

1. *For any B and $n \geq 0$, the trivial Top n-microbundle ε_B^n is the composition $B \xrightarrow{i} B \times \mathbb{R}^n \xrightarrow{p_1} B$ given by inclusion and projection. Any Top bundle isomorphic ξ^n to ε_B^n is also called a trivial Top n-microbundle.*
2. *Any vector bundle or block bundle is a Top microbundle.*
3. *If M is a Top n-manifold, then the tangent Top microbundle τ_M of M is given by the composition $M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M$, where Δ is the diagonal map.*

As in the case with Diff bundles, there are notions of the Whitney sum $\xi_1 \oplus \xi_2$ of two Top microbundles ξ_1^n and ξ_2^m, a restricted Top microbundle $\xi|_{B_0}$ on a subcomplex $B_0 \subseteq B$ for all Top microbundles ξ on B, and a Top microbundle $f^*\xi$ formed as the pullback of *the diagram*

$$\begin{array}{ccc} f^*\xi & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow[f]{} & B \end{array}$$

In fact, two homotopic maps $f, g : B_1 \to B$ yield isomorphic Top microbundles $f^*\xi$ and $g^*\xi$ over B_1. See I §11 of Steenrod [616].

As in the Diff category, there is a notion of a universal Top bundle.

Theorem B.27. *(Universal microbundle theorem) For all $n \geq 1$ there is a Top microbundle $\gamma^n : BTop_n \to ETop_n \to BTop_n$ such that, for any locally finite complex B and n-dimensional microbundle ξ over B, there is a unique homotopy class of maps $f : B \to BTop_n$ such that $f^*\gamma^n \approx \xi$.*

This result implies the main theorem of smoothing theory.

Theorem B.28. *A Top manifold M^n with $n \geq 5$ is smoothable iff there is a lift of $M \to BTop_n$ to BO_n.*

The proof of this theorem, however, requires the annulus conjecture and more. Consequently, it is true for connected open 4-manifolds, but not for topological 4-manifolds, as shown by the work of Donaldson and Freedman.

Definition B.29. *Let $i : M \to N$ be an inclusion map of Top manifolds. Suppose that there is a neighborhood U of M in N and a retraction $j : U \to M$ so that $\nu : M \xrightarrow{i} U \xrightarrow{j} M$ is a Top microbundle over M. We call ν a Cat normal microbundle in N over M. If ν is a trivial bundle, then we say that M has a trivial normal microbundle in N.*

Theorem B.30. *Let $i : M \to N$ be an inclusion of Top manifolds.*

1. *There is $q \in \mathbb{Z}_{\geq 1}$ such that $M \times \{0\} \to N \times \mathbb{R}^q$ has a normal microbundle.*
2. *If M has a normal microbundle ν in N, then the Whitney sum $\tau_M \oplus \nu$ is isomorphic to $\tau_N|_M$.*

Remark B.31. *Remarkably, according to Kister [363], microbundles are bundles. On the other hand, at this point of our discussion, we have not built enough machinery even to classify topological bundles over the circle. This problem is essentially equivalent to the annulus conjecture, which we now discuss.*

B.3.2 The annulus conjecture

At the point when Kirby proved the annulus conjecture, it was recognized to be the key to developing a theory of topological manifolds, although the precise nature of that theory was not clear at that time. For dimensions at least 6, we need to prove the product structure theorem, which we describe in the next section. The proof of the annulus conjecture in Kirby [358] relies on the principal theorems of non-simply-connected surgery. Essentially, Kirby's theorem shows that all topological manifolds are stable in the sense of Brown and Gluck [99]; i.e. their coordinate charts can be glued together by homeomorphisms that are somewhere linear. This definition is similar to the one arising in Quinn's work on homology manifolds. See Section 8.6.

Kirby used the methods of Hsiang-Shaneson [321] and Wall [666] in the study of homotopy tori, along with the Poincaré conjecture and the end theorem of Browder-Livesay-Levine [91], to prove the following.

Theorem B.32. *(Stable homeomorphism conjecture, Kirby [358]) We say that a Top homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ is stable if it can be written as a finite composition $h = h_1 \circ h_2 \circ \cdots \circ h_r$ of Top homeomorphisms, where each h_i is the identity on some open set U_i of $\mathbb{R}^n$. For all $n \geq 5$, all orientation-preserving homeomorphisms of $\mathbb{R}^n$ are stable.*

Proof. The statement can be proved using Kirby's Torus trick. Let $h\colon \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism. Immerse the punctured torus $\mathbb{T}_0 = \mathbb{T}^n \backslash \{pt\}$ into $\mathbb{R}^n$. Then h pulls back the standard PL or smooth structure to a new one on $\mathbb{T}_0$. The end of $\mathbb{T}_0$ with this structure is still PL equivalent to $\mathbb{S}^{n-1} \times \mathbb{R}$, and therefore $\mathbb{T}_0$ compactifies to a PL manifold $\mathbb{T}'$ that, after glueing in a disk, is homotopy equivalent to the standard torus $\mathbb{T}^n$. Since PL equivalences are stable, if this new manifold is PL equivalent to $\mathbb{T}^n$, we can then compare the universal cover of $\mathbb{T}'$ to h and see that h is PL on some open set, and is therefore stable.

More specifically, let $h\colon \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism and let $\beta\colon \mathbb{T}^n \backslash \{pt\} \to \mathbb{R}^n$ be a PL immersion which restricts to the identity B^n. Let W_0 be the pullback of β over h with natural maps $\beta'\colon W \to \mathbb{R}^n$ and $h_0\colon W_0 \to \mathbb{T}^n \backslash \{pt\}$. The map β' is an immersion so W_0 is a PL manifold with PL coordinate patches obtained by restricting β'. Since $\beta|_B = id$, it follows that $\beta'|_{h^{-1}(B)}$ can be considered as the identity map. Therefore h_0 is a Top homeomorphism which is identified with h on $h^{-1}(B)$. The end of W_0 is homeomorphic to $\mathbb{S}^{n-1} \times [0, \infty)$, so it is tame. The theorem of Browder-Livesay-Levine, i.e. Siebenmann's thesis for simply connected ends, implies that W_0 admits a PL boundary for $n \geq 6$. By the generalized Poincaré conjecture, this boundary must be a PL sphere. For $n = 5$, we use a theorem of Wall [666] that a PL manifold which is homeomorphic to $\mathbb{S}^4 \times \mathbb{R}$ is PL homeomorphic to $\mathbb{S}^4 \times \mathbb{R}$. The map h_0 can be extended to a homeomorphism by coning $h_1\colon W_1 \to \mathbb{T}^n$. We now apply Hsiang-Shaneson-Wall to conclude that, after passage to a finite cover, there is a PL homeomorphism $g\colon \widehat{W}_1 \to \mathbb{T}^n$ which is homotopic to $\widehat{h}_1^{-1}$. By lifting to the universal covers, we have the composition

$$\mathbb{R}^n \xrightarrow{\widetilde{g}^{-1}} \widetilde{W} \xrightarrow{\widetilde{h}_1} \mathbb{R}^n.$$

The homeomorphism $\widetilde{g}^{-1}$ is PL, and therefore stable. The homeomorphism $\widetilde{h}_1 \circ \widetilde{g}^{-1}$ is bounded because it covers a homeomorphism of $\mathbb{T}^n$ that is homotopic to the identity. Therefore, the homeomorphism $\widetilde{h}_1$ is stable. But $\widetilde{h}_1$ agrees with h on $h^{-1}(B)$, so h is stable. □

We now discuss the annulus conjecture. The general Schoenflies theorem (Brown [97], Mazur [435]) states that, if an $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ is embedded into the standard n-sphere $\mathbb{S}^n$ in a locally flat way, i.e. extends to that of a thickened sphere $\mathbb{S}^{n-1} \times [-\varepsilon, \varepsilon]$, then the pair $(\mathbb{S}^n, \mathbb{S}^{n-1})$ is Top homeomorphic to the pair $(\mathbb{S}^n, \mathbb{S}^{n-1})$, where $\mathbb{S}^{n-1}$ is the equator of $\mathbb{S}^n$. The annulus conjecture (now theorem) seems at first to be a small variant of the generalized Schoenflies theorem to a pair of spheres, but it is actually much deeper.

The annulus conjecture was proved in dimension 2 by Radó [528] and in dimension 3 by Moïse [468] and Bing [57]. The verification of the stable homeomorphism conjecture in all higher dimensions by Kirby [358] proves the annulus conjecture for $n \geq 5$. Subsequently, using controlled ideas and the methods of Freedman, Quinn [521] completed the problem in dimension 4.

Theorem B.33. *(Annulus conjecture) If $f, g : \mathbb{S}^{n-1} \to \mathbb{R}^n$ are disjoint with locally flat embeddings such that $f(\mathbb{S}^{n-1})$ is contained in the bounded component of $\mathbb{R}^n - g(\mathbb{S}^{n-1})$, then the closed region A bounded by $f(\mathbb{S}^{n-1})$ and $g(\mathbb{S}^{n-1})$ is homeomorphic to the annulus $\mathbb{S}^{n-1} \times [0, 1]$.*

Ultimately the work of Kirby and Siebenmann enables all of surgery to be developed. It is immediate that the annulus conjecture is critical: the very beginning of surgery is surgery of the 0-sphere $\mathbb{S}^0$ to make manifolds connected, i.e. form connected sums. The reason that there is a well-defined connected sum of two oriented topological manifolds is precisely the annulus conjecture.

B.3.3 The product structure theorem

Reductions of problems involving Top manifolds to bundle theory often require the product structure theorem, which relates the concordance classes of PL structure on a Top manifold M with those on $M \times \mathbb{R}$.

Theorem B.34. *(Product structure theorem) Let M^n be a Top manifold with $n \geq 5$.*

1. *(Existence) Let Σ be a PL structure on $M \times \mathbb{R}$. Then there is a PL structure Γ on M such that $\Sigma \sim \Gamma \times \mathbb{R}$.*
2. *(Uniqueness) If Γ_0 and Γ_1 are PL structures on M, and if $\Gamma_0 \times \mathbb{R} \sim \Gamma_1 \times \mathbb{R}$, then $\Gamma_0 \sim \Gamma_1$.*

Corollary B.35. *Let M^n be a Top manifold with $n \geq 5$. If $\mathcal{T}^{PL}(M)$ denotes the collection of concordance classes of PL structures on M, then the sets $\mathcal{T}^{PL}(M)$ and $\mathcal{T}^{PL}(M \times \mathbb{R})$ are in bijective correspondence.*

Remark B.36. *Following immediately from Donaldson [202], the product structure theorem fails if M^n is a 4-manifold.*

B.3.4 Transversality and handlebody decompositions

To complete our overview of the foundations of the theory of topological manifolds, we now state transversality and the existence of handlebody structures. With these tools, almost all geometric arguments from the smooth or PL categories can be transferred to the topological category.

Transversality is handled by the following. We have a Top map

$$f : (M^m, f^{-1}(X)) \to (Y^m, X^n)$$

between Top pairs m-manifolds, where X is a closed submanifold of Y. We assume that there is a normal Top microbundle ξ^n of X in Y, and in addition a normal Top microbundle ν of $f^{-1}(X)$ in M such that f embeds each fiber of ν into some fiber of ξ. In this case, we say that f is *Top transverse to ξ at ν*. We can also easily define *transversality on an open set $U \subseteq M$*, or *near a set $C \subseteq M$*.

One should however exercise some care. Normal bundles for Top submanifolds often fail to exist or fail to be unique up to isotopy. Milnor however showed that the stable normal bundle exists in a unique manner (see Novikov [484]). Marin [428] shows that the stable microbundle transversality is the correct condition. The following theorem demonstrates that topological transversality is possible. See Quinn [526] and Freedman-Quinn [254] for the missing dimension of the following result from Kirby-Siebenmann [361].

Theorem B.37. *(Topological transversality theorem) Let C and D be closed subsets of a (metrizable) Top m-manifold M and let U and V be open neighborhoods of C and D, respectively. Let ξ^n be a normal n-microbundle to a closed subset X of a space Y. Suppose that $f : M \to Y$ is a continuous map Top transverse to ξ on U at ν_0. Suppose $m \neq 4 \neq m-n$, and either $\partial M \subseteq C$ or $m-1 \neq 4 \neq m-1-n$. Then there is a homotopy $f_t : M \to Y$ with $t \in [0,1]$ of $f_0 = f$ fixing a neighborhood of $C \cup (M - V)$ so that f_1 is transverse to ξ on an open neighborhood of $C \cup D$ at a microbundle ν equal to ν_0 near C. Furthermore, if Y is a metric space with metric d, and $\varepsilon : M \to (0,\infty)$ is continuous, then we can require that $d(f_t(x), f(x)) < \varepsilon$ for all $x \in M$ and $t \in [0,1]$.*

Unlike Diff theory in which handlebody structures come from Morse theory, and unlike PL theory in which they come from regular neighborhood theory, in Top theory they are constructed in a more roundabout way.

Definition B.38. *Let W be an m-dimensional Top manifold with an m-dimensional submanifold $M \subseteq W$. A handlebody decomposition of W on M is a filtration $M = M_0 \subset M_1 \subset M_2 \subset \cdots$ with $\bigcup_{i=1}^{\infty} M_i = W$ of submanifolds such that*

1. *the closure $H_i = \mathrm{cl}(M_i - M_{i-1})$ is a clean compact submanifold of M_i in the sense that it meets the (possibly empty) boundary transversally;*
2. *there is an isomorphism $(H_i, H_i \cap M_{i-1}) \cong (\mathbb{D}^k, \partial\mathbb{D}^k) \times \mathbb{D}^{m-k}$ for some $k \in \{0, \dots, m\}$.*

These H_i are called handles, and it is assumed that the collection $\{H_i\}$ of handles is locally finite; i.e. each compact subset of W meets only finitely many handles. If a filtration is present, we say that W is a handlebody on M.

Remark B.39. *The filtration may possibly be infinite in the noncompact case.*

Theorem B.40. *Let $m \geq 6$. If W is a Cat m-manifold and M is a clean Cat m-submanifold of W, then W can be described as a Cat handlebody on M. In particular, handlebody decompositions exist for all Top manifolds of dimension at least 6.*

Remark B.41. *Not all 4-manifolds have handlebody structures. For example, Freedman's E_8-manifold does not have one. Furthermore, not all locally smooth G-manifolds have equivariant handlebody structures; the product stucture theorem also fails in this setting. See Steinberger-West [618] and Quinn [521].*

With these tools the s-cobordism theorem can be proved in Top, as well as Farrell fibration and the splitting principle of Farrell-Hsiang [222]. See Lees [394] for the relevant immersion theory. Surgery of compact Top manifolds as formulated by Wall can be executed for Top manifolds using the tools of Top handlebody theory. The chief technical problem is to make transverse the self-intersections of a framed Top immersion $f : \mathbb{S}^k \times \mathbb{R}^k \to M^{2k}$ when $k \geq 3$, and then apply the Whitney lemma to find a regular homotopy of f to an embedding when Wall's self-intersection coefficient is zero. With transversality available, the situation is no different than the PL case.

Remark B.42. *Once transversality and handlebody structures are available, the surgery is basically the same as the PL case. However, after all of the great effort showing that Top manifolds behave more or less like smooth manifolds, one should realize that there are serious differences when one moves to more complicated situations, like group actions. The equivariant product structure theorem fails, and while there is a reasonable smooth equivariant handlebody theorem, there is no topological variant.*

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